COUNTEREXAMPLES ON GENERALIZED METRIC SPACES

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Received January 25, 2006; revised April 7, 2006

Abstract. In this paper, we give counterexamples of some questions on generalized metric spaces. First we show that there exists an open sequence-covering map of a countable g-second countable space onto the sequential fan S_ω. This is a counterexample for a question posed by Y. Tanaka. Second we show that there exists a regular Fréchet space Y satisfying the following conditions: (1) Y has a point-countable cs-network and k-network of closed subsets; (2) every first countable closed subset of Y is countable; (3) Y is not locally separable and does not have any star-countable k-network. This is a counterexample for questions posed by S. Lin.

1 Introduction We assume that all spaces are regular T_1 and all maps are continuous onto. The letter $\mathbb{N}$ is the set of natural numbers. Unexplained notions and terminology are the same as in [3]. We recall some definitions.

Definition 1.1 *Let X be a space. For $x \in X$, let $\mathcal{B}_x$ be a family of subsets of X. Then $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ is called a weak-base for X [1] if it satisfies (1) every element of $\mathcal{B}_x$ contains x, (2) for $B_0, B_1 \in \mathcal{B}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subset B_0 \cap B_1$ and (3) $G \subset X$ is open iff for each $x \in G$ there exists $B \in \mathcal{B}_x$ with $B \subset G$. A space X is called g-first countable [12] if it has a weak-base $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ such that each $\mathcal{B}_x$ is countable. A space with a countable weak-base is called g-second countable [12].*

Obviously both a first countable space and a g-second countable space are g-first countable. The *sequential fan* S_ω is the space obtained by identifying the limits of countably many convergent sequences. A space is first countable iff it is g-first countable and Fréchet [1]. Hence S_ω is not g-first countable.

Definition 1.2 *Let $f : X \to Y$ be a map. Then f is called sequence-covering [11] if whenever $\{y_n\}_{n\in\omega}$ is a sequence in Y converging to $y \in Y$, there exists a sequence $\{x_n\}_{n\in\omega}$ in X converging to a point $x \in f^{-1}(y)$ such that $x_n \in f^{-1}(y_n)$. And f is called 1-sequence-covering [5] if for each $y \in Y$, there exists a point $x_y \in f^{-1}(y)$ such that whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y converging to a point $y \in Y$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to the point x_y with $x_n \in f^{-1}(y_n)$.*

In [13, Question 2.19(2)], Y. Tanaka posed the following question.

Question 1.3 *Let $f : X \to Y$ be an open map. If X is g-first countable, then so is Y ?*

It is well known that first countability is preserved by an open map. S. Lin pointed out [7] that, if a sequential space Y is a 1-sequence-covering image of a g-first countable space, then Y is g-first countable.

2000 *Mathematics Subject Classification.* Primary 54A20, 54B15, 54C10.

Key words and phrases. weak base; open map; sequence-covering map; g-first countable; Fréchet; point-countable; star-countable; cs-network; cs^*-network; k-network.

Definition 1.4 *Let $\mathcal{A}$ be a family of subsets of a set X. $\mathcal{A}$ is said to be point-countable if each point of X is contained in at most countably many elements of $\mathcal{A}$. $\mathcal{A}$ is said to be star-countable if each element of $\mathcal{A}$ intersects with at most countably many elements of $\mathcal{A}$.*

Definition 1.5 *Let $\mathcal{P}$ be a family of subsets of a space X. Then $\mathcal{P}$ is called a cs-network if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in X$ and any neighborhood U of x, there exist $P \in \mathcal{P}$ and $m \in \mathbb{N}$ such that $\{x, x_n : n \geq m\} \subset P \subset U$. $\mathcal{P}$ is called a cs^*-network if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in X$ and any neighborhood U of x, there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x, x_{n_j} : j \in \mathbb{N}\} \subset P \subset U$. $\mathcal{P}$ is called a k-network if for any compact set $K \subset X$ and an open set U with $K \subset U$, there exists a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$.*

Every *cs*-network is a cs^*-network.

In the book [6, Question 5.1.20, Question 5.2.10], S. Lin posed the following questions on a point-countable cover.

Question 1.6 *Let X be a regular Fréchet space with a point-countable cs^*-network. Is X locally separable if each first countable closed subspace of X is locally separable ?*

Question 1.7 *Let X be a regular Fréchet space with a point-countable k-network. Does X have a star-countable k-network if each first countable closed subspace of X is locally separable ?*

In this paper, we present counterexamples for these questions posed by Y. Tanaka and S. Lin.

2 Counterexamples

Example 2.1 *We show that there exists an open sequence-covering map φ of a countable g-second countable space X onto the sequential fan S_ω. Let $\mathcal{B} = \{B_k\}_{k\in\mathbb{N}}$ be a countable open base of the real line. For each $k \in \mathbb{N}$, we can take a subset $C_k \subset B_k$ such that $|C_k| = \omega$ and $C_k \cap C_{k'} = \emptyset$ for distinct $k, k' \in \mathbb{N}$. We put $C_k = \{x_{k,l}\}_{l\in\mathbb{N}}$ and $C = \bigcup_{k\in\mathbb{N}} C_k$. Note that every non-empty open set of C contains some C_k. For each $k, l \in \mathbb{N}$, let $S_{k,l}$ be a convergent sequence homeomorphic to the usual convergent sequence $S = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. We put*

$$S_{k,l} = \{y_{k,l}\} \cup \{y_{k,l}(m, n) : 1 \leq m \leq l, n \in \mathbb{N}\},$$

where $y_{k,l}$ is the limit point of $S_{k,l}$.

Consider the topological sum $C \oplus (\oplus\{S_{k,l} : k, l \in \mathbb{N}\})$. Let X be the space obtained by identifying $x_{k,l}$ and $y_{k,l}$ for each $k, l \in \mathbb{N}$. Note that a subset U of X is open in X iff $U \cap C$ is open in C and for every $x_{k,l} \in U$, $|S_{k,l} - U| < \omega$. Obviously X is a countable Hausdorff space. We observe that X is 0-dimensional. Let U be an open set of X and let $x_{k,l} \in U$. For a clopen set B of C satisfying $x_{k,l} \in B \subset U \cap C$, the set

$$V = (B \cup (\bigcup\{S_{i,j} : i, j \in \mathbb{N}, x_{i,j} \in B\})) \cap U$$

is a clopen set in X such that $x_{k,l} \in V \subset U$. Thus X is 0-dimensional, in particular it is completely regular.

Next we observe that X is g-second countable. For each $k, l, j \in \mathbb{N}$, we put

$$S^j_{k,l} = \{y_{k,l}\} \cup \{y_{k,l}(m, n) : 1 \leq m \leq l, n \geq j\}.$$

Let $x \in X$. If x is an isolated point in X, let $\mathcal{G}_x = \{\{x\}\}$. If $x = x_{k,l}$, let $\mathcal{G}_x = \{(B_j \cap C) \cup S^j_{k,l} : x_{k,l} \in B_j \in \mathcal{B}\}$. Then $\mathcal{G} = \bigcup_{x \in X} \mathcal{G}_x$ is countable and it is not difficult to show that $\mathcal{G}$ is a weak base for X. Thus X is g-second countable.

By the observations above, X is a countable completely regular space which is g-second countable.

We put $S_\omega = \{\infty\} \cup \{(m,n) : m,n \in \mathbb{N}\}$. Each point $(m,n) \in S_\omega$ is isolated. A basic open neighborhood of ∞ is of the form $V(f) = \{\infty\} \cup \{(m,n) : n \geq f(m)\}$, where $f : \mathbb{N} \to \mathbb{N}$ is a function. We define a map of X onto S_ω as follows:

$$\varphi(x) = \begin{cases} \infty & \text{if } x = x_{k,l} \\ (m,n) & \text{if } x = y_{k,l}(m,n). \end{cases}$$

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. By the definition of φ, $|S_{k,l} - \varphi^{-1}(V(f))| < \omega$ for every $k,l \in \mathbb{N}$. Hence $\varphi^{-1}(V(f))$ is open in X. Thus φ is continuous.

We show that φ is an open map. Let U be an open set of X. If $U \cap C = \emptyset$, then $\varphi(U)$ is obviously open. If $U \cap C \neq \emptyset$, then there exists $k \in \mathbb{N}$ such that $C_k = \{x_{k,l}\}_{l \in \mathbb{N}} \subset U \cap C$. For each $l \in \mathbb{N}$, let

$$\widetilde{S}_{k,l} = \{x_{k,l}\} \cup \{y_{k,l}(l,n) : n \in \mathbb{N}\}.$$

Then note $\varphi(\widetilde{S}_{k,l}) = \{\infty\} \cup \{(l,n) : n \in \mathbb{N}\}$. Hence $|\widetilde{S}_{k,l} - U| < \omega$ for each $l \in \mathbb{N}$. This implies $\varphi(U) \supset V(f)$ for some function f. Thus φ is open.

Finally we see that φ is sequence-covering. Let $K \subset S_\omega$ be a convergent sequence with the limit ∞. Then there exists $l \in \mathbb{N}$ such that

$$K \subset \{\infty\} \cup \{(m,n) : m \leq l, n \in \mathbb{N}\}.$$

Since $S_{k,l}$ $(k \in \mathbb{N})$ is homeomorphic to $\{\infty\} \cup \{(m,n) : m \leq l, n \in \mathbb{N}\}$ by the map φ, there exists a convergent sequence $K' \subset S_{k,l}$ satisfying $\varphi(K') = K$. Thus φ is sequence-covering.

Remark 2.2 *Every open map of a first countable space is sequence-covering [11]. But not every open map of a g-first countable space is sequence-covering [10, Example 3.2]. In [10, Question 3.3], the author asked whether every open map of a g-metrizable space is sequence-covering. As an application of Example 2.1, we can see that the question is negative. Every g-second countable space is g-metrizable. Recall the notations in Example 2.1 and let*

$$X' = C \cup \left(\bigcup \{\widetilde{S}_{k,l} : k,l \in \mathbb{N}\}\right) \subset X.$$

Since X' is closed in X, it is also g-second countable. Consider the restricted map $\varphi' = \varphi | X' : X' \to S_\omega$. By the same argument as in Example 2.1, the map φ' is open. Consider the convergent sequence $K = \{\infty\} \cup \{(m,n) : m = 1,2, n \in \mathbb{N}\}$ in S_ω. Then it is not difficult to check that there exists no convergent sequence K' in X' satisfying $\varphi'(K') = K$. Hence φ' is not sequence-covering.

Example 2.3 *Let P be a Bernstein set of the unit interval $I = [0,1]$. In other words, P is an uncountable set which contains no uncountable closed set of I. Let X be the space obtained from I by isolating the points of P. Obviously X has a point-countable base. Note that every open set of X containing $X - P$ is co-countable, hence X is Lindelöf. The space X was considered in [4, Example 9.4].*

Let Y be the quotient space obtained from X by collapsing the set $X - P$ to the one-point ∞. Obviously Y is regular and Fréchet. Let f be the natural map of X onto Y. Since f is a closed map and X is Lindelöf, f is compact-covering [8]. Let K be a compact subset of

Y. Take a compact subset K' of X with $f(K') = K$. Since K' is a compact space with a point-countable base, it is metrizable [2]. Hence $K' \cap P$ is countable. Therefore a compact subset of Y is a finite set or a sequence converging to ∞.

Now we observe that Y has a point-countable cs-network of closed subsets. Let $\mathcal{B}$ be a countable base of the unit interval I which is closed under the finite union. Note that every element of $\mathcal{B}$ intersects with $I - P$. Let

$$\mathcal{P} = \{\{f(p)\} : p \in P\} \bigcup \{f(B) : B \in \mathcal{B}\}.$$

Obviously $\mathcal{P}$ is a point-countable closed family in Y. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in Y converging to ∞ and let U be an open set containing $\{\infty\} \cup \{y_n\}_{n\in\mathbb{N}}$. Since f is compact-covering, there exist a sequence $\{p_n\}_{n\in\mathbb{N}} \subset P$ and a set $K \subset X - P$ such that $\{p_n\}_{n\in\mathbb{N}} \cup K$ is compact and $f(p_n) = y_n$. Since K is compact, there exist $B \in \mathcal{B}$ and $k \in \mathbb{N}$ such that $K \cup \{p_n\}_{n\geq k} \subset B \subset f^{-1}(U)$. Thus $\{\infty\} \cup \{y_n\}_{n\geq k} \subset f(B) \subset U$. Moreover $\mathcal{P}$ is a k-network for Y, because a compact subset of Y is a finite set or a sequence converging to ∞.

Let A be a first countable closed subset of Y. If $\infty \notin A$, then A is countable, because A is closed. Assume $\infty \in A$. Since ∞ is a G_δ-point in A, there exists a G_δ-set G in Y such that $G \cap A = \{\infty\}$. Since P is a Bernstein set, $Y - G$ is countable. Hence A is countable. Thus every first countable closed subset of Y is countable.

Since every neighborhood of ∞ contains uncountably many isolated points, it is not separable. Hence Y is not locally separable. It is known in [9, Corollary 2.4] that every k-space with a star-countable k-network is a σ-space, in particular every point is a G_δ-set. But the point ∞ is not a G_δ-set in Y. Therefore Y has no star-countable k-network.

Acknowledgement: The author would like to thank Shou Lin for informing the author of the questions. The author would like to also thank the referee for careful reading of the paper.

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