KADISON'S SCHWARZ INEQUALITY AND FURUTA'S THEOREM

Masatoshi Fujii, Masahiro Nakamura and Yuki Seo*

Received March 16, 2006

Abstract. We make an interpretation of n-terms arithmetic-harmonic mean inequality via Kadison's Schwarz inequality. It is inspired by a skillful proof of it due to T.Furuta. Certain allied topics are also discussed.

1. Introduction. The arithmetic-harmonic mean inequality says that

$$\frac{1}{n}\sum_{i=1}^{n} A_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} A_i^{-1}\right)^{-1}. \tag{1}$$

holds for positive operators $A_1, \cdots, A_n > 0$, i.e Recently, Furuta [3] showed one-line proof of the harmonic-arithmetic operator mean inequality with weight $\lambda = (\lambda_i)$:

Theorem 1. *Let A_i be positive invertible operators on a Hilbert space H for $i = 1, 2, \cdots, n$ and let $\lambda = (\lambda_i)$ be a weight, i.e., $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then*

$$A = \sum_{i=1}^{n} \lambda_i A_i \geq H = \left(\sum_{i=1}^{n} \lambda_i A_i^{-1}\right)^{-1} \geq 0. \tag{2}$$

As a matter of fact, he proposed the following beautiful equality:

$$A - H = \sum_{i=1}^{n} (1 - HA_i^{-1})\lambda_i A_i (1 - A_i^{-1}H). \tag{3}$$

Moreover, based on this, he gave a short proof of the following reverse inequality

$$\frac{(M+m)^2}{4Mm}\left(\sum_{i=1}^{n} \lambda_i A_i^{-1}\right)^{-1} \geq \sum_{i=1}^{n} \lambda_i A_i \tag{4}$$

if $0 < m \leq A_i \leq M$ for $i = 1, 2, \cdots, n$

On the other hand, Izumino privately informed us that the Anderson-Morley-Trapp formula

$$\frac{A+B}{2} - \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} = \frac{1}{2}(A-B)(A+B)^{-1}(A-B)$$

is available to a proof of (1) in the case of $n = 2$.

By the way, Bhagwat and Subramanian [1] presented a proof of (1). So we follow them to prove the weighted version (2) of (1): Put $A = \sum_{i=1}^{n} \lambda_i A_i$ and $H = \left(\sum_{i=1}^{n} \lambda_i A_i^{-1}\right)^{-1}$ and moreover

$$S_i = (H^{-1} - A_i^{-1})(\lambda_i A_i)(H^{-1} - A_i^{-1}) \text{ for } i = 1, \cdots, n.$$

2000 *Mathematics Subject Classification.* 47A63, 47A64 and 47B15.

Key words and phrases. Kadison's Schwarz inequality, Kantorovich inequality, noncommutative Kantorovich inequality and operator geometric mean.

Then $R = \sum S_i \geq 0$ and so

$$A - H = HRH \geq 0.$$

Incidentally, we note that it connects with (3). Actually, we have

$$\begin{aligned} A - H = HRH &= \sum_{i=1}^{n} HS_iH \\ &= \sum_{i=1}^{n} H(H^{-1} - A_i^{-1})(\lambda_i A_i)(H^{-1} - A_i^{-1})H \\ &= \sum_{i=1}^{n} (1 - HA_i^{-1})\lambda_i A_i(1 - A_i^{-1}H). \end{aligned}$$

2. Kadison's Schwarz inequality. We here apply an operator-valued inner product in $B(H) \times \cdots \times B(H)$, where $B(H)$ is the algebra of all operators acting on a Hilbert space H:

$$\langle (A_1, \cdots, A_n), (B_1, \cdots, B_n) \rangle = \sum_{i=1}^{n} B_i^* A_i.$$

In particular, for given a weight $\lambda = (\lambda_i)$, we define

$$\Phi(A_1 \oplus \cdots \oplus A_n) = \langle (A_1, \cdots, A_n), (\lambda_1, \cdots, \lambda_n) \rangle = \sum_{i=1}^{n} \lambda_i A_i. \tag{5}$$

Then Φ is a unital positive linear map on $\sum_{i=1}^{n} \oplus B(H)$.

We now cite Kadison's Schwarz inequality, see [4, Theorem 1.17]:

Theorem K. *Let Φ be a unital positive linear map of $B(H)$ into $B(K)$. Then*

$$\Phi(A^2) \geq \Phi(A)^2 \tag{6}$$

for all selfadjoint operators $A \in B(H)$ and

$$\Phi(A^{-1}) \geq \Phi(A)^{-1} \tag{7}$$

for all positive invertible $A \in B(H)$.

If Φ is as in (5), then Theorem K implies the following inequalities, in which the second one shows the arithmetic-harmonic mean inequality (2).

Theorem 2. *Let $\lambda = (\lambda_i)$ be as in Theorem 1. Then*

$$\sum_{i=1}^{n} \lambda_i A_i^2 \geq \left(\sum_{i=1}^{n} \lambda_i A_i \right)^2 \tag{8}$$

for all selfadjoint operators $A \in B(H)$ and and

$$\sum_{i=1}^{n} \lambda_i A_i^{-1} \geq \left(\sum_{i=1}^{n} \lambda_i A_i \right)^{-1}. \tag{9}$$

for all positive invertible $A \in B(H)$.

3. Integral expression. Let $F(t)$ be a selfadjoint operator-valued continuous function defined on an interval I in $\mathbb{R}$ and μ a probability measure on I. Then we take a simple funtion $G(t) = \sum_{i=1}^{n} \alpha_i \chi_{I(i)}(t)$ which approximates $F(t)$, where $\{I(i); i = 1, \cdots, n\}$ is a decomposition of I and $\alpha_i = \mu(I(i))$ for $i = 1, \cdots, n$. Applying (8) in Theorem 2, we have

$$\sum_{i=1}^{n} \alpha_i A_i^2 \geq \left(\sum_{i=1}^{n} \alpha_i A_i \right)^2 . \tag{10}$$

In other words,

$$\left(\int_I G(t) d\mu(t) \right)^2 \leq \int_I G(t)^2 d\mu(t).$$

By taking the limit, we have the following integral inequalities.

Theorem 3. *Let $F(t)$ and μ be as in above. Then*

$$\left(\int_I F(t) d\mu(t) \right)^2 \leq \int_I F(t)^2 d\mu(t). \tag{11}$$

In particular, if $F(t)$ is positive and invertible for all $t \in I$, then

$$\left(\int_I F(t) d\mu(t) \right)^{-1} \leq \int_I F(t)^{-1} d\mu(t). \tag{12}$$

The second inequality is obtained by a similar way.

4. Generalized Kantorovich inequality. It is well-known that the celebrated Kantorovich inequality is a reverse of Schwarz inequality, cf. [4, Theorem 1.23]: If A satisfies $0 < m \leq A \leq M$, then

$$\phi(A)\phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm} \tag{13}$$

holds for all states ϕ of $B(H)$.

The following theorem is obtained as a reverse of (12):

Theorem 4. *Let $F(t)$ and μ be as in above and $0 < m \leq F(t) \leq M$ for all $t \in I$. Then*

$$\phi\left(\int_I F(t) d\mu(t) \right) \phi\left(\int_I F(t)^{-1} d\mu(t) \right) \leq \frac{(M+m)^2}{4Mm} \tag{14}$$

holds for all states ϕ of $B(H)$.

We here note that a state ϕ is not replaced by a positive linear map Φ because $\Phi(X)$ and $\Phi(Y)$ don't commute in general, but it follows from [4, Theorem 1.32 (iv)] that

$$\int_I F(t)^{-1} d\mu(t) \leq \frac{(M+m)^2}{4Mm} \left(\int_I F(t) d\mu(t) \right)^{-1} . \tag{15}$$

On the other hand, the following theorem is another noncommutative version of the Kantorovich inequality [4, Theorem 1.26]:

If A satisfies $0 < m \le A \le M$, then

$$\Phi(A) \sharp \Phi(A^{-1}) \le \frac{M+m}{2\sqrt{Mm}} \tag{16}$$

holds for all unital positive linear map Φ of $B(H)$ to a unital C^*-algebra. Here $\sharp$ is the geometric mean in the sense of Kubo-Ando [5].

As a consequence, we have an integral version of a theorem due to Nakamoto and Nakamura [6].

Theorem 5. *Let $F(t)$ be as in Theorem 4. Then*

$$\int_I F(t)d\mu(t) \sharp \int_I F(t)^{-1}d\mu(t) \le \frac{M+m}{2\sqrt{Mm}}.$$

5. Another extension of Kadison's Schwarz inequality. At this end, we extend (5) by replacing weights to positive linear maps. That is, we take a family of positive linear maps $\Phi_1, \cdots, \Phi_n$ such that

$$\Phi_1 + \cdots + \Phi_n = 1, \text{ the identity map.}$$

and (5) is extended to

$$\Phi(A_1 \oplus \cdots \oplus A_n) = \langle (A_1, \cdots, A_n), (\Phi_1, \cdots, \Phi_n) \rangle = \sum_{i=1}^n \Phi_i(A_i). \tag{17}$$

Then we have the following theorem as Theorem 2:

Theorem 6. *Let Φ be as in (17). Then*

$$\sum_{i=1}^n \Phi_i(A_i^2) \ge \left(\sum_{i=1}^n \Phi_i(A_i) \right)^2 \tag{18}$$

holds for selfadjoint operators A_i and

$$\sum_{i=1}^n \Phi_i(A_i^{-1}) \ge \left(\sum_{i=1}^n \Phi_i(A_i) \right)^{-1}. \tag{19}$$

holds for positive invertible operators A_i.

Theorem 6 has the following difference reverse inequality of Theorem 6:

Theorem 7. *Let Φ be as in (17). Then*

$$(0 \le) \sum_{i=1}^n \Phi_i(A_i) - \left(\sum_{i=1}^n \Phi_i(A_i^{-1}) \right)^{-1} \le (\sqrt{M} - \sqrt{m})^2. \tag{20}$$

holds for positive invertible operators A_i with $0 < m \le A_i \le M$ for $i = 1, \cdots, n$.

Proof. Put $A = \sum_{i=1}^n \Phi_i(A_i)$ and $H = \left(\sum_{i=1}^n \Phi_i(A_i^{-1})\right)^{-1}$. Since

$$(M - A_i)(1/m - A_i^{-1}) \geq 0$$

and so

$$M + m \geq MmA_i^{-1} + A_i$$

for $i = 1, \cdots, n$, it follows that

$$M + m \geq Mm \sum_{i=1}^n \Phi_i(A_i^{-1}) + \sum_{i=1}^n \Phi_i(A_i) = M + m - MmH^{-1} - H.$$

Hence we have

$$\begin{aligned} A - H &\leq M + m - MmH^{-1} - H \\ &= (\sqrt{M} - \sqrt{m})^2 - (\sqrt{Mm}H^{-\frac{1}{2}} - H^{\frac{1}{2}})^2 \leq (\sqrt{M} - \sqrt{m})^2, \end{aligned}$$

as desired.

Finally we mention that we need the Bochner-Stieljes integral to the integral form of Theorems 6 and 7, which remains in a future paper

References

[1] K.V.Bhagwat and R.Subramanian, *Inequalities between means of positive operators*, Math. Proc. Camb. Phil. Soc., **83** (1978), 393–401.

[2] J.I.Fujii, M.Nakamura, J.E.Pečarić and Y.Seo, *Bounds for the ratio and difference between parallel sum and series via Mond-J.E.Pečarić method*, preprint.

[3] T.Furuta, *Short proof that the arithmetic mean is greater than the harmonic mean and its reverse inequality*, Math. Inequal. Appl., 8(2005), 751.

[4] T.Furuta, J.Mićić, J.E.Pečarić and Y.Seo , *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[5] F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann., 246(1980), 205-224.

[6] R.Nakamoto and M.Nakamura, *Operator mean and Kantorovich inequality*, Math. Japon., **44** (1996), 495–498.

Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
Email : mfujii@@cc.osaka-kyoiku.ac.jp

*) Tennoji Branch, Senior Highschool, Osaka Kyoiku University, Tennoji, Osaka 543-0054, Japan
Email : yukis@@cc.osaka-kyoiku.ac.jp