QUALITATIVE BEHAVIOUR OF THE FIRST-PASSAGE-TIME DENSITY OF A ONE-DIMENSIONAL DIFFUSION OVER A MOVING BOUNDARY

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Abstract. We deal with the qualitative behaviour of the first-passage-time density of a one-dimensional diffusion process X(t) over a moving boundary; in particular, we study the value that the first-passage time density takes at zero, the distribution of the maximum process, and the distribution of the first instant at which X(t) attains the maximum in an interval [0,T]. Our results generalize the analogous ones already known for Brownian motion. Some examples are reported.

1 Introduction First-passage time (FPT) problems for random processes are very relevant in a variety of biological applications (see e.g [23]), expecially when one models neural activity (see e.g. [18]), but also when one studies the dynamics of a population.
Many papers have been devoted to find the FPT density of a diffusion process over a moving boundary (see e.g. [8], [10], [24]), although the few analytical results are known only for some special boundaries. In this note we pursue a different intention: we are not concerned in finding explicit formulae for the FPT density of a diffusion process over a given boundary, but our aim is to study the qualitative behaviour of the FPT density and to investigate the distribution of the maximum process. We consider a temporally homogeneous, one-dimensional diffusion process $X(t)$ defined over the interval $I = (r_1, r_2)$, starting from $0 \in I$, and characterized by drift $b(x)$ and infinitesimal variance $\sigma^2(x)$, that is $X(t)$ is the solution of the stochastic differential equation (SDE):

$$dX(t) = b(X(t))dt + \sigma(X(t))dBt, \quad X(0) = 0 \tag{1.1}$$

where B_t is a standard Brownian motion (BM) and the functions b and σ are regular enough. If $S : [0, +\infty) \longrightarrow I$ is a continuous function with $S(0) \geq 0$, let $\tau_S = inf\{t > 0 : X(t) \geq S(t)\}$ be the FPT of $X(t)$ over the (time dependent) boundary $S(t)$, and let $f_S(t)$ denote the FPT density, i.e. the probability density function (p.d.f.) of τ_S. If $S(t) = S$ is constant, then the distribution of τ_S can be studied in terms of the maximum process $M_t = \max_{s\in[0,t]} X(s)$.

Notice that some success in the solution of FPT problems can be achieved in the special case when the process $X(t)$ can be reduced to BM, via a variable change; indeed we do it by using the techniques of [2] and [3], i.e. by combining a deterministic transformation of the process with a random time-change. A different approach, consisting of reducing the Kolmogorov equation for a diffusion to the backward equation for BM was considered in [25].

The value that the FPT density f_S takes at $t = 0$, in terms of the boundary S, is particularly interesting in various numerical methods found in the literature to estimate f_S, when S is given. Thus, for a class of one-dimensional diffusions (1.1), we study the $\lim_{t\to 0^+} f_S(t)$, in the case when $S(t)$ is increasing (locally at zero) and differentiable for $t > 0$. This generalizes the analogous result found in [21] for BM.

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We also investigate the distribution of M_T, both in the case when T is fixed, and when it is a r.v. which is independent of X. The evaluation of the tail probability $P(M_T > z)$ for fixed z, is a key point in many statistical problems (for some examples in genetics, see e.g. [7]).
Moreover, we show that the first instant at which $X(t)$ attains the maximum in the interval $[0, T]$ is a random variable with a compound arc-sine law.

The paper is organized as it follows: section 2 deals with the evaluation of the FPT density at time zero, for a given boundary; section 3 is devoted to study the distribution of the maximum process M_T.

2 Limit at zero of the first-passage-time density In this section, we deal with the limit at $t = 0$ of the FPT density of a one-dimensional diffusion process over a curved boundary; this generalizes the result for BM found in [21]. Let $X(t) \in I = (r_1, r_2)$ $(-\infty \leq r_1 \leq 0 < r_2 \leq +\infty)$ be the solution of the SDE:

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dB_t \\ X(0) = 0 \end{cases} \tag{2.1}$$

where B_t is BM. Throughout the paper, we will suppose that the drift (b) and diffusion (σ) coefficients satisfy the following conditions:

A1 $b, \sigma : I \to \mathbf{R}$ are continuous functions and a constant $K > 0$ exists, such that, for every $x, y \in I$:

$$|b(x) - b(y)| \leq K|x - y|$$

$$b^2(x) + \sigma^2(x) \leq K(1 + x^2)$$

A2 σ is a non-negative, bounded function and it is differentiable for every x belonging to the interior of I. Moreover, there exists a strictly increasing function $\rho : \mathbf{R}^+ \to \mathbf{R}$ such that $\rho(0) = 0$, $\int_{0^+} \rho^{-2}(u)du = +\infty$ and $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$, for every $x, y \in I$.

Conditions A1 and A2 ensure that there exists a unique non-explosive solution of (2.1) (see e.g. [12], [14]); A2 holds, for instance, if $\sigma(\cdot)$ is Lipschitz-continuous, or Holder-continuous of order $\geq 1/2$.

Furthermore, we will suppose that $X(t) \in I$, $\forall t \geq 0$ (sufficient conditions for this can be found e.g. in [4], [13], [14].

Let $S : \mathbf{R}^+ \longrightarrow I$ be a continuous function with $S(0) \geq 0$ and let $\tau_S = \inf\{t > 0 : X(t) \geq S(t)\}$ be the first-passage-time of $X(t)$ over the boundary $S(t)$. We denote by $F_S(t) = P(\tau_S \leq t)$ the cumulative distribution function of τ_S. As pointed out in [11], for regular enough drift and diffusion coefficients, τ_S admits a continuous density $f_S(t) = \frac{d}{dt}F_S(t)$ and in fact we will suppose that this is the case throughout the paper.
Motivated by the analogy with the definition holding for BM (see [15], [21]), we give the following:

Definition 2.1 *A continuous function $S : \mathbf{R}^+ \longrightarrow I$ is said to be an upper function for $X(t)$ if $P(\tau_S > 0) = 1$, otherwise S is said to be a lower function for $X(t)$.*

Notice that, if $S(t)$ is an upper function for $X(t)$, then $S(0) \geq 0$.

Now, we recall the Kolmogorov's test in the case of BM with $I = (-\infty, +\infty)$:

Let $S(t)$ be continuous, increasing and

$$\lim_{t \to 0^+} \frac{S(t)}{\sqrt{t}} = +\infty \tag{2.2}$$

Then S is an upper function for B_t whenever

$$\int_{0^+} \frac{S(t)}{t^{3/2}} e^{-\frac{S^2(t)}{2t}} dt < \infty \tag{2.3}$$

We also recall that the process $X(t)$ is said to be recurrent if for every $x \in I = (r_1, r_2)$ the probability of the process coming back to x infinitely often is one, or equivalently (see e.g. [13]) if for any $x,\ y \in I$, it results $P(\tau_y < \infty) = 1$ where $\tau_y \doteq \inf\{t > 0 : X(t) = y | X(0) = x\}$ is the first-hitting time of X to y when starting from x. A necessary condition for recurrence is (see e.g. [13]):

$$\liminf_{t \to \infty} X(t) = r_1 \quad \textit{and} \quad \limsup_{t \to \infty} X(t) = r_2.$$

Let us consider now the infinitesimal generator L associated to the diffusion (2.1):

$$Lh(x) = b(x)h'(x) + \frac{1}{2}h''(x)\sigma^2(x), \quad h \in C^2(I) \tag{2.4}$$

and let $u(x) \in C^2(I)$ be the solution of the problem:

$$\begin{cases} Lu(x) = 0, \ x \in I \\ u(0) = 0; \ u'(0) = 1 \end{cases} \tag{2.5}$$

Setting, for $x \in I$:

$$\xi(x) = \exp\left(-\int_0^x \frac{2b(z)}{\sigma^2(z)} dz\right)$$

it is easily seen that the function $u(x)$ is explicitly given by:

$$u(x) = \int_0^x \xi(t) dt \tag{2.6}$$

It is called the *scale* function.
If the boundaries r_1 and r_2 of I are unattainable (see e.g. [12], [13]) (for $I = (-\infty, +\infty)$ this means that diffusion $X(t)$ does not explode), the recurrence of $X(t)$ is equivalent to the conditions (see [13]):

$$\lim_{x \to r_1} u(x) = -\infty \quad \lim_{x \to r_2} u(x) = +\infty.$$

For instance, BM is recurrent, being in this case $u(x) = x$.
The function u given by (2.6) is strictly increasing, and the process $Y(t) \doteq u(X(t))$ is a local martingale; in fact, by Itô's formula it follows:

$$dY(t) = u'(u^{-1}(Y(t)))\sigma(u^{-1}(Y(t)))dB_t \tag{2.7}$$

We denote by

$$\langle Y \rangle_t = \int_0^t [u'(X(s))\sigma(X(s))]^2 ds \tag{2.8}$$

the quadratic variation of the process $Y(t)$. Then, the following holds:

Proposition 2.1 *Let $S : \mathbf{R}^+ \longrightarrow I$ be a continuous increasing function. Let us assume that the solution $X(t)$ of (2.1) is recurrent and that*

$$\langle Y \rangle_\infty = \infty \tag{2.9}$$

Moreover, we suppose that there exist two deterministic, continuous increasing functions $\alpha(t)$ and $\beta(t)$, with $\alpha(0) = \beta(0) = 0$, such that for every $t \in [0, \delta], \delta > 0$:

$$\alpha(t) \leq \langle Y \rangle_t \leq \beta(t) \tag{2.10}$$

Furthermore, set

$$\tilde{S}_\alpha(t) = u \circ S \circ \alpha^{-1}(t) \ , \ \tilde{S}_\beta(t) = u \circ S \circ \beta^{-1}(t)$$

and let us suppose that the following conditions hold:

$$\int_{0^+} \frac{\tilde{S}_\alpha(t)}{t^{3/2}} e^{-\frac{\tilde{S}^2_\beta(t)}{2t}} dt < \infty \tag{2.11}$$

$$\lim_{t \to 0^+} \frac{\tilde{S}_\beta(t)}{\sqrt{t}} = +\infty \tag{2.12}$$

Then $S(t)$ is an upper function for $X(t)$.

Proof. Let u be the function defined in (2.6) and $Y(t) = u(X(t))$, then we have:

$$\tau = \tau_S = \inf\{t > 0 : X(t) \geq S(t)\} = \inf\{t > 0 : Y(t) \geq (u \circ S)(t)\} \tag{2.13}$$

Thanking to (2.9) we can use a random time-change (see e.g. [22]), and we obtain that there exists a Wiener process $\tilde{B}_t$ such that a.s.

$$Y(t) = \tilde{B}_{\langle Y \rangle_t} \tag{2.14}$$

Thus:

$$\tau = \inf\{t > 0 : \tilde{B}_{\langle Y \rangle_t} \geq (u \circ S)(t)\} \tag{2.15}$$

The random function $\rho(t) \doteq \langle Y \rangle_t$ is an increasing process, so:

$$\rho(\tau) = \langle Y \rangle_\tau = \inf\{r > 0 : \tilde{B}_r \geq (u \circ S \circ A)(r)\} \doteq \tilde{\tau} \tag{2.16}$$

where we have denoted by $A(t)$ the "inverse" of the random function $\rho(t)$, i.e.:

$$A(t) = \inf\{s > 0 : \langle Y \rangle_s > t\} \tag{2.17}$$

and $\tilde{\tau}$ is the first-passage-time of the Wiener process $\tilde{B}$ over the random increasing function $(u \circ S \circ A)$.
Therefore $P(\tau > 0) = 1$ if and only if $P(\tilde{\tau} > 0) = 1$. This means that $S(t)$ is upper function for $X(t)$ iff $(u \circ S \circ A)(t)$ is a (random) upper function for $\tilde{B}_t$. Note that, in general, $A(t)$ is not deterministic, but it is bounded from above and below by the two deterministic functions:

$$\beta^{-1}(t) \leq A(t) \leq \alpha^{-1}(t), \ \ t \in [0, \delta] \tag{2.18}$$

By using (2.18) is now straightforward to verify that (2.11) and (2.12) imply that $u \circ S \circ A$ is an upper function for $\tilde{B}_t$, from which the result follows, thanking to the observation above. $\Box$

Remark 2.1 For a recurrent process satisfying (2.9), (2.10), (2.11) and (2.12), Proposition 2.1 gives a sufficient condition so that S is an upper function for $X(t)$, reducing the problem to that of the BM associated to $X(t)$ via the combination of the deterministic transformation $x \longrightarrow y = u(x)$ and the random time-change given by (2.14). In this way, we are able to transform a generally difficult problem into a simpler one. A different approach was followed in [25], where a space-time transformation was considered which maps the Kolmogorov equation associated to the diffusion process $X(t)$ into the backward equation for the Wiener process. This transformation jointly acts on space and time and, in some cases, it changes the nature of the boundaries of I.

Now we recall the result of Peskir, holding for BM.

Theorem 2.1 ([21]) *Let $\tilde{S} : \mathbf{R}^+ \longrightarrow \mathbf{R}$ be an upper function for B_t satisfying $\tilde{S}(0) = 0$, and let $\tilde{\tau}$ be the first-passage time of B_t over $\tilde{S}$. Assume that $\tilde{S}$ is C^1 on $(0, +\infty)$, increasing (locally at zero), and concave (locally at zero). Then, the following identities hold for the density $\tilde{f}$ of $\tilde{\tau}$:*

$$\tilde{f}(0^+) = \lim_{t \to 0^+} \left(\frac{1}{2} \frac{\tilde{S}(t)}{t^{3/2}} \phi \left(\frac{\tilde{S}(t)}{\sqrt{t}} \right) \right) = \lim_{t \to 0^+} \left(\frac{\tilde{S}'(t)}{\sqrt{t}} \phi \left(\frac{\tilde{S}(t)}{\sqrt{t}} \right) \right) \tag{2.19}$$

in the sense that if the second and third limit exist so does the first one; here $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ and the limits can take any value in $[0, +\infty]$.
Furthermore, no matter if $\tilde{S}$ is concave or not, for $\tilde{S}(0) \geq 0$, whenever the limit $\tilde{f}(0^+)$ exists (and it is finite), the following formula holds:

$$\tilde{f}(0^+) = \lim_{t \to 0^+} \frac{\Psi(\frac{\tilde{S}(t)}{\sqrt{t}})}{\int_0^t \Psi(\frac{\tilde{S}(t) - S(s)}{\sqrt{t-s}}) ds}$$

where $\Psi(x) = 1 - \Phi(x) = 1 - \int_{-\infty}^x \phi(t) dt$. $\Box$

Remark 2.2 Let $\tilde{S} : \mathbf{R}^+ \longrightarrow \mathbf{R}$ be continuous and satisfying $\tilde{S}(0) > 0$. Then, if $\tilde{S}$ is either increasing (locally at zero) or decreasing (locally at zero) $\tilde{f}(0^+)$ can only be zero (see Corollary 2.2 and Proposition 2.4 of [21]).

Definition 2.2 *We say that the diffusion process $X(t) \in I$ which is the solution of the SDE (2.1) is conjugated to BM if there exists an increasing differentiable function $v : I \longrightarrow \mathbf{R}$ with $v(0) = 0$, such that the process $Y(t) \doteq v(X(t))$ is BM.*

Notice that, if $X(t)$ is conjugated to BM, then X is recurrent.

Example 2.1 Let $X(t)$ be the solution of the SDE:

$$\begin{cases} dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))dt + \sigma(X(t))dB_t \\ X(0) = 0 \end{cases} \tag{2.20}$$

with $\sigma(\cdot) \geq 0$, and let $S(t)$ be an increasing continuous function on $[0, +\infty)$ with $S(0) = 0$. Let us suppose that for every $x \in I$ the integral:

$$v(x) = \int_0^x \frac{1}{\sigma(r)} dr \tag{2.21}$$

is convergent; by Itô's formula we obtain that $v(X(t))$ coincides with B_t, i.e. $X(t)$ is conjugated to B_t. Then:

$$\tau = \inf\{t > 0 : X(t) \geq S(t)\} = \inf\{t > 0 : v(X(t)) \geq (v \circ S)(t)\} =$$

$$= \inf\{t > 0 : B_t \geq \tilde{S}(t)\}$$

where, since v is increasing, $\tilde{S}(t) \doteq (v \circ S)(t)$ is increasing too. So, if

$$\int_{0^+} \frac{v(S(t))}{t^{3/2}} e^{-v^2(S(t))/2t} dt < +\infty \quad and \quad \lim_{t \to 0^+} \frac{v(S(t))}{\sqrt{t}} = +\infty$$

then $v \circ S$ is an upper function for B_t and therefore S is an upper function for $X(t)$. In this case, with the notations of Proposition 2.1, $Y(t) \equiv B_t$, $\rho(t)$ is deterministic, being $\rho(t) = t$. The density f of the first-passage time τ of $X(t)$ over S satisfies: $f(0^+) = \tilde{f}(0^+)$, where $\tilde{f}$ is the density of the first-passage time of B_t over $\tilde{S} = v \circ S$. Therefore, by Theorem 2.1 we get the value of $f(0^+)$ by replacing $\tilde{S}$ in (2.19) with $v \circ S$.

Now, consider for instance the boundary:

$$S(t) = v^{-1}\left(\sqrt{2t \log(1/t) + t \log\log(1/t) + ct}\right) \tag{2.22}$$

where v is the function defined by (2.21) and c is a positive constant. Notice that $\tilde{S}(t) \doteq (v \circ S)(t)$ is the upper function for B_t considered in the Example 2.3 of (Peskir, 2002), and so the boundary S given by (2.22) is an upper function for $X(t)$. We have $f(0^+) = \tilde{f}(0^+) = e^{-c/2}/\sqrt{4\pi}$ (see [21], pg. 8).

Example 2.2 (Feller process) Taking $\sigma(x) = \sqrt{x \vee 0}$ in the Example 2.1, we obtain the process $X(t) \in [0, +\infty)$ which is the solution of the SDE:

$$dX(t) = \frac{1}{4} dt + \sqrt{X(t) \vee 0}\, dB_t\ , X(0) = 0 \tag{2.23}$$

(note that, although $\sqrt{x}$ is not Lipschitz-continuous, the solution is unique because $\sqrt{x}$ is Holder-continuous (see e.g. condition A2)). We have $v(x) = 2\sqrt{x}$ and, if S is the boundary defined by (2.22), $\tilde{S}(t) = (v \circ S)(t) = 2\sqrt{S(t)}$. Thus $S(t) = [2t \log(1/t) + t \log\log(1/t) + ct]/4$ is an upper function for $X(t)$ and $f(0^+) = e^{-c/2}/\sqrt{4\pi}$.

Example 2.3 (Wright & Fisher-like process) Taking $\sigma(x) = \sqrt{x(1-x) \vee 0}$ in the Example 2.1, we obtain the SDE:

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2} X(t)\right) dt + \sqrt{X(t)(1 - X(t)) \vee 0}\, dB_t,\ X(0) = 0 \tag{2.24}$$

It is a particular case of the Wright & Fisher diffusion equation for population genetics, and it is also used in certain diffusion models for neural activity (see e.g. [18]); it can be shown (see e.g. [4], [5]) that $X(t)$ remains in the interval $[0, 1]$ for every time $t \geq 0$. As it is easy to see, $X(t)$ is conjugated to BM by means of the function $v(x) = 2\arcsin\sqrt{x}$ i.e. $v(X(t)) = 2\arcsin\sqrt{X(t)} \equiv B_t$.

Now we turn to the generalization of Theorem 2.1 to a more general diffusion process. The following holds:

Theorem 2.2 *Let $X(t)$ be the solution of the SDE (2.1) and let $S : \mathbf{R}^+ \longrightarrow I$ be an upper function for $X(t)$ satisfying $S(0) = 0$, and τ the first-passage time of $X(t)$ over S. Let us suppose that all the assumptions of Proposition 2.1 hold and that the functions $\alpha(t)$ and $\beta(t)$ (see (2.10)) are differentiable at $t = 0$. Finally, set $\tilde{S}_\alpha(t) = (u \circ S \circ \alpha^{-1})(t)$, $\tilde{S}_\beta(t) = (u \circ S \circ \beta^{-1})(t)$ and denote by $\tilde{\tau}_\alpha$, $\tilde{\tau}_\beta$ the FPT of BM over $\tilde{S}_\alpha(t)$ and $\tilde{S}_\beta(t)$, respectively. Then for the density f of τ it holds:*

$$\alpha'(0)\tilde{f}_\alpha(0) \le f(0^+) \le \beta'(0)\tilde{f}_\beta(0) \tag{2.25}$$

where $\tilde{f}_\alpha$, $\tilde{f}_\beta$ denote the densities of $\tilde{\tau}_\alpha$ and $\tilde{\tau}_\beta$, respectively.

Furthermore, assume that $S, \tilde{S}_\alpha, \tilde{S}_\beta$ are C^1 on $(0, +\infty)$, increasing (locally at zero), and concave (locally at zero). Then:

$$\alpha'(0) \cdot \lim_{t\to 0^+} \left(\frac{1}{2} \frac{\tilde{S}_\alpha(t)}{t^{3/2}} \phi\left(\frac{\tilde{S}_\alpha(t)}{\sqrt{t}} \right) \right) \le f(0^+) \le \beta'(0) \cdot \lim_{t\to 0^+} \left(\frac{1}{2} \frac{\tilde{S}_\beta(t)}{t^{3/2}} \phi\left(\frac{\tilde{S}_\beta(t)}{\sqrt{t}} \right) \right) \tag{2.26}$$

where $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$, $f(0^+) = \lim_{t\to 0^+} f(t)$.

Proof. Using the notations of Proposition 2.1 (see (2.16)), we have:

$$P\{\tau \le t\} = P\{\rho(\tau) \le \rho(t)\} = P\{\tilde{\tau} \le \rho(t)\}$$

so, by (2.10) we get

$$P\{\tilde{\tau} \le \alpha(t)\} \le P\{\tau \le t\} \le P\{\tilde{\tau} \le \beta(t)\} \tag{2.27}$$

Then, since:

$$\tilde{\tau}_\beta = \inf\{r : \tilde{B}_r \ge (u \circ S \circ \beta^{-1})(r)\},\ \tilde{\tau}_\alpha = \inf\{r : \tilde{B}_r \ge (u \circ S \circ \alpha^{-1})(r)\}$$

by (2.18) we obtain $\tilde{\tau}_\beta \le \tilde{\tau} \le \tilde{\tau}_\alpha$ a.s. and so (2.27) yields:

$$P\{\tilde{\tau}_\alpha \le \alpha(t)\} \le P\{\tau \le t\} \le P\{\tilde{\tau}_\beta \le \beta(t)\} \tag{2.28}$$

Therefore:

$$\int_0^{\alpha(t)} \tilde{f}_\alpha(s)ds \le \int_0^t f(s)ds \le \int_0^{\beta(t)} \tilde{f}_\beta(s)ds \tag{2.29}$$

By dividing all members of (2.29) by $t > 0$, and passing to the limit as $t \to 0^+$, we get:

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^{\alpha(t)} \tilde{f}_\alpha(s)ds \le f(0^+) \le \lim_{t\to 0^+} \frac{1}{t} \int_0^{\beta(t)} \tilde{f}_\beta(s)ds \tag{2.30}$$

By applying the Hospital's rule to (2.30), we finally obtain (2.25). The second part of the Theorem follows from (2.25), by using (2.19). □

Remark 2.3 In the special case when $\rho(t) = \langle Y \rangle_t$ is deterministic, $A = \rho^{-1}$ is deterministic too. Then, if $\tilde{f}$ denotes the density of $\tilde{\tau}$, it holds:

$$f(0^+) = \rho'(0^+)\tilde{f}(0^+) \tag{2.31}$$

Notice that, if $X(t)$ is conjugated to B_t, then $Y(t) = B_t$ and $\rho(t) = t$; so (2.31) gives $f(0^+) = \tilde{f}(0^+)$.

By using (2.25) and Corollary 2.2 of [21], we get:

Corollary 2.1 *Let us suppose that $X(t)$ satisfies all the assumptions of Theorem 2.2 and let $S : \mathbf{R}^+ \longrightarrow I$ be a continuous function with $S(0) \geq 0$. If there exist $\epsilon > 0$ and $\delta > 0$ such that, for every $t \in (0, \delta)$:*

$$S(t) \geq S_\epsilon(t) \doteq u^{-1}(\sqrt{-(2+\epsilon)\beta(t)\log(\beta(t))}) \tag{2.32}$$

then $f(0^+) = 0$.
If there exists $\delta > 0$ such that, for every $t \in (0, \delta)$:

$$S(t) \leq S_0(t) \doteq u^{-1}(\sqrt{-2\alpha(t)\log(\alpha(t))}) \tag{2.33}$$

then $f(0^+) = +\infty$.
Therefore S_0 and S_ϵ appear to separate those boundaries S implying $f(0^+) = 0$ from those implying $f(0^+) = +\infty$.

Proof. Inequality (2.32) implies $\tilde{S}_\beta(t) \geq \sqrt{-(2+\epsilon)t\log t}$, $t \in (0, \delta)$; then from Corollary 2.2 of [21] it follows that $\tilde{f}_\beta(0^+) = 0$ and therefore, by using (2.25), $f(0^+) = 0$. Analogously, from (2.33) it follows $\tilde{S}_\alpha(t) \leq \sqrt{-2t\log t}$, $t \in (0, \delta)$ and by Corollary 2.2 of [21] again we get $\tilde{f}_\alpha(0^+) = +\infty$ which implies, by using (2.25), that $f(0^+) = +\infty$. □

Remark 2.4 Let $S(t)$ (with $S(0) > 0$) be an upper boundary for $X(t)$. If $X(t)$ is conjugated to B_t via the function v, we have:

$$f_S(0^+) = f^B_{v\circ S}(0^+) \tag{2.34}$$

where $f^B_{v\circ S}$ denotes the density of the FPT of B_t over the boundary $v \circ S$. Since $S(0) > 0$ and v is increasing, also $(v \circ S)(0) > 0$; therefore (see also Peskir, 2002) $f^B_{v\circ S}(0^+) = 0$ and then by (2.34) we get $f_S(0^+) = 0$, too. However, even if $X(t)$ is not conjugated to B_t, when $S(0) > 0$ and the assumptions of Proposition 2.1 hold, one gets $\tilde{f}_\alpha(0^+) = \tilde{f}_\beta(0^+) = 0$, so by (2.25) it follows $f_S(0^+) = 0$, too. Instead, if $S(0) = 0$, by Peskir's result for BM (see [21]), $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ are allowed to take positive values at zero; thus, unless $\alpha'(0)$ and $\beta'(0)$ are both zero (see (2.25)), we are not able to conclude that $f_S(0^+) = 0$.
On the other hand, if $S(t)$ is an upper boundary for $X(t)$ and $f_S(0^+) > 0$, then it must be $S(0) = 0$. Indeed, by (2.25) we get $\beta'(0)\tilde{f}_\beta(0^+) > 0$ and so (because $\beta'(0) > 0$) we obtain $\tilde{f}_\beta(0^+) > 0$ i.e. the FPT density of B_t over the boundary $v \circ S \circ \beta^{-1}$ takes a positive value at zero; by this (see Peskir, 2002) it follows that $v \circ S \circ \beta^{-1}(0) = 0$ which implies $S(0) = 0$. Moreover, by using Peskir's argument we can see that, if $f_S(0^+) = 0$, then $S(0)$ is allowed to take a positive value.

Brownian motion with drift

In this subsection we treat the case of Brownian motion with drift, i.e. the process $X(t) = B_t + \mu t,\ \mu \in \mathbf{R}$. Notice that this process is not recurrent. For a boundary S, we have:

$$\tau = \tau_S = \inf\{t > 0 : B_t + \mu t \geq S(t)\} \equiv \hat{\tau} \doteq \inf\{t > 0 : B_t \geq \hat{S}(t)\}$$

where $\hat{S}(t) \doteq S(t) - \mu t$.

Remark 2.5 By using the Kolmogorov conditions, it is easy to see that $S(t)$ is an upper function for $B_t + \mu t$ if and only if $S(t)$ is an upper function for B_t.

Proposition 2.2 *Let $S(t)$ be an upper function for $B_t + \mu t$ which is increasing and concave (locally at zero). Assume that S is C^1 on $(0, +\infty)$ and $S'(t) \geq \mu$, locally at zero, and let f_0 be the FPT density of BM over $S(t)$. Then, the FPT density f_μ of $B_t + \mu t$ over S satisfies:*

$$f_\mu(0^+) = e^{\mu S(0)} \left[f_0(0^+) - \mu \lim_{t \to 0} \frac{\phi(S(t)/\sqrt{t})}{\sqrt{t}} \right] \tag{2.35}$$

Proof. By Theorem 2.1, we have:

$$\begin{aligned} f_\mu(0^+) &= \lim_{t \to 0^+} \left[\frac{\hat{S}'(t)}{\sqrt{t}} \phi\left(\frac{\hat{S}(t)}{\sqrt{t}}\right) \right] = \lim_{t \to 0^+} \left[\frac{S'(t) - \mu}{\sqrt{t}} \phi\left(\frac{S(t)}{\sqrt{t}} - \mu\sqrt{t}\right) \right] \\ &= \lim_{t \to 0^+} \left[\frac{S'(t)}{\sqrt{t}} \phi\left(\frac{S(t)}{\sqrt{t}}\right) e^{\mu S(t) - \mu^2 t/2} - \frac{\mu}{\sqrt{t}} \phi\left(\frac{S(t)}{\sqrt{t}}\right) e^{\mu S(t) - \mu^2 t/2} \right] \end{aligned}$$

from which (2.35) follows. □

Remark 2.6
(i) If $S(0) > 0$, then $f_0(0^+) = 0$ (see Remark 2.2) and so:

$$f_\mu(0^+) = -\mu e^{\mu S(0)} \cdot \lim_{t \to 0} \frac{\phi(S(t)/\sqrt{t})}{\sqrt{t}}.$$

(ii) If $S(0) = 0$, then $f_0(0^+) > 0$ and so $f_\mu(0^+) = f_0(0^+) - \mu \cdot \lim_{t \to 0} \frac{\phi(S(t)/\sqrt{t})}{\sqrt{t}}$.
(iii) For a number of boundaries S which are upper functions for B_t it holds:

$$\lim_{t \to 0} \frac{\phi(S(t)/\sqrt{t})}{\sqrt{t}} = 0 \tag{2.36}$$

and therefore $f_\mu(0^+) = e^{\mu S(0)} f_0(0^+)$. For instance, this is the case for the following functions:
(a) $S(t) = t^{1/4}$; (b) $S(t) = \sqrt{(2 + \epsilon) t \log(1/t)},\ \epsilon > 0$;
(c) $S(t) = \sqrt{2t \log(1/t)}$; (d) $S(t) = \sqrt{2t \log(1/t) + t \log\log(1/t) + ct},\ c > 0$.
All of them vanishes at zero, so from (ii) it follows that $f_\mu(0^+) = f_0(0^+)$.
However, in general (2.36) does not hold; for instance $S(t) = \sqrt{-2t \log t^{1/2 - \delta}}$ $(0 < \delta < 1/2)$ is an upper function for B_t but $\lim_{t \to 0} \frac{\phi(S(t)/\sqrt{t})}{\sqrt{t}} = \lim_{t \to 0} \frac{t^{1/2 - \delta}}{\sqrt{t}} = +\infty$. In this case, it is easy to see that both $f_0(0^+)$ and $f_\mu(0^+)$ are equal to $+\infty$.

Remark 2.7 By considering the process $Z(t) \doteq X(t) - X(0)$ and replacing $S(t)$ with $S(t) - X(0)$ the results of section 2 can be generalized to the case when $X(0) \neq 0$.

3 The maximum process In this section, we investigate the distribution of the maximum process

$$M_T = \max_{s \in [0,T]} X(s)$$

where $X(t)$ is the solution of (2.1). If $S(t) = S$ is a constant threshold, then the distribution of τ_S can be studied in terms of M_T, since for a given $T > 0$ and $S \geq 0$:

$$P(\tau_S \leq T) = P(M_T \geq S)$$

Note that, in order to make the FPT problem meaningful, we will assume that $X(t)$ is recurrent, otherwise τ_S may be infinite with positive probability.

We start from the case when T is a fixed, deterministic quantity.
Unlike BM, for general stochastic processes closed formulae for $P(M_t \leq z)$ are not available; in certain applications one is satisfied with the determination of the tail behaviour of $P(M_T > z)$ for some fixed $T > 0$. For instance, when $X(t)$ is a Gaussian process with stationary increments, under certain conditions, it holds for $z \to +\infty$ ([6]):

$$P(M_T > z) \sim const \cdot z^{\beta} \Psi\left(\frac{z}{\sigma_X(T)}\right) \tag{3.1}$$

where β is a positive constant, $\Psi(x) = P(W > x)$ is the tail distribution of a standard Gaussian random variable W, and $\sigma_X^2(t)$ is the variance function of $X(t)$. In particular, if $X(t)$ is BM (see e.g. [16]):

$$P(M_T > z) = 2\Psi\left(\frac{z}{\sqrt{T}}\right) \tag{3.2}$$

For a diffusion satisfying our assumptions, the following holds (for the proof see [1]):

Theorem 3.1 ([1]) *Let be given $T > 0$ and let us assume that the solution $X(t)$ of (2.1) is recurrent and that*

$$\langle Y \rangle_\infty = \infty \tag{3.3}$$

where the process $Y(t)$ is defined by (2.7). Moreover, with the notations of the Proposition 2.1, we suppose that there exist two deterministic, continuous increasing functions $\alpha(t)$ and $\beta(t)$, with $\alpha(0) = \beta(0) = 0$, such that for every $t < T$:

$$\alpha(t) \leq \langle Y \rangle_t \leq \beta(t) \tag{3.4}$$

Then, for any $z > 0$:

$$2\Phi\left(\frac{u(z)}{\sqrt{\beta(T)}}\right) - 1 \leq P(M_T \leq z) \leq 2\Phi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right) - 1 \tag{3.5}$$

or, equivalently:

$$2\Psi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right) \leq P(M_T > z) \leq 2\Psi\left(\frac{u(z)}{\sqrt{\beta(T)}}\right) \tag{3.6}$$

where $\Phi(x) = 1 - \Psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$ □

If the quadratic variation $\langle Y \rangle_t = \rho(t)$ of $Y(t)$ is deterministic (this is e.g. the case when $Y(t)$ is an integral process with a deterministic integrand – see Remark 3.1), we have: $\rho(t) = \alpha(t) \equiv \beta(t)$ and (3.5), (3.6) become:

$$P(M_T \leq z) = 2\Phi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right) - 1 \tag{3.5$'$}$$

$$P(M_T > z) = 2\Psi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right) \tag{3.6$'$}$$

Note the affinity between (3.6') and (3.2)).
Moreover, if $X(t)$ is conjugated to BM by means of the function v, then $Y(t) = v(X(t)) \equiv B_t$, being $\langle Y \rangle_t = \alpha(t) = \beta(t) = t$, so using (3.2) we obtain:

$$P(M_T \leq z) = 2\Phi\left(\frac{v(z)}{\sqrt{T}}\right) - 1 \tag{3.5$''$}$$

$$P(M_T > z) = 2\Psi\left(\frac{v(z)}{\sqrt{T}}\right) \tag{3.6$''$}$$

Note that $\lim_{x\to+\infty} \frac{\Psi(x)}{e^{-x^2/2}} = 0$, so $P(M_T > z)$ decays more fastly then $e^{-\frac{v^2(z)}{2T}}$, as $z \to \infty$. If $X(t) \equiv B_t$, then (3.6$''$) with $v(z) = z$, is the well-known formula (3.2) for the distribution of the maximum of BM.

As an application of the above results, we consider now a jump-diffusion process $\tilde{X}(t)$ that is obtained as a superposition of a simple–diffusion process $X(t)$ and a homogeneous Poisson process $N(t)$. We suppose that $N(t)$ is independent of B_t, and jumps of amplitude $\epsilon > 0$ can occur at exponentially distributed time–intervals, with rate $\lambda > 0$. The following holds:

Proposition 3.1 ([1]) *For $\epsilon > 0$, let us consider the process $\tilde{X}(t)$ which is the solution of the jump-diffusion equation:*

$$d\tilde{X}(t) = b(\tilde{X}(t))dt + \sigma(\tilde{X}(t))dB_t + \epsilon dN(t), \ \tilde{X}(0) = 0$$

where $N(t)$ is a homogeneous Poisson process with rate $\lambda > 0$, which is independent of B_t. Let $X(t)$ be the simple-diffusion process obtained from the equation above disregarding the jumps (i.e. $X(t)$ solves the SDE $dX(t) = b(X(t))dt + \sigma(X(t))dB_t$, $X(0) = 0$) and let us suppose that $X(t)$ satisfies all the assumptions of Theorem 2.1. Then, it holds:

$$P\left(\max_{t\in[0,T]} \tilde{X}(t) > z\right) = e^{-\lambda T} \sum_{n=0}^{\infty} a_n \frac{(\lambda T)^n}{n!}$$

where

$$a_n = \begin{cases} P(M_T \geq z - n\epsilon) & \text{if } 0 \leq n \leq [z/\epsilon] \\ 1 & \text{if } n > [z/\epsilon] \end{cases},$$

$M_T = \max_{s\in[0,T]} X(s)$ and $P(M_T > z - n\epsilon)$ can be estimated by using (3.6). In particular, if $X(t)$ is conjugated to BM by means of the function v, we get from (3.6"):

$$P\left(M_T > z - n\epsilon\right) = 2\Psi\left(\frac{v(z - n\epsilon)}{\sqrt{T}}\right)$$

□

Turning to diffusions, we go to study the asymptotics of $P(M_T > z)$, as $z \to +\infty$, in the case when T is a random variable independent of the process X. We say that T has regularly varying tails with index $\nu \geq 0$, and we will write $T \in RV(\nu)$, if $P(T > t) = L(t)t^{-\nu}$, where $L(\cdot)$ is a function slowly varying at $+\infty$ i.e. $\lim_{x,y\to+\infty} L(x)/L(y) = 1$.

Theorem 3.2 ([9]) *If $\tilde{B}_t$ is BM and Λ is a nonnegative random variable such that $\Lambda \in RV(\mu)$, then as $z \to \infty$:*

$$P\left(\max_{s\in[0,\Lambda]} \tilde{B}_s > z\right) = P\left(\Lambda^{1/2} \max_{s\in[0,1]} \tilde{B}_s > z\right) \sim E\left(\max_{s\in[0,1]} \tilde{B}_s\right)^{2\mu} P(\Lambda > z^2) \tag{3.7}$$

where

$$E\left(\max_{s\in[0,1]} \tilde{B}_s\right)^{2\mu} = \frac{2^\mu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\mu\right).$$

□

Theorem 3.3 ([1]) *Let $X(t)$ be the solution of the SDE (2.1), and let us suppose that all the assumptions of Theorem 2.1 are satisfied. Moreover, let us assume that the functions $\alpha^{-1}(t)$ and $\beta^{-1}(t)$ are regularly varying at $+\infty$ with index $\gamma > 0$. Then, if $T \in RV(\nu)$, for $z \to +\infty$:*

$$L(\alpha^{-1}(z^2))(\alpha^{-1}(z^2))^{-\nu} \leq P(\rho(T) > z^2) \leq L(\beta^{-1}(z^2))(\beta^{-1}(z^2))^{-\nu} \tag{3.8}$$

Moreover:

$$a\mathcal{E}_1 L_a(z^2) z^{-2\gamma\nu} \leq P(M_T > z) \leq b\mathcal{E}_1 L_b(z^2) z^{-2\gamma\nu} \tag{3.9}$$

where $\mathcal{E}_1 = \frac{2^{\gamma\nu}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\gamma\nu\right)$, a and b are suitable positive constants and L_a, L_b are functions slowly varying at ∞. □

Remark 3.1 A special case in which $\rho(t)$ is deterministic occurs when the diffusion $X(t)$ is an integral process with deterministic integrand, i.e. $X(t) = \int_0^t \bar{\sigma}(s)dB_s$, where $\bar{\sigma}(\cdot) > 0$ is a deterministic function and $\rho(t) = \int_0^t \bar{\sigma}^2(s)ds$ behaves like t^γ, $t \to \infty$. Under these assumptions, $X(t)$ turns out to be a square integrable martingale which is a Gaussian centered process with variance function $\sigma_X^2(t) = \rho(t)$.
For such a process, if T is given and fixed, by (3.6') with $u(z) = z$, we obtain:

$$P(M_T > z) = 2\Psi\left(\frac{z}{\sqrt{\int_0^T \bar{\sigma}^2(s)ds}}\right) \tag{3.10}$$

The instant at which $X(t)$ attains its maximum

Let us consider now, for T given and fixed, the first instant θ at which $X(t)$ attains its maximum value in the interval $[0,T]$, i.e. $X(\theta) = \max_{t\in[0,T]} X(t) = M_T$. Notice that θ is not a stopping time. As it is well-known ([19]) when $X(t) \equiv B_t$, the distribution of θ follows the arc–sine law, that is:

$$P(\theta \leq t) = \frac{2}{\pi}\arcsin\sqrt{\frac{t}{T}},\ 0 < t < T \tag{3.11}$$

This corresponds to the probability density:

$$f_\theta(t) = \frac{1}{\pi\sqrt{t(T-t)}},\ 0 < t < T$$

and it results $E(\theta) = T/2$.
A similar arc–sine law holds for the maximum of the Brownian bridge, i.e. conditioned BM (see [20]).

Let $X(t)$ be a diffusion satisfying the assumptions of Theorem 2.1; first we suppose that the quadratic variation $\rho(t)$ of the local martingale $Y(t)$ associated to $X(t)$, is deterministic. Then, using our notations (see section 2):

$$u(M_T) = \max_{t\in[0,T]} u(X(t)) = \max_{t\in[0,T]} \tilde{B}_{\rho(t)} = \max_{s\in[0,\rho(T)]} \tilde{B}_s$$

So,

$$\tilde{B}_{\rho(\theta)} = u(X(\theta)) = \max_{t\in[0,\rho(T)]} \tilde{B}_t \tag{3.12}$$

and therefore $\rho(\theta)$ obeys the arc–sine law:

$$P(\rho(\theta) \le t) = \frac{2}{\pi} \arcsin\sqrt{\frac{t}{\rho(T)}},\quad t \in (0, \rho(T))$$

From this it follows:

$$P(\theta \le t) = P(\rho(\theta) \le \rho(t)) = \frac{2}{\pi} \arcsin\sqrt{\frac{\rho(t)}{\rho(T)}},\quad t \in [0, T] \tag{3.13}$$

In particular, if $X(t)$ is conjugated to BM, then $\rho(t) = t$ and so θ follows the arc–sine law. If $\rho(t)$ is not deterministic, recalling that $\alpha(t) \le \rho(t) \le \beta(t)$, we get:

$$\max_{t\in[0,\alpha(T)]} \tilde{B}_t \le u(M_T) \le \max_{t\in[0,\beta(T)]} \tilde{B}_t \tag{3.14}$$

If we denote by $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$, the first instant at which $\tilde{B}_t$ attains its maximum in the interval $[0, \alpha(T)]$ and in the interval $[0, \beta(T)]$, respectively, we obtain:

$$\tilde{\theta}_\alpha \le \rho(\theta) \le \tilde{\theta}_\beta \tag{3.15}$$

Thus:

$$\frac{2}{\pi} \arcsin\sqrt{\frac{t}{\beta(T)}} \le P(\rho(\theta) \le t) \le \frac{2}{\pi} \arcsin\sqrt{\frac{t}{\alpha(T)}},\quad t \in (0, \alpha(T)) \tag{3.16}$$

and therefore

$$\frac{2}{\pi} \arcsin\sqrt{\frac{\alpha(t)}{\beta(T)}} \le P(\theta \le t) \le \frac{2}{\pi} \arcsin\sqrt{\frac{\beta(t)}{\alpha(T)}},\ 0 < t < \beta^{-1}(\alpha(T))$$

In particular, $E(\tilde{\theta}_\alpha) \le E(\rho(\theta)) \le E(\tilde{\theta}_\beta)$ and so $\frac{1}{2}\alpha(T) \le E(\rho(\theta)) \le \frac{1}{2}\beta(T)$. In the following,

we report a few examples of diffusion processes for which the results of this section apply (for more see [1]).

Example 3.1 (Ornstein-Uhlenbeck process) Let us consider the process $X(t)$ which is the solution of the SDE:

$$dX(t) = -bX(t)dt + \sigma dB_t, \; X(0) = X_0 \tag{3.17}$$

where b and σ are positive constants. The explicit solution of (3.17) is (see e.g. [17]):

$$X(t) = e^{-bt}U(t) \tag{3.18}$$

where $U(t) = X_0 + \int_0^t \sigma e^{bs} dB_s$. Setting $Y(t) = \int_0^t \sigma e^{bs} dB_s$, and using a random time–change, we can write:

$$U(t) = X_0 + \tilde{B}_{\rho(t)} \tag{3.19}$$

where

$$\rho(t) = \langle Y \rangle_t = \frac{\sigma^2}{2b}\left(e^{2bt} - 1\right) \tag{3.20}$$

In this case the quadratic variation of the process $Y(t)$ associated to $X(t)$ is deterministic. Since $\max_{s\in[0,T]} e^{-bs}U(s) \leq \max_{s\in[0,T]} U(s)$, we have:

$$P(M_T > z) = P\left(\max_{s\in[0,T]} X(s) > z\right) = P\left(\max_{s\in[0,T]} e^{-bs}U(s) > z\right) \leq \tag{3.21}$$

$$\leq P\left(\max_{s\in[0,T]} U(s) > z\right) = P\left(\max_{s\in[0,T]} \tilde{B}_{\rho(s)} > z - X_0\right) = P\left(\max_{t\in[0,\rho(T)]} \tilde{B}_t > z - X_0\right)$$

If T is given and fixed, from (3.21) we get:

$$P(M_T > z) \leq 2\Psi\left(\frac{z - X_0}{\sqrt{\rho(T)}}\right) \tag{3.22}$$

Now, let us suppose that T has tail which decays at an exponential rate, in the following way:

$$P(T > z) = L(z)(\rho(z))^{-\nu}, \; z \to \infty \tag{3.23}$$

with $\rho(t)$ given by (3.20). Then, as $z \to \infty$:

$$P(\rho(T) > z) = P(T > \rho^{-1}(z)) = L(\rho^{-1}(z))z^{-\nu}$$

and so $\rho(T) \in RV(\nu)$. By using (3.21) and (3.7), we obtain:

$$P\left(\max_{s\in[0,T]} U(s) > z\right) \sim E\left(\max_{s\in[0,1]} \tilde{B}_s\right)^{2\nu} P\left(\rho(T) > (z - X_0)^2\right) \tag{3.24}$$

Thus:

$$P\left(\max_{s\in[0,T]} U(s) > z\right) \sim \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} + \nu\right) L\left(\rho^{-1}((z - X_0)^2)\right)(z - X_0)^{-2\nu} \tag{3.25}$$

Finally, (3.21) implies that, for $z \to \infty$:

$$P\left(\max_{s\in[0,T]} X(s) > z\right) \leq \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} + \nu\right) L\left(\rho^{-1}((z - X_0)^2)\right)(z - X_0)^{-2\nu} \tag{3.26}$$

that is the rate of decay of $P(M_T > z)$, as $z \to \infty$, is at least $z^{-2\nu}$.

Example 3.2 (Feller process) Let $X(t)$ be the process considered in Example 2.2 (see section 2). Recall that $X(t)$ is conjugated to Brownian motion by means of the function $v(x) = 2\sqrt{x}$ i.e. $v(X(t)) = 2\sqrt{X(t)} \equiv B_t$. Thus, if T is given and fixed we obtain:

$$P(M_T > z) = P\left(\max_{s\in[0,T]} X(s) > z\right) = \tag{3.27}$$

$$P\left(\max_{s\in[0,T]} v(X(s)) > v(z)\right) = P\left(\max_{s\in[0,T]} B_s > 2\sqrt{z}\right) = 2\Psi\left(\frac{2\sqrt{z}}{\sqrt{T}}\right)$$

If $T \in RV(\nu)$, by using Theorem 3.2, we get for $z \to \infty$:

$$P(M_T > z) = P\left(\max_{s\in[0,T]} B_s > 2\sqrt{z}\right) \sim \tag{3.28}$$

$$\sim \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\nu\right) P(T > 4z) = \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\nu\right) L(4z)(4z)^{-\nu}$$

and so the rate of decay of $P(M_T > z)$ for $z \to \infty$, is $z^{-\nu}$. For what concerns θ, it follows the arc–sine law, since $X(t)$ is conjugated to BM.

Example 3.3 (Wright & Fisher-like process) Let $X(t)$ be the the process considered in Example 2.3 (see section 2). Recall that $X(t)$ is conjugated to BM by means of the function $v(x) = 2\arcsin\sqrt{x}$ i.e. $v(X(t)) = 2\arcsin\sqrt{X(t)} \equiv B_t$. Let T be given and fixed, then by (3.6") we get:

$$P(M_T > z) = 2\Psi\left(\frac{2\arcsin\sqrt{z}}{\sqrt{T}}\right) \tag{3.29}$$

i.e. the first-passage time of $X(t)$ through the threshold z has density:

$$f_z(t) = \frac{2\arcsin\sqrt{z}}{t^{3/2}}\ \phi\left(\frac{2\arcsin\sqrt{z}}{t^{1/2}}\right) \tag{3.30}$$

Clearly, θ obeys the arc–sine law. In this case it is meaningless to consider the asymptotics of $P(S_T > z)$ for $z \to \infty$, since $X(t)$ is forced to remain confined in the interval $[0, 1]$.

Example 3.4 The usefulness of the results of Theorem 3.1 for T fixed, and Theorem 3.3 for $T \in RV(\nu)$, relies on the fact that the function $\alpha(t)$ is close enough to $\beta(t)$. Here we show an example of diffusion satisfying all the preceding assumptions, for which this holds. Let be $\sigma > 0$, $\epsilon > 0$, and consider the SDE:

$$dX(t) = \frac{\epsilon\sigma^2\sin(2X(t))}{2(1+\epsilon\cos^2(X(t))}dt + \sigma dB_t \tag{3.31}$$

As easily seen, $X(t)$ is recurrent, and (see also [2]):

$$\rho(t) = \langle Y\rangle_t = \int_0^t \frac{(1+\epsilon\cos^2 x)^2}{(1+\epsilon)^2}\sigma^2 ds \tag{3.32}$$

from which it follows:

$$\alpha(t) \doteq \frac{\sigma^2 t}{(1+\epsilon)^2} \le \langle Y\rangle_t \le \sigma^2 t \doteq \beta(t) \tag{3.33}$$

Of course, if $\epsilon \simeq 0$, then $\beta(t) \simeq \alpha(t)$.

Example 3.5 (A temporally non-homogeneous SDE) Let $V(t)$ be the solution of the SDE:

$$\begin{cases} dV(t) = -\frac{V(t)}{1-t}dt + dB_t \ , \ 0 \leq t \leq 1 \\ V(0) = V(1) = 0 \end{cases} \tag{3.34}$$

The diffusion $V(t)$ is the Brownian bridge, i.e. BM conditioned to take the value 0 at time $t = 1$. The explicit solution of (3.34) is:

$$V(t) = (1-t) \int_0^t \frac{1}{1-s} dB_s \tag{3.45}$$

Set

$$X(t) = \frac{V(t)}{1-t} \ , \ 0 \leq t \leq 1 \tag{3.46}$$

The diffusion $X(t)$ turns out to be a local martingale with quadratic variation

$$\langle X \rangle_t = \rho(t) = \frac{t}{1-t}, \ 0 \leq t \leq 1.$$

So, by a random time–change it results $X(t) = \tilde{B}(\frac{t}{1-t})$, for a suitable BM $\tilde{B}$.

Now, for $T \in (0,1)$ given and fixed and $z > 0$, we get from (3.10):

$$P(M_T > z) = P\left(\max_{t \in [0,T]} X(t) > z\right) = 2\Psi\left(\frac{z}{\sqrt{\frac{T}{1-T}}}\right) \tag{3.47}$$

If τ' denotes the first-passage time of $V(t)$ over the stright line $y = z(1-t)$, i.e. $\tau' = \inf\{t > 0 : V(t) \geq z(1-t)\}$, then:

$$P(M_T > z) = P(\tau' \leq T) \tag{3.48}$$

The first instant θ at which $X(t)$ attains its maximum has distribution (from (3.13)):

$$P(\theta \leq t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t(1-T)}{T(1-t)}} \ , \ 0 \leq t < T < 1 \tag{3.49}$$

Remark 3.2 The results of this section can be generalized to the case when $X(0) \neq 0$; in fact, by setting $Z(t) = X(t) - X(0)$, we have $Z(0) = 0$ and $\max_{s \in [0,T]} X(s) = X(0) + \max_{s \in [0,T]} Z(s)$.

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