

THE RUIN PROBABILITY FOR THE STORAGE PROCESS WITH LARGE SCALE DEMANDS

MAKOTO DOI

Received January 14, 2006; revised May 18, 2006

ABSTRACT. In this paper we consider the ruin probability for a storage process. This process has two phases, the inflow phase and the outflow one, and the switchover of which is controlled by a certain storage level. In these phases the storage increases or decreases at each rate dependent on the present level. Furthermore the large scale demand for the system may happen in both phases according to Poisson process. The limiting probability distribution for the storage level and the ruin probability are given by the solution of a system of renewal equations.

1 Introduction The storage process considered in this paper is that arising from stochastic models for queues, inventories, dams, insurance risk, nursing-insurance risk and so on. It is most important problem to get the ruin probability for the process because, for example, the insurance company wishes to avoid the ruin.

This paper investigates the ruin probability for the storage process with upper boundary which has two phases, called as the inflow and outflow phases, and the switch over of these phases is controlled by a certain storage level.

In the inflow phase the storage is increasing and in the outflow phase the storage is decreasing. Assume that the storage increases or decreases at each rate dependent on the present phase and level, and that the inflow has two different increasing rates. In both phases the large scale demand for the system may occur according to Poisson process. We present the analytical solution for the steady-state probabilities of storage levels and the ruin probability incurred the first epoch at which the storage level drops down below the zero level.

For this process Doi and Ōsawa [1] have studied numerically on the steady-state probabilities of storage levels and Doi [2] has got the mean ruin time. For the simple process with increasing rate, Doi [3] has studied on the mean ruin time and Doi, Nagai and Ōsawa [4] have got the ruin probability.

2 The Mathematical Model Let $X(t)$ be the storage level at time t . Assume that it has boundaries L and zero, that is, $0 \leq X(t) \leq L$. We define a time interval in which $X(t)$ is increasing as an inflow phase, and during this phase $X(t)$ has an inflow rate $\alpha_1(x)$ given that $X(t) = x$ for $0 < x < L$. We also define a time interval in which $X(t)$ is decreasing as an outflow phase having a rate $\alpha_0(x)$ ($l < x < L$). Once $X(t)$ reaches the upper bound L , it remains at the level in a certain period whose length is exponentially distributed with parameter ν_L . Immediately after this period, the phase changes to the outflow one and $X(t)$ is controlled according to the outflow rate $\alpha_0(x)$ given that $X(t) = x$ for $l < x < L$.

Throughout these phases, the large scale demand for the system may occur according to the Poisson process, that is, the inter-occurrence time has an exponential distribution

2000 *Mathematics Subject Classification.* 37A50, 46N30.

Key words and phrases. Ruin probability, storage process, large scale demands.

with parameter λ . Let the amount of each demand has a distribution function $F(x)$ with the density function $f(x)$ having the finite mean.

There are two cases for the switch over from the outflow phase to the inflow one. First, if $X(t)$ decreases to the level l continuously, the phase instantaneously changes to the inflow one. Second, if $X(t)$ drops down into a domain $(0, l)$ because of a large scale demand for the system, the phase instantaneously changes to the inflow one. In two cases stated above the system can be switched without any loss of time. When the demand larger than the present level occurs, the storage becomes empty and the system is ruined. If a large scale demand happens in the inflow phase, the inflow phase is continued except the case of large scale demand dropping down below the zero level. Once the ruin occurs, $X(t)$ remains at the level zero in a certain period according to an exponential distribution with parameter ν_0 . Immediately after that period, the inflow phase begins.

There are many applications analyzed by the mathematical model described in this section, for example, production-inventory problems, production systems with shocks, finite capacity queueing-inventory problem, M/G/1 queueing systems with removable server (H.C.Tijms [5]), insurance and nursing insurance risk problems. For the mean ruin time for this process Doi [2] has studied.

For this process, in the next section, we define the Markov process and constitute the integro-differential equations.

3 Integro-Differential Equations For the model above, the states of the storage process are classified into four categories:

$$\left\{ \begin{array}{ll} (\xi(t), X(t)) = (1, x) & \text{if the process is in inflow phase and the storage level is} \\ & x \text{ at time } t, (0 < x < L) , \\ (\xi(t), X(t)) = (0, x) & \text{if the process is in outflow phase and the storage level is} \\ & x \text{ at time } t, (l < x < L) . \\ (\xi(t), X(t)) = 0 & \text{if the storage process is ruined at time } t , \\ (\xi(t), X(t)) = L & \text{if the storage is full at time } t , \\ & \text{where } \xi(t) \text{ indicates the present phase.} \end{array} \right.$$

Thus we constitute the Markov process $\{(\xi(t), X(t)) : t \geq 0\}$.

Now, we define its probability distribution for $i = 0, 1$ and $t \geq 0$:

$$\begin{aligned} p_t(i, x) &= P[(\xi(t), X(t)) = (i, x)], \\ P_t(0) &= P[(\xi(t), X(t)) = 0], \\ P_t(L) &= P[(\xi(t), X(t)) = L]. \end{aligned}$$

We have the following Kolmogorov's forward equations with respect to $p_t(i, x)$.

$$\begin{aligned} &\text{For } p_t(1, x) \ (0 < x < L) \\ (1) \quad &\frac{\partial p_t(1, x)}{\partial t} + \alpha_1(x) \frac{\partial p_t(1, x)}{\partial x} = -\lambda \left\{ p_t(1, x) - \int_0^{L-x} p_t(1, x+y) dF(y) \right\} \\ &\quad + \lambda I_{(0,l)}(x) \left\{ P_t(L) f(L-x) + \int_0^{L-x} p_t(0, x+y) dF(y) \right\}, \\ &\text{and for } p_t(0, x) \ (l < x < L) \\ (2) \quad &\frac{\partial p_t(0, x)}{\partial t} - \alpha_0(x) \frac{\partial p_t(0, x)}{\partial x} = -\lambda \{ p_t(0, x) - P_t(L) f(L-x) \} \end{aligned}$$

$$- \int_0^{L-x} p_t(0, x + y) dF(y) \}$$

where $I_{(0,l)}(x) = 1$ if $x \in (0, l)$; $= 0$ otherwise.

Assuming that $\lim_{t \rightarrow \infty} p_t(i, x) = p(i, x)$ ($i = 0, 1$), $\lim_{t \rightarrow \infty} P_t(0) = P_0$ and $\lim_{t \rightarrow \infty} P_t(L) = P_L$ exist, we consider the steady state of this process.

Hence we have the following integro-differential equations from (1) and (2).

For $p(1, x)$ ($0 < x < L$)

$$(3) \quad \alpha_1(x) \frac{dp(1, x)}{dx} = -\lambda \{ p(1, x) - \int_0^{L-x} p(1, x + y) dF(y) \} + \lambda I_{(0,l)}(x) \{ P_L \cdot f(L - x) + \int_0^{L-x} p(0, x + y) dF(y) \},$$

and for $p(0, x)$ ($l < x < L$)

$$(4) \quad \alpha_0(x) \frac{dp(0, x)}{dx} = \lambda \{ p(0, x) - P_L \cdot f(L - x) - \int_0^{L-x} p(0, x + y) dF(y) \}$$

To solve these equations, we need the boundary conditions:

$$(5) \quad \nu_0 P_0 = \lambda \int_0^L \{ I_{(l,L)}(y) p(0, y) + p(1, y) \} \{ 1 - F(y) \} dy + \lambda P_L \{ 1 - F(L) \},$$

$$(6) \quad (\nu_L + \lambda) P_L = \alpha_1(L-) p(1, L-),$$

$$(7) \quad \nu_L P_L = \alpha_0(L-) p(0, L-),$$

$$(8) \quad p(0, l+) + p(1, l-) = \int_0^{L-l} \{ p(0, l + u) + p(1, l + u) \} dF(u) + P_L \cdot f(L - l).$$

Note that we suppose $p(0, x) = 0$ for $0 < x < l$.

4 Analytical Solutions In this section, similarly in [1] and [2], we take the outflow rate as $\alpha_0(x) = \alpha_0$ for $l < x < L$ and the inflow rates, according to the storage level, as $\alpha_1(x) = \alpha_1$ for $l < x < L$ and $\alpha_1(x) = \alpha_2$ for $0 < x < l$ ($\alpha_1 < \alpha_2$). First we have the following Theorem concerning $p(0, x)$ in the outflow phase.

Theorem 1 *If we take $\alpha_0(x) = \alpha_0$ for $l < x < L$ then $p(0, x)$ is obtained as follows:*

$$(9) \quad p(0, x) = B(x) + \int_0^{L-x} e^{\delta_0 y} B(x + y) dM_0(y)$$

where

$$(10) \quad B(x) = A_0(x) - \frac{\lambda}{\alpha_0} \int_0^{L-x} A_0(x + y) \{ 1 - F(y) \} dy,$$

$$(11) \quad A_0(x) = \frac{1}{\alpha_0} \{ \nu_L + \lambda F(L - x) \} P_L,$$

$$(12) \quad M_0(y) = \sum_{n=1}^{\infty} H_0^{2n*}(y), H_0(y) = \frac{\lambda}{\alpha_0} \int_0^y e^{-\delta_0 y} \{ 1 - F(y) \} dy$$

and δ_0 is the unique solution of the equation

$$(13) \quad \int_0^\infty \frac{\lambda}{\alpha_0} e^{-\delta_0 x} \{1 - F(x)\} dx = 1.$$

($H_0^{2n*}(y)$ is the $2n$ -th fold convolution of $H_0(y)$.)

Proof

To evaluate the right hand side of (4), we note the following relation.

$$(14) \quad \begin{aligned} -p(0, x) + \int_0^{L-x} p(0, x+y) dF(y) \\ = \frac{d}{dx} \int_0^{L-x} p(0, x+y) \{1 - F(y)\} dy \end{aligned}$$

Using this relation we rewrite (4) as:

$$(15) \quad p(0, x) = A_0(x) - \frac{\lambda}{\alpha_0} \int_0^{L-x} p(0, x+y) \{1 - F(y)\} dy$$

where

$$(16) \quad A_0(x) = p(0, L-) + \frac{\lambda}{\alpha_0} F(L-x) P_L.$$

Note that

$$(17) \quad p(0, L-) = \frac{\nu_L}{\alpha_0} P_L.$$

Since it is not the proper renewal function, we need the Tijms' method [5].

Using δ_0 defined by (13), we define the distribution function:

$$(18) \quad H_0(x) = \begin{cases} \frac{\lambda}{\alpha_0} \int_0^x e^{-\delta_0 y} \{1 - F(y)\} dy & (x > 0), \\ 0 & (x \leq 0). \end{cases}$$

Then we have the standard renewal equation from (15) with concern to $e^{\delta_0 x} p(0, x)$ by use of δ_0 and $H_0(x)$ as follows:

$$(19) \quad \begin{aligned} e^{\delta_0 x} p(0, x) &= e^{\delta_0 x} A_0(x) - \int_0^{L-x} e^{\delta_0(x+y)} p(0, x+y) dH_0(y) \\ &= e^{\delta_0 x} B(x) + \int_0^{L-x} e^{\delta_0(x+y)} p(0, x+y) dH_0^{2*}(y). \end{aligned}$$

From the solution of this equation we have (9). \square

Next we have the following Theorem concerning $p(1, x)$ ($l < x < L$) in the inflow phase.

Theorem 2 *If we take $\alpha_1(x) = \alpha_1$ for $l < x < L$ then $p(1, x)$ is obtained as follows:*

$$(20) \quad p(1, x) = \frac{\nu_L + \lambda}{\alpha_1} \left\{ 1 + \int_0^{L-x} e^{\delta_1 y} dM_1(y) \right\} P_L,$$

where

$$(21) \quad M_1(y) = \sum_{n=1}^{\infty} H_1^{n*}(y), \quad H_1(y) = \frac{\lambda}{\alpha_1} \int_0^y e^{-\delta_1 y} \{1 - F(y)\} dy$$

and δ_1 is the unique solution of the equation

$$(22) \quad \int_0^\infty \frac{\lambda}{\alpha_1} e^{-\delta_1 x} \{1 - F(x)\} dx = 1.$$

Proof

We have the relation:

$$(23) \quad -p(1, x) + \int_0^{L-x} p(1, x + y)dF(y) = \frac{d}{dx} \int_0^{L-x} p(1, x + y)\{1 - F(y)\}dy.$$

Then (3) ($l < x < L$) is reduced to

$$(24) \quad p(1, x) = p(1, L-) + \frac{\lambda}{\alpha_1} \int_0^{L-x} p(1, x + y)\{1 - F(y)\}dy.$$

We define the distribution function:

$$(25) \quad H_1(x) = \begin{cases} \frac{\lambda}{\alpha_1} \int_0^x e^{-\delta_1 x}\{1 - F(y)\}dy & (x > 0), \\ 0 & (x \leq 0). \end{cases}$$

Then we have the standard renewal equation from (24) with concern to $e^{\delta_1 x}p(1, x)$ ($l < x < L$).

$$(26) \quad e^{\delta_1 x}p(1, x) = e^{\delta_1 x}p(1, L-) + \int_0^{L-x} e^{\delta_1(x+y)}p(1, x + y)dH_1(y).$$

Thus we have (20). Note that

$$p(1, L-) = \frac{\nu_L + \lambda}{\alpha_1} P_L.$$

□

Next we have $p(1, x)$ for $0 < x < l$.

Theorem 3 *If we take $\alpha_1(x) = \alpha_2$ for $0 < x < l$ then $p(1, x)$ is obtained as follows:*

$$(27) \quad p(1, x) = A_1(x) + \int_0^{l-x} e^{\delta_2 y} A_1(x + y)dM_2(y),$$

where

$$(28) \quad A_1(x) = p(1, l-) - \frac{\lambda}{\alpha_2} P_L \{F(L - l) - F(L - x)\} - \frac{\lambda}{\alpha_2} \int_x^l \int_l^L \{p(0, u) + p(1, u)\}dF(u - v)dv,$$

$$(29) \quad M_2(y) = \sum_{n=1}^{\infty} H_2^{n*}(y), H_2(y) = \frac{\lambda}{\alpha_2} \int_0^y e^{-\delta_2 y}\{1 - F(y)\}dy$$

and δ_2 is the unique solution of the equation

$$(30) \quad \int_0^{\infty} \frac{\lambda}{\alpha_2} e^{-\delta_2 x}\{1 - F(x)\}dx = 1.$$

Proof

Note that

$$\begin{aligned}\int_0^{L-x} p(1, x+y)dF(y) &= \int_0^{l-x} p(1, x+y)dF(y) + \int_l^L p(1, u)dF(u-x), \\ \int_0^{L-x} p(0, x+y)dF(y) &= \int_l^L p(0, u)dF(u-x).\end{aligned}$$

We rewrite (3) as

$$\begin{aligned}\alpha_2 \frac{dp(1, x)}{dx} &= -\lambda \left\{ p(1, x) - \int_0^l p(1, x+y)dF(y) \right\} \\ &\quad + \lambda [P_L \cdot f(L-x) + \int_l^L \{p(0, u) + p(1, u)\}dF(u-x)].\end{aligned}$$

In the same manner of obtaining (26) we have the following.

$$e^{\delta_2 x} p(1, x) = e^{\delta_2 x} A_1(x) + \int_0^{l-x} e^{\delta_2(x+y)} p(1, x+y)dH_2(y).$$

From this equation we have (27). \square

5 The ruin probability In order to express $p(1, l-)$ in (28) by P_L , we use (8) and (9). Thus we have

$$\begin{aligned}(31) \quad p(1, l-) &= \int_0^{L-l} \{p(0, l+u) + p(1, l+u)\}dF(u) + P_L \cdot f(L-l) \\ &\quad - \left[A_0(l+) - \frac{\lambda}{\alpha_0} \int_0^{L-l} A_0(l+y)\{1-F(y)\}dy + \int_0^{L-l} e^{\delta_0 y} A_0(l+y)dM_0(y) \right. \\ &\quad \left. - \frac{\lambda}{\alpha_0} \int_0^{L-l} e^{\delta_0 y} \int_0^{L-(l+y)} A_0(l+u)\{1-F(u)\}dudM_0(y) \right].\end{aligned}$$

Since the all probabilities are expressed by P_L , we have the following theorem for the ruin probability.

Theorem 4 *If we set*

$$p^*(0, x) = \frac{p(0, x)}{P_L}, p^*(1, x) = \frac{p(1, x)}{P_L} \text{ and } P_0^* = \frac{P_0}{P_L},$$

then the ruin probability is obtained as

$$(32) \quad P_0 = \frac{\lambda P_L}{\nu_0} \left[\int_0^L \{I_{(l, L)}(y)p^*(0, y) + p^*(1, y)\}\{1-F(y)\}dy + 1 - F(L) \right],$$

where

$$(33) \quad P_L = \left[\int_0^L \{p^*(0, x) + p^*(1, x)\}dx + P_0^* + 1 \right]^{-1}.$$

Proof

From (5) P_0^* is obtained by $p^*(0, x)$ and $p^*(1, x)$ as follows:

$$P_0^* = \frac{\lambda}{\nu_0} \left[\int_0^L \{I_{(l,L)}(y)p^*(0, y) + p^*(1, y)\} \{1 - F(y)\} dy + 1 - F(L) \right].$$

Since we have

$$P_L = \left[\int_0^L \{I_{(l,L)}(x)p^*(0, x) + p^*(1, x)\} dx + P_0^* + 1 \right]^{-1},$$

we obtain (32). \square

6 Concluding remarks We have discussed the ruin probability with large scale demands under the steady state condition. Next we will need to have the ruin probability in finite time.

7 Acknowledgement The author would like to thank referees for constructive comments.

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Faculty of Science, Tokai University
 Hiratsuka Kanagawa, 259-1292 JAPAN
 e-mail: msdoi@sm.u-tokai.ac.jp