SOME RESULTS ON RANDOM FLIGHTS

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Received April 11, 2006

Abstract. We consider in this paper random flights in $\mathbb{R}^d$ performed by a particle changing direction of motion at Poisson times. Directions are uniformly distributed on spheres S_1^d. For the position $(X_1(t), ..., X_d(t))$ we obtain the conditional characteristic function

$$E\left\{e^{i\sum_{k=1}^d \alpha_k X_k(t)} \mid N(t) = n\right\}$$

and related density $p_n(x_1, ..., x_d; t)$ in terms of $(n+1)-$fold integrals of products of Bessel functions. These integrals can be worked out in simple terms for spaces of dimension $d = 2$ and $d = 4$. In these two cases also the unconditional distribution is determined in explicit form. We point out that a strict connection between these types of motions with infinite directions and the equation of damped waves holds only for $d = 2$. The related motion with random velocity in $\mathbb{R}^3$ is analyzed and its distribution derived.

We consider a particle initially located at the origin of a frame of reference of the space $\mathbb{R}^d, d \geq 2$. The particle initially chooses the direction of its motion with uniform law on the surface of the hypersphere $\partial S_{ct}^d = \{(x_1, ..., x_d) : \sum_{k=1}^d x_k^2 = 1\}$, that is

$$f(\theta_1, ..., \theta_{d-2}, \phi) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \cdots \sin\theta_{d-2}, \tag{1}$$

where $0 \leq \theta_j \leq \pi,\ j = 1, ..., d-2,\ 0 \leq \phi \leq 2\pi$. The changes of direction of motion are governed by a homogeneous Poisson process and therefore at each Poisson event occurs the particle changes direction according to the uniform law (1).

We assume that the particle moves with a constant velocity c and its position $(X_1(t), ..., X_d(t))$ at time t, when the number $N(t)$ of Poisson events in the interval $(0, t)$ is equal to n, can therefore be written as

$$\begin{aligned}
X_d(t) &= c\sum_{j=1}^{n+1}(s_j - s_{j-1}) \sin\theta_{1,j} \sin\theta_{2,j} \cdots \sin\theta_{d-2,j} \sin\phi_j \\
X_{d-1}(t) &= c\sum_{j=1}^{n+1}(s_j - s_{j-1}) \sin\theta_{1,j} \sin\theta_{2,j} \cdots \sin\theta_{d-2,j} \cos\phi_j \\
&\cdots \\
X_2(t) &= c\sum_{j=1}^{n+1}(s_j - s_{j-1}) \sin\theta_{1,j} \cos\theta_{2,j} \\
X_1(t) &= c\sum_{j=1}^{n+1}(s_j - s_{j-1}) \cos\theta_{1,j},
\end{aligned} \tag{2}$$

2000 *Mathematics Subject Classification.* 60K99, 62G30.

Key words and phrases. Bessel functions, random flights, uniform distribution on hyperspheres, hyperspherical coordinates.

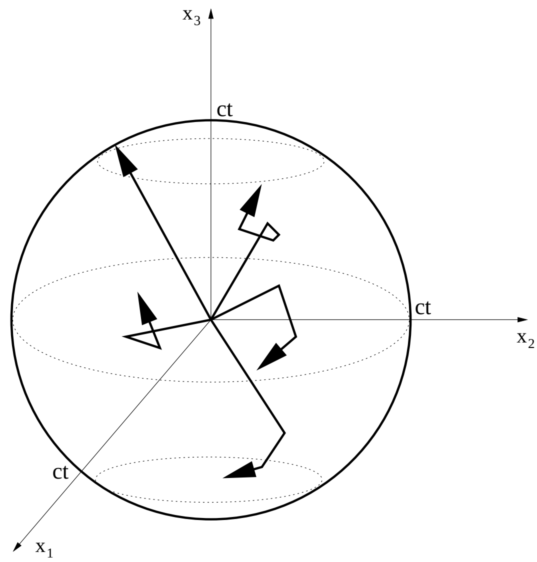

Figure 1: Sample paths of the random flight in $\mathbb{R}^3$.

where by $s_j,\ j = 1, ..., n$ we indicate the instants of occurrence of the Poisson events with $s_0 = 0,\ s_{n+1} = t$.

The sample paths described by the moving particle appear as straight lines with sharp turns and look like polygonals made up of randomly oriented segments of random length (see Figure 1).

Our more general results are stated in the next theorem.

Theorem 1 *The conditional characteristic function of* $(X_1(t), ..., X_d(t))$ *is given by*

$$E\left\{e^{i\sum_{k=1}^{d}\alpha_k X_k(t)} \mid N(t) = n\right\}$$

$$= \frac{n!}{t^n}\left\{2^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)\right\}^{n+1}\int_0^t ds_1 \cdots \int_{s_{n-1}}^t ds_n \prod_{j=1}^{n+1} \frac{J_{\frac{d}{2}-1}\left(c(s_j - s_{j-1})\sqrt{\sum_{k=1}^d \alpha_k^2}\right)}{\left(c(s_j - s_{j-1})\sqrt{\sum_{k=1}^d \alpha_k^2}\right)^{\frac{d}{2}-1}}, \tag{3}$$

The related distribution function for the absolutely continuous component of the vector position $(X_1(t), ..., X_d(t))$ *is*

$$\begin{aligned} &p_n(x_1, ..., x_d; t) \\ &= \frac{Pr\left\{\bigcap_{k=1}^d (X_k(t) \in dx_k) \,|N(t) = n\right\}}{\prod_{k=1}^d dx_k} \\ &= (2\pi)^{-\frac{d}{2}}\left[2^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)\right]^{n+1} \frac{n!}{t^n\left(\sqrt{\sum_{k=1}^d x_k^2}\right)^{\frac{d}{2}-1}} \cdot \\ &\quad \cdot \int_0^\infty \rho^{\frac{d}{2}} J_{\frac{d}{2}-1}\left(\rho\sqrt{\sum_{k=1}^d x_k^2}\right) d\rho \int_0^t ds_1 \cdots \int_{s_{n-1}}^t ds_n \prod_{j=1}^{n+1} \frac{J_{\frac{d}{2}-1}\left(c\rho(s_j - s_{j-1})\right)}{\left(c\rho(s_j - s_{j-1})\right)^{\frac{d}{2}-1}}, \end{aligned} \tag{4}$$

for $n \geq 1, d \geq 2$.

The general formula (4) shows that the probability distribution of the position $\underline{X}(t) = (X_1(t), ..., X_d(t))$ is isotropic, but its explicit expression can be obtained only in two cases, that is for $d = 2$ and $d = 4$.

If the length of the displacements of the particle are constant, we obtain the initial version of the problem of random flights studied by Pearson, Kluyver, Rayleigh and Watson. In this case, and assuming that at time t the $(n+1)-$th step has been completed, formula (4) becomes

$$p_n(x_1, ..., x_d)$$

$$= \frac{2^{\left(\frac{d}{2}-1\right)n-1}\pi^{-\frac{d}{2}}\left[\Gamma\left(\frac{d}{2}\right)\right]^{n+1}}{\left(a\sum_{k=1}^{d} x_k^2\right)^{\frac{d}{2}-1}} \int_0^\infty \rho^{-\left(\frac{d}{2}-1\right)n+1} J_{\frac{d}{2}-1}\left(\rho\sqrt{\sum_{k=1}^{d} x_k^2}\right)\left(J_{\frac{d}{2}-1}(\rho a)\right)^{n+1} d\rho, \tag{5}$$

for $n \geq 1$.

For $d = 2$ and $d = 4$ the density (4) can be evaluated by means of the following integrals (a sort of semigroup-type property of Bessel functions)

$$\int_0^a x^\mu (a-x)^\nu J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2\pi}\Gamma(\mu+\nu+1)} a^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(a), \tag{6}$$

with $Re\ \mu > -\frac{1}{2}, Re\nu > -\frac{1}{2}$, and

$$\int_0^a \frac{J_\mu(z) J_\nu(a-z)}{z(a-z)} dz = \left(\frac{1}{\mu} + \frac{1}{\nu}\right)\frac{J_{\mu+\nu}(a)}{a}, \tag{7}$$

with $Re\ \mu > 0, Re\ \nu > 0$ (see Gradstheyn-Ryzhik formula 6.533(2) and formula 6.581(3)). The integrals (6) and (7) permit us to evaluate the distribution (4) respectively for $d = 2$ and $d = 4$. Therefore we get that

$$p_n(x_1, x_2, x_3, x_4; t) = \frac{n(n+1)}{\pi^2 (ct)^{2n+2}}\left(c^2t^2 - \sum_{k=1}^{4} x_k^2\right)^{n-1}, \tag{8}$$

for $n \geq 1$ and $(x_1, x_2, x_3, x_4) \in S_{ct}^4$, and

$$p_n(x_1, x_2; t) = \frac{n}{2\pi (ct)^n}\left(c^2t^2 - x_1^2 - x_2^2\right)^{\frac{n}{2}-1}, \tag{9}$$

for $n \geq 1$ and $(x_1, x_2) \in S_{ct}^2$.

We observe that the distribution (8) for $n = 1$ becomes the uniform distribution in the hypersphere S_{ct}^4, while the distribution (9) for $n = 2$ coincides with the uniform law inside the disc S_{ct}^2. We note, also, that the above distributions tend to zero on the edge of the sphere (for $n > 1$ for $d = 4$ and for $n > 2$ for $d = 2$).

In order to have a closer insight into the behavior of the distribution (8) we give the following result

$$E\left\{\left(\sqrt{\sum_{k=1}^{4} X_k^2(t)}\right)^m \middle| N(t) = n\right\} = n(n+1)(ct)^m \frac{\Gamma(n)\Gamma\left(\frac{m+4}{2}\right)}{\Gamma\left(\frac{m+4}{2}+n\right)}, \tag{10}$$

that for $m = 1$, as $n \to \infty$, tends to zero.

From (8) we obtain the law of the absolutely continuous part of the unconditional distribution

$$
\begin{aligned}
p(x_1, x_2, x_3, x_4; t) &= \frac{Pr\left\{\bigcap_{k=1}^{4}\left(X_k(t) \in dx_k\right)\right\}}{\prod_{k=1}^{4} dx_k} \\
&= \frac{\lambda}{c^4 t^3 \pi^2} e^{-\frac{\lambda}{c^2 t}\sum_{k=1}^{4} x_k^2} \left\{2 + \frac{\lambda}{c^2 t}\left(c^2t^2 - \sum_{k=1}^{4} x_k^2\right)\right\},
\end{aligned} \tag{11}
$$

for $(x_1, x_2, x_3, x_4) \in S_{ct}^4$. The distribution (11) has almost the form of a truncated four-dimensional Gaussian distribution with independent components. As $\lambda, c \to \infty$ e $\frac{c^2}{\lambda} \to 1$ we obtained that

$$
\lim_{n\to\infty} p(x_1, x_2, x_3, x_3; t) = \frac{e^{-\sum_{k=1}^{4} x_k^2}}{\pi^2 t^2}
$$

that represents the distribution of Brownian motion in $\mathbb{R}^4$.

It's hard to obtain explicit probability distribution also for $N(t) = 1$ for arbitrary values of the dimension d. In fact we have that

$$
p_1(x_1, ..., x_d; t) = \frac{\Gamma^2\left(\frac{d}{2}\right)}{2t\pi^{\frac{d+1}{2}}\sqrt{\sum_{k=1}^{d} x_k^2}} \frac{1}{\Gamma(\frac{d}{2} - \frac{1}{2})} \int_{\frac{t}{2} - \frac{\sqrt{\sum_{k=1}^{d} x_k^2}}{2c}}^{\frac{t}{2} + \frac{\sqrt{\sum_{k=1}^{d} x_k^2}}{2c}} \frac{(\sin v)^{d-3} ds_1}{(cs_1)^{d-2}(c(t-s_1))}, \tag{12}
$$

where

$$
\cos v = \frac{(c(t-s_1))^2 + \left(\sqrt{\sum_{k=1}^{d} x_k^2}\right)^2 - (cs_1)^2}{2c(t-s_1)\sqrt{\sum_{k=1}^{d} x_k^2}}
$$

and $(x_1, ..., x_d) \in S_{ct}^d = \left\{(x_1, ..., x_d) : \sum_{k=1}^{d} x_k^2 < c^2t^2\right\}$. In particular for $d = 3$ formula (11) yields

$$
p_1(x_1, x_2, x_3; t) = \frac{\log\left(\frac{ct+\sqrt{x_1^2+x_2^2+x_3^2}}{ct-\sqrt{x_1^2+x_2^2+x_3^2}}\right)}{\pi(2ct)^2\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad (x_1, x_2, x_3) \in S_{ct}^3. \tag{13}
$$

Of course for $d = 2, 4$ we reobtain from (12) the distribution (8) and (9) for $n = 1$.

We can extract from (11) the probability law of motions described by the projection of random flights onto lower spaces (see Figure 2). In particular, we studied a moving particle in $\mathbb{R}^3$ with random velocity with the following components

$$
v_{x_1} = c\sin\theta_1 \sin\theta_2 \sin\phi, \;\; v_{x_2} = c\sin\theta_1 \sin\theta_2 \cos\phi, \;\; v_{x_3} = c\sin\theta_1 \cos\theta_2
$$

and intensity $c\sin\theta, \theta \in (0, \pi)$. Therefore the projection of the four-dimensional motion onto $\mathbb{R}^3$ has the following density function

$$
p^4(x_1, x_2, x_3; t) = \frac{\sqrt{\lambda\pi}e^{-\lambda t}}{c^3t^2\pi^2\sqrt{t}} \sum_{k=0}^{\infty} \left\{\frac{\lambda}{c^2 t}\left(c^2t^2 - \sum_{k=1}^{3} x_k^2\right)\right\}^{k-\frac{1}{2}} \frac{2k+2}{2k-1} \frac{1}{\Gamma\left(k - \frac{1}{2}\right)}, \tag{14}
$$

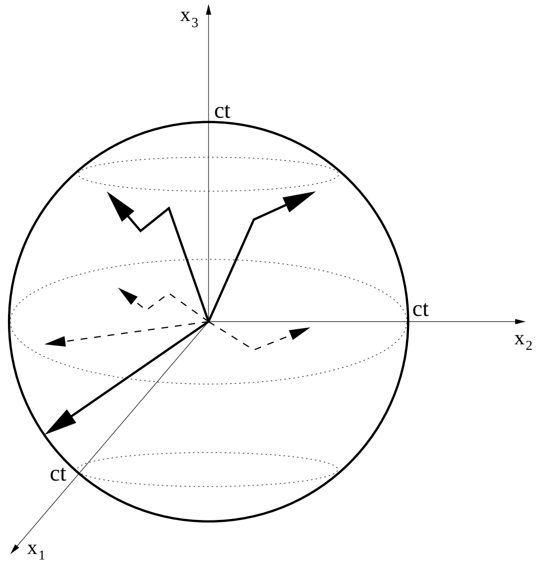

Figure 2: Sample paths in $\mathbb{R}^3$ and their projections on the equatorial plane.

for $(x_1, x_2, x_3) \in S^3_{ct}$.

While in $\mathbb{R}^2$ we have that

$$p(x_1, x_2; t) = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2-(x_1^2+x_2^2)}}}{\sqrt{c^2t^2-(x_1^2+x_2^2)}}, \tag{15}$$

is a solution to the equation of planar, damped waves also called two-dimensional telegraph equation, namely

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c^2 \left\{ \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right\} p,$$

it seems the analogous relationships do not hold for random flights in $\mathbb{R}^4$ and for their projections onto $\mathbb{R}^3, \mathbb{R}^2, \mathbb{R}$.

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