CATEGORY OF HYPER BCK-ALGEBRAS

H. HARIZAVI, J. MACDONALD AND R. A. BORZOOEI

Received October 15, 2005

Abstract. In this paper we first define the category of hyper BCK-algebras. After that we show that the category of hyper BCK-algebras is connected, factorisable and has equalizers, coequalizers, products, coproducts, intersection and kernel. It is a consequence that this category is complete and cocomplete and hence has pullbacks and pushouts.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki[5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras)was introduced in 1934 by F. Marty [8] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [7], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. In [3], R. A. Borzooei and H. Harizavi introduced the notion of regular congruence relation on hyper BCK-algebras and constract a quotient hyper bck-algebra. Now we follow [1, 3, 7] and introduce the category of hyper BCK-algebra and obtain some results, as mentioned in the abstract.

2. Preliminaries

Definition 2.1. [7] By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$.
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

In any hyper BCK-algebra H, $0 \circ 0 = \{0\}$, $0 \ll x$, $x \ll x$, $0 \circ x = \{0\}$, $x \circ y \ll x$, $x \circ 0 = \{x\}$, for all $x, y \in H$. Let I be a nonempty subset of a hyper BCK-algebra H. Then I is said to be, a *hyper BCK-ideal* of H if $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$,

2000 *Mathematics Subject Classification.* 06F35, 03G25.
Key words and phrases. hyper BCK-algebra, BCK-algebra, equalizers, product, coproduct, pullbacks, pushouts, intersection, kernels, factorisable.

a *hyper BCK-subalgebra* of H if $x \circ y \subseteq I$ for all $x, y \in I$, *reflexive* if $x \circ x \subseteq I$ for all $x \in H$. Let H be a hyper BCK-algebra, Θ be an equivalence relation on H and $A, B \subseteq H$. Then, we write $A\Theta B$ if there exist $a \in A$ and $b \in B$ such that $a\Theta b$, we write $A\bar{\Theta}B$ if for all $a \in A$ there exist $b \in B$ such that $a\Theta b$ and for all $b \in B$ there exist $a \in A$ such that $a\Theta b$, Θ is called a *congruence relation* on H if $x\Theta y$ and $x^{'}\Theta y^{'}$ then $x \circ x^{'}\bar{\Theta} y \circ y^{'}$ for all $x, y, x', y'a \in H$, Θ is called a *regular relation* on H if $x \circ y\Theta\{0\}$ and $y \circ x\Theta\{0\}$ then $x\Theta y$ for all $x, y \in H$. Let H and H' are two hyper BCK-algebras and $f : H \longrightarrow H'$ be a map. Then f is said to be *a homomorphism* of hyper BCK-algebras if $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in H$, and it is easy to show that, $f(0) = 0'$.

Theorem 2.2. [3] *Let Θ be a regular congruence relation on hyper BCK-algebra H and let*

$$I = [0]_\Theta,\ I_x = [x]_\Theta \quad and \quad \frac{H}{I} = \{I_x : x \in H\}$$

Then, $\frac{H}{I}$ with hyperoperation "$\circ$" and hyperorder "$\ll$" which is defined as follows is a hyper BCK-algebra which is called quotient hyper BCK-algebra.

$$I_x \circ I_y = \{I_z : z \in x \circ y\} \quad , \quad I_x \ll I_y \Longleftrightarrow I \in I_x \circ I_y$$

3. Category of hyper BCK-algebras

Definition 3.1. The class of all hyper BCK-algebras and homomorphisms between hyper BCK-algebras with usual composition of mappings forms a category called the *category of hyper BCK-algebras* and denoted by $\mathcal{HBCK}$.

Theorem 3.2. *Let $f \in Hom(H, K)$ in the $\mathcal{HBCK}$. Then the following statements are equivalent:*
(i) f is injective,
(ii) f is monic.

Proof. $(i) \Longrightarrow (ii)$ The proof is straightforward.
$(ii) \Longrightarrow (i)$ Let $f \in Hom(H, K)$ be monic morphism in $\mathcal{HBCK}$. It is enough to prove that $\mathrm{Ker}f = \{0\}$ where $\mathrm{Ker}f = \{x \in H : f(x) = 0\}$. It is easy to prove that $\mathrm{Ker}(f)$ is a hyper BCK-ideal of H. Now, let $i, j : \mathrm{Ker}f \longrightarrow H$ be two maps such that i be inclusion map and $j(x) = 0$, for all $x \in \mathrm{Ker}f$. Then it is clear that $i, j \in Hom(Ker(f), H)$ and $f \circ i = f \circ j = 0$. Since f is left cancelable, then $i = j$. Hence, $x = i(x) = j(x) = 0$ for all $x \in \mathrm{Ker}f$ and so $\mathrm{Ker}f = \{0\}$. Therefore, f is injective. □

Theorem 3.3. $\{0\}$ *is a zero object in $\mathcal{HBCK}$.*

Proof. The set $\{0\}$ trivially forms a hyper BCK-algebra. Hence, $\{0\} \in\mathcal{HBCK}$. Let $H \in\mathcal{HBCK}$. Since $g : H \longrightarrow \{0\}$ with $g(x) = 0$, for all $x \in H$, is an unique morphism from H into $\{0\}$ in $\mathcal{HBCK}$ and $f(0) = 0$ for each morphism f in $\mathcal{HBCK}$, then the sets $Hom(\{0\}, H)$ and $Hom(H, \{0\})$ are singleton. Hence, $\{0\}$ is the zero object in $\mathcal{HBCK}$. □

Theorem 3.4. *$\mathcal{HBCK}$ is connected.*

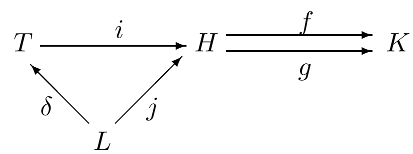
Proof. Let H and K are two objects in $\mathcal{HBCK}$. Since $0_{HK} : H \longrightarrow K$ with $0_{HK}(x) = 0$, for all $x \in H$, is a morphism in $\mathcal{HBCK}$, then $\mathrm{Mor}(H, K) \neq \phi$. Therefore, $\mathcal{HBCK}$ is connected. □

Theorem 3.5. *$\mathcal{HBCK}$ has equalizers.*

Proof. Let $f, g \in Hom(H, K)$ and $T = \{x \in H : f(x) = g(x)\}$. Since $f(0) = 0 = g(0)$, then $0 \in T \neq \phi$. Let $x, y \in T$, then $f(x) = g(x)$ and $f(y) = g(y)$. Hence,

$$f(x \circ y) = f(x) \circ f(y) = g(x) \circ g(y) = g(x \circ y)$$

and so $x \circ y \in T$. Therefore, T is a hyper BCK-subalgebra of H and so $T \in \mathcal{HBCK}$. Now, we show that T together with the inclusion morphism $i : T \longrightarrow H$ is the equalizer of f and g in $\mathcal{HBCK}$. It is clear that $f \circ i = g \circ i$. Let $j \in Hom(L, H)$ such that $f \circ j = g \circ j$. Then, $f(j(x)) = g(j(x))$, for all $x \in L$, and so $\mathrm{Im}(j) \subseteq T$. Hence, we can define a morphism $\delta : L \longrightarrow T$ by $\delta(x) = j(x)$, for all $x \in L$. It is clear that $\delta \in Hom(L, T)$ and $i \circ \delta = j$. So the following diagram is commutative.



Since i is monic, then δ is an unique morphism in $\mathcal{HBCK}$ such that the above diagram is commutative. Therefore, $\mathcal{HBCK}$ has equalizers. □

Lemma 3.6. *Let $H \in \mathcal{HBCK}$ and $\{\Theta_i : i \in I\}$ is a nonempty family of regular congruence relation on H, then $\bigcap_{i \in I} \Theta_i$ is a regular congruence relationon H.*

Proof. The proof is straightforward. □

Lemma 3.7. *Let $f \in Hom(H, K)$ in $\mathcal{HBCK}$ and relation R_f associated with f on H is defined as follows:*

$$xR_f y \Longleftrightarrow f(x) = f(y)$$

Then, R_f is a regular congruence relation on H.

Proof. Let $a, b, x \in H$ and $aR_f b$. Then, $f(a) = f(b)$. Since f is a homomorphism, then $f(a \circ x) = f(b \circ x)$ and $f(x \circ a) = f(x \circ b)$. Hence by the definition of R_f, $a \circ x \overline{R}_f b \circ x$ and $x \circ a \overline{R}_f x \circ b$ and so R_f is a congruence relation on H. Now, let $a, b \in H, a \circ bR_f\{0\}$ and $b \circ aR_f\{0\}$. Then, there are $s \in a \circ b$ and $t \in b \circ a$ such that $sR_f 0$ and $tR_f 0$. Hence, $f(s) = 0 = f(t)$. Thus, $0 = f(s) \in f(a \circ b) = f(a) \circ f(b)$ and $0 = f(t) \in f(b \circ a) = f(b) \circ f(a)$. This implies that $f(a) \ll f(b)$ and $f(b) \ll f(a)$. Hence by (HK4), $f(a) = f(b)$ and so $aR_f b$. Therefore, R_f is a regular congruence relation on H. □

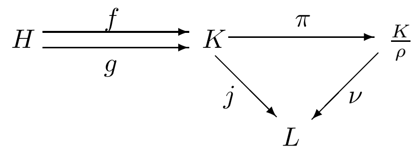
Theorem 3.8. *$\mathcal{HBCK}$ has coequalizers.*

Proof. Let $f, g \in Hom(H, K)$ in $\mathcal{HBCK}$ and

$$\sum = \{\Theta : \Theta \text{ is a regular congruence relation on } K \text{ such that } f(a)\Theta g(a), \text{ for all } a \in H\}$$

Since $K \times K \in \sum$, then $\sum \neq \phi$. Now, let $\rho = \bigcap_{\Theta \in \sum} \Theta$. By Lemma 3.6, ρ is a regular congruence relation on K. It is clear that $f(a)\rho g(a)$, for all $a \in H$. Hence, $\rho \in \sum$ and so ρ is the minimal element of $\sum$ with respect to $\subseteq$. Now consider the quotient hyper BCK-algebra $\frac{K}{\rho}$ and the canonical epimorphism $\pi : K \longrightarrow \frac{K}{\rho}$. Since $f(a)\rho g(a)$ for all $a \in K$, then $\pi(f(a)) = \pi(g(a))$ for all $a \in K$. Hence, $\pi \circ f = \pi \circ g$. Now, let $j \in Hom(K, L)$ such that $j \circ f = j \circ g$ and let $\nu : \frac{K}{\rho} \longrightarrow L$ is defined by $\nu([a]_\rho]) = j(a)$ for all $a \in K$. First, we show that ν is well-defined. Let $a, b \in K$ and $[a]_\rho = [b]_\rho$. Since by Lemma 3.7, the relation R_j associated with j is a regular congruence relation on K and $j(f(a)) = j(g(a))$, then $R_j \in \sum$. On the other hand, since ρ is a minimal element of $\sum$, hence $\rho \subseteq R_j$. Since

$a\rho b$, then aR_jb and so $j(a) = j(b)$. Hence the map ν is well-defined. It is clear that ν is a homomorphism and this implies that $\nu \in Hom(\frac{K}{\rho}, L)$. Moreover, it easy to check that the following diagram is commutative.



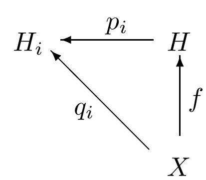
Since π is epic, then ν is an unique morphism such that the above diagram is commutative. Therefore, $\mathcal{HBCK}$ has coequalizers. □

Theorem 3.9. *$\mathcal{HBCK}$ has products.*

Proof. Let $\{(H_i, \circ_i, 0_i)\}_{i\in I}$ be a family of objects in $\mathcal{HBCK}$ and H be all of functions $f : I \longrightarrow \bigcup_{i\in I} H_i$ such that $f(i) \in H_i$, for all $i \in I$. The function $0 : I \longrightarrow \bigcup_{i\in I} H_i$, which is defined by $0(i) = 0_i$, is called the zero element of H. Let " $\circ$ " be a hyperoperation on H which is defined as follows:

$$(f \circ g)(i) = f(i) \circ_i g(i), \quad \forall f, g \in H, \forall i \in I$$

We can check that $(H, \circ, 0)$ forms a hyper BCK-algebra and so $H \in \mathcal{HBCK}$. For each $i \in I$, there exists a natural projection $p_i : H \longrightarrow H_i$ which is defined by $p_i(f) = f(i)$, for all $f \in H$. We can see that the map p_i, for all $i \in I$, is a homomorphism and so for all $i \in I$, $p_i \in Hom(H, H_i)$ in $\mathcal{HBCK}$. We claim that object H together with the morphisms $\{p_i\}_{i\in I}$ is a product of the family $\{(H_i, \circ_i, 0_i)\}_{i\in I}$. Let $(H', \circ', 0')$ be an object of $\mathcal{HBCK}$ and let $\{q_i \in Hom(H', H_i) : i \in I\}$ be a family of morphisms in $\mathcal{HBCK}$. For any $x \in X$, let $f_x : I \longrightarrow \bigcup_{i\in I} H_i$ be a function which is defined by $f_x(i) = q_i(x)$ for all $i \in I$. Then we can define the function $f : H' \longrightarrow H$ by $f(x) = f_x$, for all $x \in H'$. Now, it is easy to prove that $f \in Hom(H', H)$ and $p_i \circ f = q_i$, for all $i \in I$. Thus the following diagram is commutative,



Moreover, f is an unique morphism such that the above diagram is commutative. Therefore, the object H together with the morphisms $\{p_i\}_{i\in I}$ is the product of the family $\{H_i\}_{i\in I}$ in $\mathcal{HBCK}$. □

Corollary 3.10. *$\mathcal{HBCK}$ has pullbacks.*

Proof. By Theorems 3.5 and 3.9, $\mathcal{HBCK}$ has equalizers and products. Therefore, $\mathcal{HBCK}$ has pullbacks by [[1], Theorem 3.7]. □

Theorem 3.11. *$\mathcal{HBCK}$ is complete.*

Proof. By Theorems 3.9 and 3.5, $\mathcal{HBCK}$ has products and equalizers. Therefore, $\mathcal{HBCK}$ is complete by [[1], Theorem 6.2]. □

Corollary 3.12. *Let $\mathcal{I}$ be a small category and let $F : \mathcal{I} \longrightarrow \mathcal{HBCK}$ be a functor. Then F has a Limit.*

Proof. Let $F : \mathcal{I} \longrightarrow \mathcal{HBCK}$ be a functor from a small category. By Theorem 3.11, $\mathcal{HBCK}$ is complete. Therefore, F has a Limit by [[1], the definition of Limit]. □

Theorem 3.13. *$\mathcal{HBCK}$ has coproducts.*

Proof. Let $\{(H_i, \circ_i, 0)\}_{i\in I}$ be a family of objects in $\mathcal{HBCK}$. Without of loss of generality, we can let $H_i \cap H_j = \{0\}$, for all $i, j \in I$ and $i \neq j$. Hence, we can assume that $H_i \cap H_j = \{0\}$, for all $i, j \in I$ and $i \neq j$. Now, let "$\circ$" be a hyperoperation on $H = \bigcup_{i\in I} H_i$ which is defined as follows:

$$x \circ y = \begin{cases} x \circ_i y & \text{if } x, y \in H_i, \text{ for some } i \in I \\ \{x\} & \text{otherwise,} \end{cases}$$

for all $x, y \in H$. Now, we prove that $(H, \circ, 0)$ forms a hyper BCK-algebra.

(HK1) (1) If $x, y, z \in H_i$, the proof is clear.

(2) If $x \in H_i$, $y, z \in H_j$ and $i \neq j$, then

$$(x \circ z) \circ (y \circ z) = \{x\} \circ (y \circ z) = \{x\} \ll \{x\} = x \circ y.$$

(3) If $x, y \in H_i$, $z \in H_j$ and $i \neq j$, then

$$(x \circ z) \circ (y \circ z) = \{x\} \circ \{y\} = x \circ y \ll x \circ y$$

(4) If $x, z \in H_i$, $y \in H_j$ and $i \neq j$, then

$$(x \circ z) \circ (y \circ z) = (x \circ z) \circ \{y\} = x \circ z \ll \{x\} = x \circ y$$

The other cases are the same to the one of the above cases.

(HK2) (1) If $x, y, z \in H_i$, the proof is clear.

(2) If $x \in H_i$, $y, z \in H_j$ and $i \neq j$, then

$$(x \circ y) \circ z = \{x\} = (x \circ z) \circ y$$

(3) If $x, y \in H_i$, $z \in H_j$ and $i \neq j$, then

$$(x \circ y) \circ z = x \circ y = (x \circ z) \circ y$$

(4) If $x, z \in H_i$, $y \in H_j$ and $i \neq j$, then

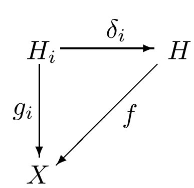
$$(x \circ y) \circ z = x \circ z = (x \circ z) \circ y$$

The other cases are the same to the one of the above cases.

(HK3) The proof is straightforward.

(HK4) Let $x, y \in H$ be such that $x \ll y$ and $y \ll x$. Then $x \in H_i$ and $y \in H_j$ for some $i, j \in I$. If $i = j$, the proof is clear. Now we let $i \neq j$. In this case, $0 \in x \circ y = \{x\}$ and $0 \in y \circ x = \{y\}$ which implies that $x = y$.

Therefore, $(H, \circ, 0)$ is a hyper BCK-algebra and so $H \in \mathcal{HBCK}$. Let $(H', \circ', 0')$ be an object of $\mathcal{HBCK}$, $\{g_i \in Hom(H_i, H') : i \in I\}$ be a family of morphisms in $\mathcal{HBCK}$ and for each $i \in I$, $\delta_i \in Hom(H_i, H)$ be a inclusion map. Since, for any $x \in H$ there exists $i \in I$ such that $x \in H_i$. Hence, we can define $f : H \longrightarrow H'$ by $f(x) = g_i(x)$, where $x \in H_i$. It is easy to check that f is an unique morphism in $\mathcal{HBCK}$ such that the following diagram is commutative.



Hence, object H together with the morphisms $\{\delta_i\}_{i\in I}$ is a coproduct of the family $\{(H_i, \circ_i, 0_i)\}_{i\in I}$. Therefore, $\mathcal{HBCK}$ has coproducts. □

Corollary 3.14. *$\mathcal{HBCK}$ has pushouts.*

Proof. By Theorems 3.8 and 3.13, $\mathcal{HBCK}$ has coequalizers and coproducts. Therefore, $\mathcal{HBCK}$ has pushouts by [[1], the dual of Theorem 3.7]. □

Theorem 3.15. *$\mathcal{HBCK}$ is co-complete.*

Proof. By Theorems 3.13 and 3.8, $\mathcal{HBCK}$ has coproducts and coequalizers. Therefore, $\mathcal{HBCK}$ is co-complete by [[1], the dual of Theorem 6.2]. □

Corollary 3.16. *Let $\mathcal{I}$ be a small category and let $F : \mathcal{I} \longrightarrow \mathcal{HBCK}$ be a functor. Then F has a Co-Limit.*

Proof. Let $F : \mathcal{I} \longrightarrow \mathcal{HBCK}$ be a functor from a small category. By Theorem 3.15 $\mathcal{HBCK}$ is co-complete. Therefore, F has a Co-Limit by [[1], the dual of the definition of Co-Limit]. □

Theorem 3.17. *$\mathcal{HBCK}$ is factorisable.*

Proof. Let $f \in Hom(H, K)$ in $\mathcal{HBCK}$ and R_f be the regular congruence relation on H associated with f in Lemma 3.7. Let $\nu : \frac{H}{R_f} \longrightarrow K$ is defined by $\nu([a]_{R_f}) = f(a)$. By the proof of Theorem 3.8, ν is well-defined and $\nu \in Hom(\frac{H}{R_f}, K)$. It is clear that ν is a monic. Now, since $\pi : H \longrightarrow \frac{H}{R_f}$ is an epic and $f = \nu \circ \pi$, hence $\mathcal{HBCK}$ is factorisable. □

Theorem 3.18. *[3] Let I be a reflexive hyper BCK-ideal of H and relation Θ on H is defined as follows:*

$$x\Theta y \iff x \circ y \subseteq I \text{ and } y \circ x \subseteq I$$

for all $x, y \in H$. Then Θ is a regular congruence relation on H and $I = [0]_\Theta$.

Theorem 3.19. *Let $f \in Hom(H, K)$ in $\mathcal{HBCK}$. Then the following statements are hold:*
(i) If f is onto, then f is epic.
(ii) If f is epic and $Im(f)$ is a reflexive hyper BCK-ideal of K, then f is onto.

Proof. (i) The proof is strightforward.

(ii) Let $f \in Hom(H, K)$ be an epic in $\mathcal{HBCK}$ and relation Θ on K is defined as followes:

$$x\Theta y \iff x \circ y \subseteq Im(f) \text{ and } y \circ x \subseteq Im(f)$$

By Theorem 3.14, Θ is a regular congruence relation on H and $\mathrm{Im}(f) = [0]_\Theta$. Hence $\frac{K}{\mathrm{Im}(f)}$ is well-defined. Now, let $\pi, g : K \longrightarrow \frac{K}{\mathrm{Im}(f)}$ are two maps such that π is the canonical epimorphism and $g(x) = \mathrm{Im(f)}$, for all $x \in K$. It is clear that $\pi, g \in Hom(K, \frac{K}{\mathrm{Im}(f)})$ and $\pi \circ f = \mathrm{Im}(f) = g \circ f$. Since f is a right cancellable, then $\pi = g$. Hence, $(\mathrm{Im}(f))_x = \pi(x) = g(x) = \mathrm{Im}(f) = [0]_\Theta$ for all $x \in K$, and so $x\Theta 0$. Thus by the definition of Θ, $K \subseteq \mathrm{Im}(f)$ and so $\mathrm{Im}(f) = K$. Therefore, f is onto. □

Theorem 3.20. *Let $f \in Hom(H, K)$ in $\mathcal{HBCK}$ and let $Im(f)$ be a reflexive hyper BCK-ideal of K. Then the following statements are equivalent :*
(i)f is a bimorphism,
(ii)f is an isomorphism.

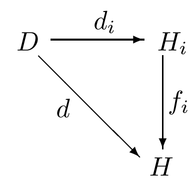
Proof. (i)$\Longrightarrow$ (ii) Let $f : H \longrightarrow K$ be a bimorphism in the category $\mathcal{HBCK}$ and let $\mathrm{Im}(f)$ be a reflexive hyper BCK-ideal of K. Then by Theorems 3.3 and 3.4 f is injective and onto. Now, let $k \in K$. Then there exists unique $h \in H$ such that $f(h) = k$. Hence we can define The function $g : K \longrightarrow H$ by $g(k) = h$. Let $k_1, k_2 \in K$ and $h_1, h_2 \in H$ such that $f(h_1) = k_1, f(h_2) = k_2$. Then, $g(k_1 \circ k_2) = \bigcup_{k \in k_1 \circ k_2} g(k) = \bigcup_{f(h) \in f(h_1) \circ f(h_2)} h = \bigcup_{h \in h_1 \circ h_2} h = h_1 \circ h_2 = g(k_1) \circ g(k_2)$. Hence, g is a homomorphism. Thus, g is a morphism of the category

$\mathcal{HBCK}$. Moreover, it easy to check that $g \circ f = \mathrm{Id}_H$ and $f \circ g = \mathrm{Id}_K$. Therefore, f is an isomorphism.

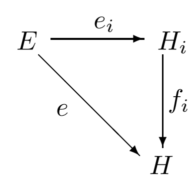
(ii)$\Longrightarrow$ (i) The proof is strightforward. □

Theorem 3.21. *$\mathcal{HBCK}$ has intersections.*

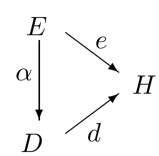
Proof. Let H and $\{H_i\}_{i\in I}$ be a family of hyper BCK-algebras, $\{f_i\}_{i\in I} \subseteq Hom(H_i, H)$ and let $D = \bigcap_{i\in I} f_i(H_i)$. Since $f_i(H_i)$ is a subalgebra of H for each $i \in I$, then D is also subalgebra of H. Let $x \in D$. Since f_i is monic for each $i \in I$, then by Theorem 3.2, it is injective. Thus for each $i \in I$ there exists an unique element $x_i \in H_i$ such that $x = f_i(x_i)$. Hence for each $i \in I$, we can define the map $d_i : D \longrightarrow H_i$ by $d_i(x) = x_i$. It is clear that $d_i \in Hom(D, H_i)$. Now, let $d : D \longrightarrow H$ be inclusion map. Since $f_i \circ d_i(x) = f_i(d_i(x)) = f_i(x_i) = x = d(x)$, for all $x \in D$ and $i \in I$. Then, $f_i \circ d_i = d$ for each $i \in I$ and so the following diagram is commutative.



Now, let $e \in Hom(E, H)$ such that the following diagram is commutative.



Let $x \in E$. Since $e(x) = f_i(e_i(x)) \in f_i(H_i)$ for each $i \in I$, then $e(x) \in D$. Hence, we can define the morphism $\alpha : E \longrightarrow D$ by $\alpha(x) = e(x)$. It is clear that $d \circ \alpha = e$. Since d is monic, then $\alpha \in Hom(E, D)$ is unique and the following diagram is commutative.



Hence, (D, H) is the intersection of the family of the subobjects $\{(H_i, f_i)\}_{i\in I}$ in $\mathcal{HBCK}$. Therefore, the category $\mathcal{HBCK}$ has intersections. □

Theorem 3.22. *$\mathcal{HBCK}$ has kernels.*

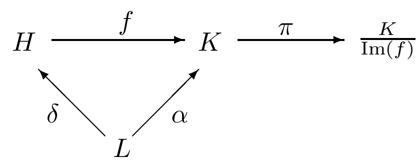
Proof. Let $f \in Hom(H, K)$ and $0 : H \longrightarrow K$ be the zero morphism in $\mathcal{HBCK}$. It is clear to prove that the equalizer of f and 0 is the kernel of f. Now, since $\mathcal{HBCK}$ has equalizers, then $\mathcal{HBCK}$ has kernels. □

Theorem 3.23. *Let $f \in Hom(H, K)$ be monic in $\mathcal{HBCK}$ and $Im(f)$ be a reflexive hyper BCK-ideal of K. Then f is a kernel.*

Proof. Let $f \in Hom(H, K)$. Since $\mathrm{Im}(f)$ is a reflexive hyper BCK-ideal of K, then by the proof of Theorem 3.15(ii), $\frac{K}{\mathrm{Im}(f)}$ is well-defined and $[0]_\Theta = Im(f)$. Now, let $\pi : K \longrightarrow \frac{K}{\mathrm{Im}(f)}$ be the canonical epimorphism. Since $f(x) \circ 0 = \{f(x)\} \subseteq Im(f)$ and $0 \circ f(x) = \{0\} \subseteq Im(f)$, then $f(x)\Theta 0$ and so $[f(x)]_\Theta = [0]_\Theta$. Hence $\pi \circ f(x) = \pi(f(x)) = [f(x)]_\Theta = [0]_\Theta$ and this implies that $\pi \circ f = 0$. Now, let $\alpha \in Hom(L, K)$ such that $\pi \circ \alpha = 0$. Since $\pi(\alpha(x)) = [0]_\Theta$, then $[\alpha(x)]_\Theta = [0]_\Theta = Im(f)$ and so $\alpha(x) \in \mathrm{Im}(f)$, for all $x \in L$. Hence, there exists $z \in H$ such that $\alpha(x) = f(z)$. Since f is monic then by Theorem 3.2, f is injective. Hence for each $x \in L$ there exists an unique $z \in H$ such that $\alpha(x) = f(z)$. Thus we can define the function $\delta : L \longrightarrow H$ by $\delta(x) = z$. Let $x, y \in L$ and $z, w \in H$ such that $\delta(x) = z$ and $\delta(y) = w$. Then,

$$\begin{aligned}\delta(x \circ y) &= \bigcup_{t\in x\circ y} \delta(t) = \bigcup_{f(s)=\alpha(t),\ t\in x\circ y} s \\ &= \bigcup_{f(s)=\alpha(t)\in\alpha(x)\circ\alpha(y)} s = \bigcup_{f(s)\in f(z)\circ f(w))} s \\ &= \bigcup_{f(s)\in f(z\circ w)} s = \bigcup_{s\in z\circ w} s = z\circ w \\ &= \delta(x)\circ\delta(y)\end{aligned}$$

Hence, δ is a homomorphism and so $\delta \in Hom(L, H)$. It is clear that $f \circ \delta = \alpha$. Since f is monic, then δ is the unique morphism in $\mathcal{HBCK}$ such that the following diagram is commutative.



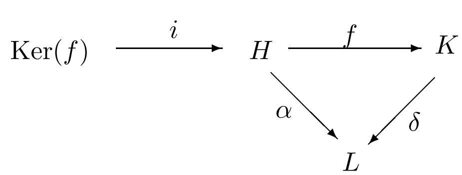
Therefore, f is a kernel. □

Theorem 3.24. *Let $f \in Hom(H, K)$ be epic in $\mathcal{HBCK}$ and $Im(f)$ be a reflexive hyper BCK-ideal of K. Then f is a cokernel.*

Proof. Let $f \in Hom(H, K)$ and $\mathrm{Ker}(f) = \{x \in H : f(x) = 0\}$. Let $i : \mathrm{Ker}(f) \longrightarrow H$ be inclusion map. It is clear that $i \in Hom(\mathrm{Ker}(f), H)$ and $f \circ i = 0$. Now, let $\alpha \in Hom(H, L)$ such that $\alpha \circ i = 0$. Since f is epic and $\mathrm{Im}(f)$ is a reflexive hyper BCK-ideal of K, then by Theorem 3.15(ii) f is onto. Let $k \in K$. Then there exists $h \in H$ such that $f(h) = k$. Let $\delta : K \longrightarrow L$ is defined by $\delta(k) = \alpha(h)$. First, we show that δ is well-defined. Let $k_1 = k_2 \in K$, $f(h_1) = k_1$ and $f(h_2) = k_2$ for some $h_1, h_2 \in H$. Then, $f(h_1) = f(h_2)$. Hence, $h_1 \circ h_2, h_2 \circ h_1 \subseteq \mathrm{Ker}(f)$ and so $\alpha(h_1 \circ h_2) = \alpha(h_2 \circ h_1) = \{0\}$. Thus, $\alpha(h_1) \ll \alpha(h_2)$ and $\alpha(h_2) \ll \alpha(h_1)$ and so $\alpha(h_1) = \alpha(h_2)$. Therefore, $\delta(k_1) = \delta(k_2)$ and so δ is well-defined. Now, let $k_1, k_2 \in K$, $f(h_1) = k_1$ and $f(h_2) = k_2$ for some $h_1, h_2 \in H$. Since f and α are homomorphism, then

$$\begin{aligned}\delta(k_1 \circ k_2) &= \delta(f(h_1)\circ f(h_2)) = \delta(f(h_1\circ h_2)) \\ &= \bigcup_{h\in h_1\circ h_2} \delta(f(h)) = \bigcup_{h\in h_1\circ h_2} \alpha(h)) \\ &= \alpha(h_1\circ h_2) = \alpha(h_1)\circ\alpha(h_2) \\ &= \delta(k_1)\circ\delta(k_2).\end{aligned}$$

Hence δ is a homomorphism and so $\delta \in Hom(K, L)$. Since f is epic, then δ is the unique morphismin in the category $\mathcal{HBCK}$ such that the following diagram is commutative.



Therefore, f is a cokernel. □

References

[1] T. S. Blyth, Categories, Longman, London New York (1987).
[2] R. A. Borzooei, M. Bakhshi, *Some Results on Hyper BCK-algebras*, Quasigroups and Related Systems, Vol. **11**, (2004) 9-24.
[3] R. A. Borzooei, H. Harizavi, *Regular congruence Relations on Hyper BCK-algebras*, Scientiae Mathematicae Japonicae, Vol. **61**, No. 1(2005), 83-97.
[4] P. Corsini, *Prolegramena of hyper group theory*, Aviani Editor, 1993.
[5] Y. Imai, K. Iséki, *On axiom systems of propositional calculi XIV*, Proc. Japan Academy, **42** (1966) 19-22.
[6] Y. B. Jun, X. L. Xin, E. H. Roh and M. M. Zahedi, *Strong hyper BCK-ideals of hyper BCK-algebra*, Mathematicae Japonicae, Vol. **51**, No. 3(2000), 493-498.
[7] Y. B. Jun, M. M. Zahedi, X. L. Xin, R.A. Borzooei, *On hyper BCK-algebra*, Italian Journal of Pure and Applied Mathematics, No.**10** (2000), 127-136.
[8] F. Marty, *Sur une generalization de la notion de groups*, 8th congress Math. Scandinaves, Stockhholm, (1934), 45-49.
[9] J. Marty, Y. B. Jun, *BCK-algebra*, Kyungmoonsa, Seoul, Korea, 1994.

H. Harizavi, R. A. Borzooei, Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: harizavi@hamoon.usb.ac.ir
E-mail address: borzooei@hamoon.usb.ac.ir

J. Macdonald, Department of Mathematics, University of British Columbia, Vancouver, Canada
E-mail address: johanm@math.ubc.ca