ON QUASI-λ-NUCLEARITY

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Abstract. We Introduce new type of maps between normed spaces, namely, p-quasi-λ-nuclear map. We prove that the composition of a q-quasi-λ-nuclear map $(0 < q \leq 1)$ with a p-quasi-λ-nuclear map $(0 < p \leq 1)$ is a pseudo-λ-nuclear map. Also we prove that for a nuclear G_∞-space a linear map T between normed spaces is p-quasi-λ-nuclear iff it is q-quasi-λ-nuclear.

1 Basic Concepts. For two sequences of scalars $x = (x_n)$ and $y = (y_n)$ we write $x_n = O(y_n)$ if there is a $\rho > 0$ such that $x_n \leq \rho y_n$ for all $n \in \mathbf{N}$.

A set A of sequences of non-negative real numbers is called a **Köthe set**, if it satisfies the following conditions:

1. For each pair of elements $a, b \in A$ there is $c \in A$ with $a_n = O(c_n)$ and $b_n = O(c_n)$.
2. For every integer $r \in N$ there exists $a \in A$ with $a_r > 0$.

The space of all sequences $x = (x_n)$ such that

$$p_a(x) := \sum_n |x_n| \, a_n < +\infty$$

for all $a \in A$, is called the **Köthe space**, $\lambda(A)$, generated by A[3].

A Köthe set P will be called a **power set of infinite type** if it satisfies the following conditions:

1. For each $a \in P$, $0 < a_n \leq a_{n+1}$ for all n.
2. For each $a \in P$, there exists $b \in P$ such that $a_n^2 = O(b_n)$.

A Köthe space of the form $\lambda(P)$ where P is a power set of infinite type is called a G_∞**-space** or a **smooth sequence space of infinite type**[9].

Let $\alpha = (\alpha_n)$ be an unbounded non-decreasing sequence of positive real numbers. Then $P_\infty = \{(k^{\alpha_n}) \colon k \in \mathbf{N}\}$ is countable Köthe set. The corresponding Köthe space $\Lambda_\infty(\alpha) = \lambda(P_\infty)$ is called the **power series of infinite type**[9].

Theorem 1.1. (Grothendieck-Pietsch criterion for nuclearity) [9] *A Köthe space $\lambda(A)$ is nuclear if and only if for every $a \in A$, there is $b \in A$ such that $(a_n/b_n) \in \ell_1$.*

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Let E and F be two arbitrary normed spaces. A linear map T from E into F is called a **nuclear** map if there are sequences (a_n), (y_n) in E' and F respectively, with

$$\sum_n ||a_n||\,||y_n|| < +\infty \text{ such that } T(x) = \sum_n \langle x, a_n\rangle y_n,$$

and a **p-quasi-nuclear** map if there is a sequence (a_n) in E' with

$$\sum_n ||a_n||^p < +\infty \text{ such that } ||T(x)||^p \leq \sum_n |\langle x, a_n\rangle|^p [5].$$

In the rest of this paper, letter λ stands for a fixed sequence space contained in ℓ_1.

A linear map T of a normed space E into a normed space F is called a **pseudo-λ-nuclear map** if there exist a sequence (α_n) in λ and a bounded sequences (a_n) and (y_n) in E' and F respectively such that $Tx = \sum_n \alpha_n \langle x, a_n\rangle y_n$, for all x in E, and a **quasi-λ-nuclear map** if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that $||Tx|| \leq \sum_n |\alpha_n|\,|\langle x, a_n\rangle|$, for all x in E[1][6].

A linear map T of a normed space E into a normed space F is called a **2-quasi-λ-nuclear map** if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \left(\sum_n |\alpha_n|\,|\langle x, a_n\rangle|^2\right)^{1/2},$$

for all x in E[8].

2 Main results.

To proceed in our work, we introduce the following definition:

Definition 2.1. *For $0 < p < +\infty$, a linear map T of a normed space E into a normed space F is called a* **p-quasi-λ-nuclear map** *if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that*

$$||Tx|| \leq \left(\sum_n |\alpha_n|\,|\langle x, a_n\rangle|^p\right)^{1/p},$$

for all x in E.

Let $\mathcal{N}(E,F)$, $\mathcal{QN}_p(E,F)$, $\mathcal{P}\lambda\mathcal{N}(E,F)$, and $\mathcal{Q}\lambda\mathcal{N}_p(E,F)$, denote the collection of all nuclear, p-quasi-nuclear, pseudo-λ-nuclear, and p-quasi-λ-nuclear maps, respectively, between normed spaces E and F. It is an easy matter to see the following proposition.

Proposition 2.1. *If $T \in \mathcal{Q}\lambda\mathcal{N}_p(E,F)$, then $T \in \mathcal{QN}_p(E,F)$*

Let $\mathbf{B}(E,F)$ denotes the collection of all bounded linear map between normed spaces E and F. Then we have the following proposition.

Proposition 2.2. *Let E, F and G be normed spaces. Let T and S be a linear maps from E into F and from F into G respectively. Then*

1. *If $T \in \mathbf{B}(E,F)$ and $S \in \mathcal{P}\lambda\mathcal{N}(F,G)$, then $ST \in \mathcal{P}\lambda\mathcal{N}(E,G)$.*
2. *If $T \in \mathcal{P}\lambda\mathcal{N}(E,F)$ and $S \in \mathbf{B}(F,G)$, then $ST \in \mathcal{P}\lambda\mathcal{N}(E,G)$.*
3. *If $T \in \mathbf{B}(E,F)$ and $S \in \mathcal{Q}\lambda\mathcal{N}_p(F,G)$, then $ST \in \mathcal{Q}\lambda\mathcal{N}_p(E,G)$.*
4. *If $T \in \mathcal{Q}\lambda\mathcal{N}_p(E,F)$ and $S \in \mathbf{B}(F,G)$, then $ST \in \mathcal{Q}\lambda\mathcal{N}_p(E,G)$.*

Our next result indicates the relationship between r-quasi-λ-nuclear and s-quasi-λ-nuclear maps.

Theorem 2.1. *If $0 < r < s < +\infty$, then r-quasi-λ-nuclear maps are s-quasi-λ-nuclear.*

Proof. Suppose that $0 < r < s < +\infty$ and $T : E \to F$ is a r-quasi-λ-nuclear map between normed spaces E and F. Then there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx||^r \leq \sum\nolimits_n |\alpha_n| \, |\langle x, a_n \rangle|^r.$$

Let $q = \frac{s}{s-r}$. Then one can show that

$$||Tx|| \leq \left(\sum\nolimits_n |\alpha_n|\right)^{\frac{1}{rq}} \left(\sum\nolimits_n |\alpha_n| \, |\langle x, a_n \rangle|^s\right)^{\frac{1}{s}}.$$

Let $\beta = (\sum_n |\alpha_n|)^{\frac{1}{sq}}$ and $b_n = \beta \, a_n$. Then

$$||Tx|| \leq \left(\sum\nolimits_n |\alpha_n| \, |\langle x, b_n \rangle|^s\right)^{1/s}.$$

Since $(\alpha_n) \in \lambda$ and (b_n) is a bounded sequence in E', T is a s-quasi-λ-nuclear map. ∎

The relationship between p-quasi-nuclear and p-quasi-ℓ_1-nuclear maps is given by the following result:

Proposition 2.3. *A linear map T from a normed space E into a normed space F is p-quasi-nuclear if and only if it is p-quasi-ℓ_1-nuclear.*

The following result is direct consequence of Proposition 2.3 and Theorem 2.1.

Corollary 2.1. [5] *If $0 < r < s < +\infty$, then r-quasi-nuclear maps are s-quasi-nuclear.*

The following known results are crucial in proving our next result.

Proposition 2.4. [8] *Each quasi-λ-nuclear map $T : E \to F$ between normed spaces E and F is also pseudo-λ-nuclear if it is regarded as a map from E into a Banach space $\ell_\infty(I)$ in which F is embedded.*

Proposition 2.5. *[5] If T is a bounded linear map from a normed space E into a Banach space F, then the following conditions are equivalent:*

1. *T is a 2-quasi-nuclear map.*
2. *T factors through the diagonal map $D_\mu : \ell_\infty \to \ell_2$ for some $\mu \in \ell_2$, that is, there are two bounded linear maps S_1 from E into ℓ_∞ and S_2 from ℓ_2 into F such that $T = S_2 D_\mu S_1$.*

Theorem 2.2. *Suppose that $0 < q \leq 1$ and $0 < p \leq 2$. If $T : E \to F$ is a q-quasi-λ-nuclear map between normed spaces E and F and if S is a p-quasi-λ-nuclear map from F into a Banach space G, then ST is a pseudo-λ-nuclear map.*

Proof. Since S is a p-quasi-λ-nuclear map. Then by Theorem 2.1, S is 2-quasi-λ-nuclear. By Proposition 2.1 and Proposition 2.5, S can be factored through a diagonal map $D_\mu : \ell_\infty \to \ell_2$ for some $\mu \in \ell_2$, that is, there are two bounded linear maps S_1 from F into ℓ_∞ and S_2 from ℓ_2 into G such that $S = S_2 D_\mu S_1$. Since $q \leq 1$, by Theorem 2.1, T is quasi-λ-nuclear. By using Proposition 2.2 and Proposition 2.4, we get the pseudo-λ-nuclearity of ST. ∎

The following result is direct consequence of Proposition 2.3 and Theorem 2.2.

Corollary 2.2. *Suppose that $0 < q \leq 1$ and $0 < p \leq 2$. If $T : E \to F$ is a q-quasi-nuclear map between normed spaces E and F and if S is a p-quasi-nuclear map from F into a Banach space G, then ST is a pseudo-nuclear map.*

The Grothedieck-Pietsch criterion for nuclearity plays a major rule for proving our last result. Before we start our arguments, we introduce the following remark.

Remark. If $\lambda = \lambda(P_0)$ is a G_∞-space, then for any $k \in \mathbf{N}$ and $a \in P_0$, there is $b \in P_0$ such that $(a_n)^k = O(b_n)$.

Theorem 2.3. *Suppose that $\lambda = \lambda(P_0)$ is a nuclear G_∞-space and $0 < q \leq p < +\infty$, a bounded linear map between normed spaces is a q-quasi-λ-nuclear map if and only if it is a p-quasi-λ-nuclear map.*

Proof. The "if" part condition follows from Theorem 2.1. To prove the only "if part", let $T : E \to F$ be a p-quasi-λ-nuclear map between normed spaces E and F. Then there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \left(\sum\nolimits_n |\alpha_n| \, |\langle x, a_n\rangle|^p\right)^{\frac{1}{p}}.$$

Then we have

$$||Tx|| \leq \left(\sum\nolimits_n |\alpha_n|^{\frac{q}{p}} \, |\langle x, a_n\rangle|^q\right)^{\frac{1}{q}}.$$

To finish our proof it is enough to show that $(|\alpha_n|^{\frac{1}{p}}) \in \lambda$. Since $q/p > 0$ we choose $k \in \mathbf{N}$ such that $1/k < q/p$. So $q/p = 1/k + t$ for some $t > 0$. Hence $|\alpha_n|^{q/p} = |\alpha_n|^{1/k} \, |\alpha_n|^t$. Let $a \in P_0$ be given, by Grothendieck-Pietsch criterion for nuclearity, we choose $b \in P_0$ such that $(a_n/b_n) \in \ell_1$. Since $b \in P_0$, choose $c \in P_0$ and $\rho > 0$ such that

$$b_n \leq \rho \, (c_n)^{1/k} \text{ for all } n \in \mathbf{N}.$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} |\alpha_n|^{\frac{q}{p}} \, a_n &= \sum_{n=1}^{\infty} |\alpha_n|^t \, |\alpha_n|^{\frac{1}{k}} \, \frac{a_n}{b_n} \, b_n \\ &\leq \rho \sum_{n=1}^{\infty} |\alpha_n|^t \, |\alpha_n|^{\frac{1}{k}} \, c_n^{\frac{1}{k}} \frac{a_n}{b_n}. \end{aligned}$$

Since $(\alpha_n) \in \lambda(P_0)$, we have (α_n) and $(\alpha_n \, c_n)$ are in ℓ_1. So there exist $\gamma > 0$ and $\beta > 0$ such that

$$|\alpha_n| \leq \gamma \text{ and } |\alpha_n| c_n \leq \beta \text{ for all } n \in \mathbf{N}.$$

Therefore

$$|\alpha_n|^t \leq \gamma^t \text{ and } |\alpha_n|^{\frac{1}{k}} \, c_n^{\frac{1}{k}} \leq \beta^{\frac{1}{k}} \text{ for all } n \in \mathbf{N}.$$

Hence

$$\sum_{n=1}^{\infty} |\alpha_n|^{\frac{q}{p}} a_n \leq \rho \, \gamma^t \, \beta^{\frac{1}{k}} \sum_{n=1}^{\infty} \frac{a_n}{b_n} < +\infty.$$

Therefore $(|\alpha_n|^{\frac{q}{p}}) \in \lambda(P_0)$ and hence T is q-quasi-λ-nuclear. ■

Applying Theorem 2.3, the following corollary is resulted:

Corollary 2.3. [7]*Suppose that $\lambda = \lambda(P_0)$ is a nuclear G_∞-space, a bounded linear map between normed spaces is a quasi-λ-nuclear map if and only if it is a 2-quasi-λ-nuclear map.*

In this section, we give some examples to show that the converse of our main previous results are not true in general. For $p > 0$, we give an example of a sequence space λ and a linear map T such that T is a p-quasi-nuclear map which is not a p-quasi-λ-nuclear map.

Example 2.1. *Define a map $D : \ell_1 \to \ell_p$ by $Dx = (x_n/3^n)$. Then D is a p-quasi-nuclear map which is not a p-quasi-$\Lambda_\infty(n)$-nuclear map.*

Proof. To show that D is a p-quasi-nuclear map, let $a_n = e_n/3^n$. Then

$$||Dx||_p^p = \left|\left|\left(\frac{x_n}{3^n}\right)\right|\right|_p^p = \sum_n \left|\frac{x_n}{3^n}\right|^p = \sum_n |\langle x, a_n\rangle|^p .$$

Since (a_n) is a sequence in ℓ_∞ with $\sum_n ||a_n||_\infty^p < +\infty$, D is a p-quasi-nuclear map. To show that D is not a p-quasi-$\Lambda_\infty(n)$-nuclear map, define a map $A : \ell_p \to \ell_1$ by putting $Ax = (x_n/3^n)$. Then A is quasi-nuclear, and hence 2-quasi-nuclear. By Proposition 2.5, A can be factored through D_μ for some $\mu \in \ell_2$, that is, there are bounded linear maps $S_2 : \ell_p \to \ell_\infty$, $D_\mu : \ell_\infty \to \ell_2$, and $S_1 : \ell_2 \to \ell_1$ such that $A = S_1 D_\mu S_2$. If we assume that D is p-quasi-$\Lambda_\infty(n)$-nuclear, then by Theorem 2.3, D is quasi-$\Lambda_\infty(n)$-nuclear and hence by Proposition 2.2, S_2D is quasi-$\Lambda_\infty(n)$-nuclear. Therefore by Proposition 2.4, S_2D is pseudo-$\Lambda_\infty(n)$-nuclear. Thus by Proposition 2.2, AD is pseudo-$\Lambda_\infty(n)$ -nuclear. Since $AD : \ell_1 \to \ell_1$ is given by $ADx = (x_n/9^n)$ and AD is pseudo-$\Lambda_\infty(n)$-nuclear, we have $(1/9^n) \in \Lambda_\infty(n)$, which is a contradiction. So A is not 2-quasi-$\Lambda_\infty(n)$-nuclear. ■

Now for $0 < r < s \leq 2$, we give an example of a sequence space λ and a linear map T such that T is a s-quasi-λ-nuclear map which is not r-quasi-λ-nuclear. To achieve that we need the following definitions and results. For two normed spaces E and F and for integers $r \geq 0$, $\mathcal{A}_r(E, F)$ denotes the collection of all finite rank linear maps A from E into F whose range is at most r-dimensional.

Definition 2.2. *[4; P. 120] Let T be a linear map from a normed space E into a normed space F. The r**-th approximation number** $\alpha_r(T)$ of T is defined to be* $\inf\{||T - A|| : A \in \mathcal{A}_r(E, F)\}$.

Definition 2.3. *[4; P. 144] Let B be an arbitrary bounded subset in a normed space E with closed unit ball U. The infimum of all $\delta > 0$ for which there is a linear subspace F of E with dimension at most n such that $B \subset \delta U + F$ is called the n**-th diameter** of B and is denoted by $d_n(B)$.*

Definition 2.4. *[see 8] Let $T : E \to F$ be a bounded linear map between normed spaces E and F with closed unit balls U and V respectively. The n-th diameter of T, denoted by $d_n(T)$, is defined to be $d_n(T(U))$.*

Lemma 2.1. *[2, P. 23] Suppose that T is a linear map from a normed space E into a normed space F. Then $d_n(T) \leq \alpha_n(T) \leq \sqrt{n}\, d_n(T)$.*

Lemma 2.2. *[2, P. 23] Suppose that T is a compact map from a Banach space X into a Banach space F. Then $\alpha_n(T) = \alpha_n(T')$, where T' is the dual map of T.*

To this end, we have furnished the necessary back ground to give our desired example.

Example 2.2. *Let* $P = \left\{ \left(n^{\ln(kn)} \right) : \, k = 1, 2, \dots \right\}$*, and* $0 < r < s \le 2$*. Define the map* D *from* ℓ_2 *into* ℓ_s *by* $Dx = (\alpha_n{}^{\frac{1}{s}} x_n)$ *where*

$$\alpha_n = \frac{1}{n^{\frac{s}{r}} \, n^{\ln(n^{\frac{s}{r}})}}.$$

Then D *is a s-quasi-*$\lambda(P)$*-nuclear map which is not r-quasi-*$\lambda(P)$*-nuclear.*

Proof. It is clear that $\lambda(P)$ is a nuclear Köthe space which is which is subset of ℓ_1. Let $k \in N$ be given. Then

$$\frac{n^{\ln(kn)}}{n^{\frac{s}{r}} \, n^{\ln(n^{\frac{s}{r}})}} = O\left(\frac{1}{n^{\frac{s}{r}}} \right).$$

Therefore for any $k \in \mathbf{N}$ we have

$$\sum\nolimits_n \frac{n^{\ln(kn)}}{n^{\frac{s}{r}} \, n^{\ln(n^{\frac{s}{r}})}} < +\infty.$$

So we get $(\alpha_n) \in \lambda(P)$. Since

$$\begin{aligned} ||Dx||_s^s &= \sum_n |\alpha_n||x_n|^s \\ &= \sum_n |\alpha_n||\langle x, e_n\rangle|^s, \end{aligned}$$

and (e_n) is bounded sequence in ℓ_2, we have s-quasi-λ-nuclearity of D. If we assume that D is a r-quasi-$\lambda(P)$-nuclear map. Then there exist a sequence $(\beta_n) \in \lambda(P)$ and a bounded sequence (a_n) in ℓ_2 with $||a_n|| \le 1$ for each $n \in \mathbf{N}$ such that

$$||Dx||^r \le \sum\nolimits_n |\beta_n| \, |\langle x, a_n\rangle|^r.$$

Let $\gamma_n = \sum_{m=n}^{\infty} |\beta_m|$. Then one can show that $\gamma = (\gamma_n) \in \lambda(P)$. Let

$$M_n = \{x \in \ell_2 : \, \langle x, a_i\rangle = 0, \, i = 1, 2, \dots, n\}.$$

If $x \in M_n$, then

$$||Dx||^r \le \sum\nolimits_{m=n}^{\infty} |\beta_m| \, |\langle x, a_m\rangle|^r \le \gamma_n \sup\nolimits_n ||a_n||^r \, ||x||^r.$$

Hence, $D(U \cap M_n) \subseteq \gamma_n^{\frac{1}{r}} V$ where U and V are the unit balls of ℓ_2 and ℓ_s respectively. Therefore

$$D'(V^\circ) \subseteq \gamma_n^{\frac{1}{r}} U^\circ + M_n^{\perp},$$

which gives $d_n(D') \le \gamma_n^{\frac{1}{r}}$. Since D is compact, by Lemma 2.2 we have $\alpha_n(D) = \alpha_n(D')$. Also by Lemma 2.1, we have $\alpha_n(D') \le \sqrt{n}\, d_n(D')$. Let i be the inclusion map from ℓ_s into ℓ_2, Then it is clear that $\alpha_n(iD) \le \alpha_n(D)$. Hence we have $\alpha_n(iD) = \alpha_n^{\frac{1}{s}}$. So we have

$$\alpha_n(iD) \le \alpha_n(D) = \alpha(D') \le \sqrt{n}\, d_n(D'),$$

and hence

$$\alpha_n^{\frac{1}{s}} \le \sqrt{n} \gamma_n^{\frac{1}{r}}.$$

Therefore

$$\alpha_n^{\frac{r}{s}} \leq n^{\frac{r}{2}} \gamma_n \leq n \, \gamma_n.$$

But

$$\alpha_n^{\frac{r}{s}} = \frac{1}{n \, n^{\ln(n)}}.$$

So we have $\frac{1}{n^{\ln(n)}} \leq \gamma_n$. Since $(\frac{1}{n^{\ln(n)}}) \notin \lambda(P)$, we have $(\gamma_n) \notin \lambda(P)$, which is a contradiction. Therefore D is not a r-quasi-$\lambda(P)$-nuclear map. ■

Problems.

Q#1 Does Theorem 2.3 still valid for any G_∞-space $\lambda(P)$ which is not nuclear?

Q#2 Does Theorem 2.2 still valid for any $p > 2$?

Q#3. Assume that $2 < r < s < +\infty$. Is it possible to find a sequence space λ which is proper subset of ℓ_1 and a linear map T between normed spaces E and F such that T is s-quasi-λ-nuclear which is not r-quasi-λ-nuclear?

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