BCK/BCI-BIALGEBRAS

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Abstract. The notion of BCK/BCI-bialgebras and sub-bialgebras is introduced, and related properties are investigated. A characterization of $X = pI(X_1) \uplus pI(X_2)$ is provided.

1 Introduction. A BCK/BCI-algebra is an important calss of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. Bialgebraic structures, for example, bisemigroups, bigroups, bigroupoids, biloops, birings, bisemirings, binear-rings, etc., are discussed in [6]. In this paper, we consider bialgebraic structures in BCK/BCI-algebras. We introduced the notion of BCK/BCI-bialgebras and sub-bialgebras, and investigate several properties. Using the notion of a commutative bigroup, we construct the concept of $X = pI(X_1) \uplus pI(X_2)$, and vice versa.

2 Preliminaries. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. In a BCK-algebra X, the following identity holds.

(a1) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A BCK-algebra X is said to be *positive implicative* if it satisfies the following identity:

$$(\forall x, y, z \in X)\,((x * y) * z = (x * y) * (x * z)).$$

A positive implicative BCK-algebra will be written by piBCK-algebra for short. A BCK-algebra X is s said to be *commutative* if $x * (x * y) = y * (y * x)$ for all $x, y \in X$. A commutative BCK-algebra will be written by cBCK-algebra for short. A BCI-algebra X is said to be *p-semisimple* if its p-radical is trivial. In a p-semisimple BCI-algebra X, we have the following axioms:

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(a2) $(\forall x, y \in X)$ $(x * (0 * y) = y * (0 * x))$,

(a3) $(\forall x \in X)$ $(0 * (0 * x) = x)$.

We refer the reader to the book [5] for further information regarding BCK/BCI-algebras.

3 BCK/BCI-bialgebras

Definition 3.1. Let $X = (X, *, \oplus, 0)$ be an algebra of type $(2, 2, 0)$. Then $X = (X, *, \oplus, 0)$ is called a *BCK-bialgebra* (resp. *BCI-bialgebra*) if there exists two distinct proper subsets X_1 and X_2 of X such that

(i) $X = X_1 \cup X_2$.

(ii) $(X_1, *, 0)$ is a BCK-algebra (resp. BCI-algebra).

(iii) $(X_2, \oplus, 0)$ is a BCK-algebra (resp. BCI-algebra).

Denote by $X = K(X_1) \uplus K(X_2)$ (resp. $X = I(X_1) \uplus I(X_2)$) the BCK-bialgebra (resp. BCI-bialgebra). If $(X_1, *, 0)$ is a BCK-algebra (resp. BCI-algebra) and $(X_2, \oplus, 0)$ is a BCI-algebra (resp. BCK-algebra), then we say that $X = (X, *, \oplus, 0)$ is a BCKI-bialgebra (resp. BCIK-bialgebra), and denoted by $X = K(X_1) \uplus I(X_2)$ (resp. $X = I(X_1) \uplus K(X_2)$).

Example 3.2. (1) Let $X = \{0, a, b, c, d\}$ and consider two proper subsets $X_1 = \{0, a, b\}$ and $X_2 = \{0, a, c, d\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

$\oplus$	0	a	c	d
0	0	0	0	0
a	a	0	a	0
c	c	c	0	0
d	d	c	a	0

Then $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are BCK-algebras. Hence $(X, *, \oplus, 0)$ is a BCK-bialgebra, i.e., $X = K(X_1) \uplus K(X_2)$.

(2) Let $X = \mathbb{R}^+ \cup \{0, a, b, c\}$ where $\mathbb{R}^+$ is the set of all positive real numbers. Define two binary operations '$*$' and '$\oplus$' as follows:

$$(\forall x, y \in \mathbb{R}^+ \cup \{0\}) \, (x * y = \max\{x - y, 0\})$$

and

$\oplus$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $(X_1 := \mathbb{R}^+ \cup \{0\}, *, 0)$ and $(X_2 := \{0, a, b, c\}, \oplus, 0)$ are BCK-algebras. Hence $(X, *, \oplus, 0)$ is a BCK-bialgebra, i.e., $X = K(X_1) \uplus K(X_2)$.

(3) Let $X = \{0, a, b, c, d\}$ and consider two proper subsets $X_1 = \{0, a, b\}$ and $X_2 = \{0, a, c, d\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

$\oplus$	0	a	c	d
0	0	0	c	c
a	a	0	c	c
c	c	c	0	0
d	d	c	a	0

Then $(X_1, *, 0)$ is a BCK-algebra and $(X_2, \oplus, 0)$ is a BCI-algebra. Hence $(X, *, \oplus, 0)$ is a BCKI-bialgebra, i.e., $X = K(X_1) \uplus I(X_2)$.

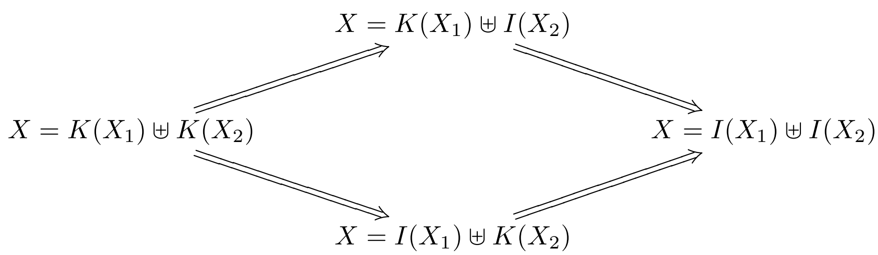
(4) Let $X = \{0, a, b, c, d, e, x, y, z\}$ and consider two proper subsets $X_1 = \{0, a, b, c, d, e\}$ and $X_2 = \{0, x, y, z\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b	c	d	e
0	0	0	0	0	d	d
a	a	0	a	0	e	d
b	b	b	0	0	d	d
c	c	b	a	0	e	d
d	d	d	d	d	0	0
e	e	d	e	d	a	0

$\oplus$	0	x	y	z
0	0	z	y	x
x	x	0	z	y
y	y	x	0	z
z	z	y	x	0

Then $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are BCI-algebras. Hence $(X, *, \oplus, 0)$ is a BCI-bialgebra, i.e., $X = I(X_1) \uplus I(X_2)$.

Proposition 3.3. *We have*



Proof. Since every BCK-algebra is a BCI-algebra, it is straightforward. □

Note that any BCI-algebra need not be a BCK-algebra. Hence the converse of Proposition 3.3 is not true in general.

Definition 3.4. Let $X = K(X_1) \uplus K(X_2)$ (resp. $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, $X = I(X_1) \uplus I(X_2)$). A subset $H(\neq \emptyset)$ of X is called a *sub-bialgebra* of X if there exist subsets H_1 and H_2 of X_1 and X_2, respectively, such that

(i) $H_1 \neq H_2$ and $H = H_1 \cup H_2$,

(ii) $(H_1, *, 0)$ is a subalgebra of $(X_1, *, 0)$,

(iii) $(H_2, \oplus, 0)$ is a subalgebra of $(X_2, \oplus, 0)$.

Example 3.5. Let X be a BCK-bialgebra in Example 3.2(1) and let $H_1 = \{0, a\}$ and $H_2 = \{0, c\}$. Then $H_1 \neq H_2$ and H_1 (resp. H_2) is a subalgebra of X_1 (resp. X_2). Hence $H = \{0, a, c\}$ is a sub-bialgebra of X. We can easily check that $(H = \{0, a, c\}, \oplus, 0)$ is a BCK-algebra. Note also that $H_3 = \{0, d\}$ is a subalgebra of X_2 and $H_1 \neq H_3$. Thus $G = \{0, a, d\}$ is a sub-bialgebra of X. We can easily check that $(G = \{0, a, d\}, \oplus, 0)$ is not a BCK-algebra.

Remark 3.6. Let L be a sub-bialgebra of a BCK-bialgebra $(X, *, \oplus, 0)$. Then L may not be a BCK-algebra under $*$ or $\oplus$ as seen in Example 3.5.

We provide a characterization of a sub-bialgebra.

Theorem 3.7. *Let $X = K(X_1) \uplus K(X_2)$ (resp. $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, $X = I(X_1) \uplus I(X_2)$) and let H be a nonempty subset of X. Then H is a sub-bialgebra of X if and only if there exist two proper subsets X_1 and X_2 of X such that*

(i) *$X = X_1 \cup X_2$, where $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are BCK-algebras (resp. $(X_1, *, 0)$ is a BCK-algebra and $(X_2, \oplus, 0)$ is a BCI-algebra, $(X_1, *, 0)$ is a BCI-algebra and $(X_2, \oplus, 0)$ is a BCK-algebra, $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are BCI-algebras),*

(ii) *$(H \cap X_1, *, 0)$ is a subalgebra of $(X_1, *, 0)$,*

(iii) *$(H \cap X_1, \oplus, 0)$ is a subalgebra of $(X_1, \oplus, 0)$.*

Proof. We prove it for the case $X = K(X_1) \uplus K(X_2)$. For other cases, we can have desired results by the similar method. Assume that H is a sub-bialgebra of X. Then $(H, *, \oplus, 0)$ is a BCK-bialgebra. Hence there exist two distinct proper subsets H_1 and H_2 of H such that

- $H = H_1 \cup H_2$,
- $(H_1, *, 0)$ and $(H_2, \oplus, 0)$ are BCK-algebras.

Taking $H_1 = H \cap X_1$ and $H_2 = H \cap X_2$ imply that $(H_1 = H \cap X_1, *, 0)$ and $(H_2 = H \cap X_2, \oplus, 0)$ are subalgebras of $(X_1, *, 0)$ and $(X_2, \oplus, 0)$, respectively. Conversely, Let H be a nonempty subset of a BCK-bialgebra $(X, *, \oplus, 0)$ satisfying conditions (i), (ii) and (iii). It is sufficient to show that $(H \cap X_1) \cup (H \cap X_2) = H$. Now,

$$\begin{aligned}(H \cap X_1) \cup (H \cap X_2) &= \big((H \cap X_1) \cup H\big) \cap \big((H \cap X_1) \cup X_2\big) \\ &= \big((H \cup H) \cap (X_1 \cup H)\big) \cap \big((H \cup X_2) \cap (X_1 \cup X_2)\big) \\ &= \big(H \cap (X_1 \cup H)\big) \cap \big((H \cup X_2) \cap X\big) \\ &= H \cap (H \cup X_2) \\ &= H.\end{aligned}$$

This completes the proof. □

Denote by $X = piK(X_1) \uplus cK(X_2)$ the $X = K(X_1) \uplus K(X_2)$ in which $(X_1, *, 0)$ is a positive implicative BCK-algebra and $(X_2, \oplus, 0)$ is a commutative BCK-algebra. Denote by $X = iK(X_1) \uplus cK(X_2)$ the $X = K(X_1) \uplus K(X_2)$ in which $(X_1, *, 0)$ is an implicative BCK-algebra and $(X_2, \oplus, 0)$ is a commutative BCK-algebra. Note that

$$X = iK(X_1) \uplus cK(X_2) \Rightarrow X = piK(X_1) \uplus cK(X_2) \Rightarrow X = K(X_1) \uplus K(X_2),$$

but the converse is not true in general. In fact, in Example 3.2(1), we can see that the implication

$$X = K(X_1) \uplus K(X_2) \Rightarrow X = piK(X_1) \uplus cK(X_2)$$

does not hold.

Example 3.8. Let $X = \{0, x, y, a, b, c\}$ and consider two subsets $X_1 = \{0, a, b, c\}$ and $X_2 = \{0, x, y\}$ of X with Cayley tables as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

$\oplus$	0	x	y
0	0	0	0
x	x	0	0
y	y	x	0

It is easy to check that $X = piK(X_1) \uplus cK(X_2)$, but $X \neq iK(X_1) \uplus cK(X_2)$.

Lemma 3.9. [3] *A BCK-algebra X is positive implicative if and only if it satisfies the following identity:*

$$(\forall x, y \in X)\,(x * y = (x * y) * y).$$

Lemma 3.10. [3] *A BCK-algebra X is commutative if and only if it is a semilattice with respect to $\wedge$.*

Using Lemmas 3.9 and 3.10, we provide a condition for $X = K(X_1) \uplus K(X_2)$ to be $X = piK(X_1) \uplus cK(X_2)$.

Theorem 3.11. *Let $X = K(X_1) \uplus K(X_2)$. Then $X = piK(X_1) \uplus cK(X_2)$ if and only if the following conditions are true.*

(i) $(\forall x, y \in X)\ (x * y = (x * y) * y)$,

(ii) *X_2 is a semilattice with respect to $\wedge_\oplus$ which is given by*

$$(\forall a, b \in X_2)\,(a \wedge_\oplus b = b \oplus (b \oplus a)).$$

Lemma 3.12. [3] *A BCK-algebra X is commutative if and only if it satisfies the following identity:*

$$(\forall x, y \in X)\,(A(x) \cap A(y) = A(\wedge y)),$$

where $A(x)$ is the initial section of x.

Applying Lemmas 3.9 and 3.12, we have a characterization of $X = piK(X_1) \uplus cK(X_2)$.

Theorem 3.13. *Let $X = K(X_1) \uplus K(X_2)$. Then $X = piK(X_1) \uplus cK(X_2)$ if and only if the following conditions are true.*

(i) $(\forall x, y \in X)\ (x * y = (x * y) * y)$,

(ii) $(\forall a, b \in X_2)\ (A(a) \cap A(b) = A(a \wedge_\oplus b))$.

Definition 3.14. [4] A set $(G, +, \bullet)$ with two binary operations $+$ and $\bullet$ is called a *bigroup* if there exists two proper subsets G_1 and G_2 of G such that $G = G_1 \cup G_2$, $(G_1, +)$ is a group, and $(G_2, \bullet)$ is a group. If both $(G_1, +)$ and $(G_2, \bullet)$ are commutative, then we say that $(G, +, \bullet)$ is a *commutative bigroup*.

Denote by $X = pI(X_1) \uplus pI(X_2)$ the $X = I(X_1) \uplus I(X_2)$ in which $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are p-semisimple BCI-algebras.

Lemma 3.15. [1] *A BCI-algebra X satisfies the identity*

$$(\forall x, y \in X)\,(x * (x * y) = y)$$

if and only if it has a sum $+$ and $(X, +)$ is a commutative group.

Lemma 3.16. [2] *In a BCI-algebra X, the following are equivalent.*

(i) $(\forall x, y \in X)\ (x * (x * y) = y)$.

(ii) *X is p-semisimple.*

Theorem 3.17. *If $X = pI(X_1) \uplus pI(X_2)$, then X has operations $+$ and $\bullet$ so that $(X, +, \bullet)$ is commutative bigroup.*

Proof. If $X = pI(X_1) \uplus pI(X_2)$, then $X = X_1 \cup X_2$, and $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are p-semisimple BCI-algebras. By means of Lemmas 3.15 and 3.16, X has two operations $+$ and $\bullet$ so that $(X, +)$ and $(X, \bullet)$ are commutative groups, in which $+$ and $\bullet$ are given by $x + y = x * (0 * y)$ and $x \bullet y = x \oplus (0 \oplus y)$ for all $x, y \in X$. Hence $(X, +, \bullet)$ is a commutative bigroup. □

Theorem 3.18. *Let $(G, +, \bullet)$ be a commutative bigroup. If we define operations $*$ and $\oplus$ on G as follows:*

$$(\forall x, y \in G)\,(x * y = x - y) \text{ and } (\forall a, b \in G)\,(a \oplus b = a \bullet b^{-1}),$$

then $G = pI(G_1) \uplus pI(G_2)$ for some $G_1, G_2 \subseteq G$.

Proof. If $(G, +, \bullet)$ is a commutative bigroup, then $G = G_1 \cup G_2$ for some $G_1, G_2 \subseteq G$, and $(G_1, +)$ and $(G_2, \bullet)$ are (commutative) groups. It is easy to prove that $(G_1, *, 0)$ and $(G_2, \oplus, 0)$ are p-semisimple BCI-algebras. Hence $G = pI(G_1) \uplus pI(G_2)$. □

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