EXTENSIONS OF FINITE COMMUTATIVE HYPERGROUPS

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Abstract. The purpose of this paper is to investigate extension problems for the category of finite commutative hypergroups. In fact, sufficiently many extensions will be provided by applying the notion of a field of finite commutative hypergroups. Moreover, the duality of such extensions will be studied via fields of finite commutative hypergroups.

1 Introduction Let H and L be finite commutative hypergroups. A finite commutative hypergroup K is called an extension of L by H if the sequence

$$1 \to H \to K \to L \to 1$$

is exact, i.e. if the quotient hypergroup K/H is isomorphic to L. Here, the notions of subhypergroup, quotient hypergroup and isomorphism between hypergroups are taken from [B-H], a source from which all the elementary knowledge needed in the sequel will be taken.

In the previous papers [H-J-K-K] and [K-I] we constructed extensions $K(H, G, \alpha)$ and $K(\hat{H}, \hat{G}, \hat{\alpha})$ for a regular action α of a finite abelian group G on a finite commutative hypergroup H which satisfies by definition the exact sequence :

$$1 \to H^{\alpha} \to K(H, G, \alpha) \to K(G) \to 1$$

and

$$1 \to K(\hat{G}) \to K(\hat{H}, \hat{G}, \hat{\alpha}) \to \hat{H}^{\hat{\alpha}} \to 1.$$

respectively. Here, $K(G)$ [resp. $K(\hat{G})$] denotes the class hypergroup [resp. the character hypergroup] of G and H^{α} [resp. $\hat{H}^{\hat{\alpha}}$] denotes the orbital hypergroup by the action α [resp. $\hat{\alpha}$] of G on H [resp. on the dual signed hypergroup $\hat{H}$ of H]. The ways of constructing $K(H, G, \alpha)$ and $K(\hat{H}, \hat{G}, \hat{\alpha})$ are different. The former depends on the theory of operator algebras, and the latter depends on representation theory. However, observing the results of the two constructions we found that $K(H, G, \alpha)$ and $K(\hat{H}, \hat{G}, \hat{\alpha})$ have a common structure as hypergroups which we express in terms of fields of finite commutative hypergroups.

In the course of the paper, for two finite commutative hypergroups H and L we give an explicit definition of a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ of finite commutative hypergroups and show that every such field φ gives rise to an extension $K(H, \varphi, L)$ of L by H as described in Theorem 1. This extension turns out to be a generalization of both the extensions $K(H, G, \alpha)$ and $K(\hat{H}, \hat{G}, \hat{\alpha})$ above. Moreover, we shall introduce the dual $\hat{\varphi} : \hat{H} \ni \chi \longmapsto Z(\chi) \subset \hat{L}$ of the field φ and show in Theorem 3 that the extension $K(\hat{L}, \hat{\varphi}, \hat{H})$ of $\hat{H}$ by $\hat{L}$ is isomorphic to the dual of $K(H, \varphi, L)$.

It is an important problem to determine the extensions of hypergroups in order to understand their full structure. At this stage we can only establish a useful characterization of the extensions obtained by fields of commutative hypergroups as is done in Theorem 2. To find all extensions of finite commutative hypergroups remains a promising task.

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2 Preliminaries We recall some notions and facts on finite commutative hypergroups from Wildberger's paper [W]. $K := (K, A)$ is called a finite commutative *signed hypergroup* if the following conditions (1)'(6) are satisfied.

(1) A is a $*$-algebra over $\mathbb{C}$ with the unit c_0.

(2) $K=\{ c_0, c_1, \ \ldots \ , c_n \}$ is a linear basis of A.

(3) $K^*=K$.

(4) $c_i c_j = \sum_{k=0}^{n} n_{ij}^k c_k$, where n_{ij}^k is a real number such that

$$c_i^* = c_j \iff n_{ij}^0 > 0 \text{ and } c_i^* \neq c_j \iff n_{ij}^0 = 0.$$

(5) $\sum_{k=0}^{n} n_{ij}^k = 1$ for any i, j.

(6) $c_i c_j = c_j c_i$ for any i, j.

In the case that $n_{ij}^k \geq 0$ for any i, j, k, $K = (K, A)$ is called a finite commutative *hypergroup*. We often denote $*$-algebra A of (K, A) by $A(K)$.

The *weight* of an element $c_i \in K$ is defined by $w(c_i) := (n_{ij}^0)^{-1}$ where $c_j = c_i^*$, and the *total weight* of K is given by $w(K) := \sum_{i=0}^{n} w(c_i)$.

For a finite commutative signed hypergroup K a function χ on K is called a *character* of K if

$$\chi(c_i)\chi(c_j) = \sum_{k=0}^{n} n_{ij}^k \chi(c_k) \qquad \text{whenever} \quad c_i c_j = \sum_{k=0}^{n} n_{ij}^k c_k.$$

The set $\hat{K}$ of all characters of K also becomes a finite commutative signed hypergroup, and the duality $\hat{\hat{K}} \cong K$ holds in the sense of isomorphisms between signed hypergroups.

3 Fields of finite commutative hypergroups Let $H = \{h_0, h_1, \ldots, h_n\} \subset A(H)$ and $L = \{\ell_0, \ell_1, \ldots, \ell_m\} \subset A(L)$ be finite commutative hypergroups. We assume that for each element $\ell \in L$ the subset $H(\ell)$ of H satisfies the following conditions:

(1) (subhypergroup condition) $H(\ell)$ is a subhypergroup of H for each $\ell \in L$ with $H(\ell_0) = \{h_0\}$ and $H(\ell^*) = H(\ell)$.

(2) (regularity condition) If $\ell_i \ell_j = \sum_{k=0}^{m} n_{ij}^k \ell_k$, $[H(\ell_i)H(\ell_j)] \supset H(\ell_k)$ holds for all k such that $n_{ij}^k \neq 0$ where $[H(\ell_i)H(\ell_j)]$ is the subhypergroup of H generated by $H(\ell_i)$ and $H(\ell_j)$.

We denote the correspondence $L \ni \ell \longmapsto H(\ell) \subset H$ by φ and call it the *field of finite commutative hypergroups based on* L.

Let $e(\ell)$ denote the Haar measure of the subhypergroup $H(\ell)$ of H for $\ell \in L$. Then, condition (2) implies that

(3) $e(\ell_i)e(\ell_j) \leq e(\ell_k)$ for all k such that $n_{ij}^k \neq 0$.

Given a field $\varphi; L \ni \ell \longmapsto H(\ell) \subset H$ we put

$$K(H,\varphi,L) := \{he(\ell) \otimes \ell \in A(H) \otimes A(L) \ ; \ h \in H, \ell \in L\}.$$

Then we obtain the following

Theorem 1. $K(H,\varphi,L)$ is a finite commutative hypergroup which is an extension of L by H.

Proof. The set $Q(\ell) := \{he(\ell); h \in H\}$ is a finite commutative hypergroup which is isomorphic to the quotient hypergroup $H/H(\ell)$ of H by $H(\ell)$. Therefore, different elements of $K(H,\varphi,L)$ are linearly independent in $A(K(H,\varphi,L)) = \bigoplus_{j=0}^m A(Q(\ell_j)) \otimes \mathbb{C} \cdot \ell_j$. It is easy to see that the elements of $K(H,\varphi,L)$ form a linear basis of the $*$-algebra $A(K(H,\varphi,L))$.

Next we examine the product of $K(H,\varphi,L)$. For all $h_p, h_q \in H$ and all $\ell_i, \ell_j \in L$ we have

$$\begin{aligned}(h_p e(\ell_i) \otimes \ell_i)(h_q e(\ell_j) \otimes \ell_j) &= h_p h_q e(\ell_i) e(\ell_j) \otimes \ell_i \ell_j \\ &= h_p h_q e(\ell_i) e(\ell_j) \otimes \sum_{k=0}^m n_{ij}^k \ell_k \\ &= \sum_{k=0}^m n_{ij}^k h_p h_q e(\ell_i) e(\ell_j) e(\ell_k) \otimes \ell_k,\end{aligned}$$

hence the product of $K(H,\varphi,L)$ is well-defined.

In order to verify $*$-operation we compute

$$\begin{aligned}(he(\ell_i) \otimes \ell_i)(he(\ell_i) \otimes \ell_i)^* &= hh^* e(\ell_i) e(\ell_i)^* \otimes \ell_i \ell_i^* \\ &= hh^* e(\ell_i) \otimes \sum_{k=0}^m n_i^k \ell_k \\ &= \sum_{k=0}^m n_i^k hh^* e(\ell_i) e(\ell_k) \otimes \ell_k.\end{aligned}$$

From this formula we conclude that the structure constant at $h_0 \otimes \ell_0$ is $n_i^0 / w(he(\ell_i))$, and $w(he(\ell_i) \otimes \ell_i) = w(he(\ell_i)) w(\ell_i)$.

It is easy to check that the structure constant at $h_0 \otimes \ell_0$ of the product

$$(h_p e(\ell_i) \otimes \ell_i)(h_q e(\ell_j) \otimes \ell_j)$$

vanishes provided $(h_p e(\ell_j) \otimes \ell_j) \neq (h_q e(\ell_i) \otimes \ell_i)^*$.

Altogether we have shown that $K(H,\varphi,L)$ is a finite commutative hypergroup.

Now let $e(H)$ be the Haar measure of H. Then

$$Q := \{(e(H) \otimes \ell_0)(he(\ell_i) \otimes \ell_i); h \in H, \ell_i \in L\} = \{e(H) \otimes \ell_i; \ell_i \in L\}$$

is isomorphic to $K(H,\varphi,L)/H \cong L$, i.e. $K(H,\varphi,L)$ is an extension of L by H.

[Q.E.D.]

We observe that $H \times L := \{h \otimes \ell; h \in H, \ell \in L\}$ is an extension of L by H with $A(H \times L) = A(H) \otimes A(L)$ and $H(\ell) = \{h_0\}$ for all $\ell \in L$. Here we consider the map $\psi : h \otimes \ell \longmapsto he(\ell) \otimes \ell$ from $H \times L$ onto $K(H, \varphi, L)$ which induces a linear map ψ from $A(H) \otimes A(L)$ onto the $*$-subalgebra $A(K(H, \varphi, L))$ of $A(H) \otimes A(L)$.

In this way we obtain the following characterization theorem on extensions arising from a field of finite commutative hypergroups.

Theorem 2. The map ψ is an $A(H)$-module map from $A(H) \otimes A(L)$ onto the $*$-subalgebra $A(K(H, \varphi, L))$ of $K(H, \varphi, L)$ such that $\psi(h_0 \otimes \ell) = e(\ell) \otimes \ell$ where $e(\ell)$ is the Haar measure of some subhypergroup $H(\ell)$ of H satisfying $e(\ell_0) = \{h_0\}$, $e(\ell^*) = e(\ell)$ and $e(\ell_i)e(\ell_j) \leq e(\ell_k)$ for all k such that $n_{ij}^k \neq 0$, $\ell_i \ell_j = \sum_{k=0}^m n_{ij}^k \ell_k$. Conversely, if an extension K of L by H satisfies the above condition, then $K = \psi(H \times L)$ is equal to $K(H, \varphi, L)$ defined by a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$.

Proof. It is clear that the map ψ defines a linear map from $A(H) \otimes A(L)$ onto $A(K(H, \varphi, L))$ such that

$$(h_p \otimes \ell_0)\varphi(h \otimes \ell) = h_p he(\ell) \otimes \ell = \varphi((h_p \otimes \ell_0)(h \otimes \ell))$$

for all $h_p \in H$, which implies that ψ is an $A(H)$-module map. Now we see that the map ψ satisfies the conditions described in the theorem.

Suppose that the map ψ from $A(H) \otimes A(L)$ onto the $*$-subalgebra $A(K)$ of $A(H) \otimes A(L)$ satisfies the conditions of the theorem. Since $\psi(h_0 \otimes \ell) = e(\ell) \otimes \ell$, it is easy to see that

$$K = \psi(H \times L) = \{(he(\ell) \otimes \ell); h \in H, \ \ell \in L\} = K(H, \varphi, L).$$

[Q.E.D.]

Remark 1. If $H(\ell) = \{h_0\}$ for all $\ell \in L$, $K(H, \varphi, L)$ is equal to $H \times L$.

Remark 2. If $H(\ell_0) = \{h_0\}$ and $H(\ell) = H$ for all $\ell \in L$ such that $\ell \neq \ell_0$, then $K(H, \varphi, L) = H \vee L$ which is the *hypergroup join* of H and L ([B-H], p.59).

Remark 3. If $H(\ell_0) = \{h_0\}$ and $H(\ell) = W$ for all $\ell \in L$ such that $\ell \neq \ell_0$, where W is a subhypergroup of H, then, $K(H, \varphi, L) = S(Q \times L; Q \to H)$ which is a hypergroup obtained by substituting $Q := H/W$ in $Q \times L$ by H in the sense of Voit [V].

Remark 4. In this section we constructed the finite commutative hypergroup $K(H, \varphi, L)$ for two finite commutative hypergroups H and L. In a similar way we can also construct the finite commutative *signed* hypergroup $K(H, \varphi, L)$ for two finite commutative *signed* hypergroups H and L.

4 The dual of a field of finite commutative hypergroups

For two finite commutative hypergroups H and L let $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ be a field of finite commutative hypergroups based on L. We denote the annihilator $A(\hat{H}, H(\ell))$ of $H(\ell)$ by $X(\ell)$ for $\ell \in L$. Then the family $\{X(\ell) \subset \hat{H} \ ; \ \ell \in L\}$ satisfies the following conditions:

(i) $X(\ell)$ is a signed subhypergroup of $\hat{H}$ for each $\ell \in L$ such that $X(\ell_0) = \hat{H}$ and $X(\ell^*) = X(\ell)$.

(ii) $X(\ell_i) \cap X(\ell_j) \subset X(\ell_k)$ holds for all k such that $n_{ij}^k \neq 0$ where $\ell_i \ell_j = \sum_{k=0}^m n_{ij}^k \ell_k$.

We call the correspondence $L \ni \ell \longmapsto X(\ell) \subset \hat{H}$ the *adjoint* of the field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ and denote it by φ_*.

For each $\chi \in \hat{H} = \{\chi_0, \chi_1, \cdots, \chi_n\}$ we denote the subset $\{\ell \in L \ ; \ \chi \in X(\ell)\}$ of L by $Y(\chi)$. Then it is easy to see that conditions (i) and (ii) yield the following conditions:

(iii) $Y(\chi)$ is a subhypergroup of L for each $\chi \in \hat{H}$ such that $Y(\chi_0) = L$ and $Y(\chi^*) = Y(\chi)$.

(iv) $Y(\chi_i) \cap Y(\chi_j) \subset Y(\chi_k)$ for all k such that $m_{ij}^k \neq 0$, where $\chi_i \chi_j = \sum_{k=0}^n m_{ij}^k \chi_k$.

Here we note that condition (iii) follows from (ii) and condition (iv) follows from (i).

By this procedure we have produced the *dual adjoint* field $\hat{H} \ni \chi \longmapsto Y(\chi) \subset L$ which will be denoted by $\hat{\varphi}_*$.

For each $\chi \in \hat{H}$, take the annihilator $A(\hat{L}, Y(\chi))$ of $Y(\chi)$ and denote it by $Z(\chi)$.

Thus we obtain the field $\hat{\varphi} : \hat{H} \ni \chi \longmapsto Z(\chi) \subset \hat{L}$ which we call the *dual* of the field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$.

Consequently we have a finite commutative signed hypergroup

$$K(\hat{L}, \hat{\varphi}, \hat{H}) = \{\rho e(\chi) \otimes \chi \ ; \ \rho \in \hat{L}, \ \chi \in \hat{H}\}.$$

Lemma In the above situation we get

(1) For each $\chi \in \hat{H}$ and $\ell \in L$, $\ell \in Y(\chi)$ if and only if $\chi \in X(\ell)$.

(2) For each $\chi \in \hat{H}$ and the Haar measure $e(\ell)$ of $H(\ell)$,

$$\chi(e(\ell)) = \begin{cases} 1 & \text{if } \chi \in X(\ell) \\ 0 & \text{if } \chi \notin X(\ell). \end{cases}$$

(3) For each $\ell \in L$ and the Haar measure $e(\chi)$ of $Z(\chi)$,

$$e(\chi)(\ell) = \begin{cases} 1 & \text{if } \ell \in Y(\chi) \\ 0 & \text{if } \ell \notin Y(\chi). \end{cases}$$

(4) For each $\chi \in \hat{H}$ and $\ell \in L$, we have $\chi(e(\ell)) = e(\chi)(\ell)$.

Proof. (1) follows immediately from the definition of $Y(\chi)$. (2) and (3) are obtained by the property of the Haar measure of subhypergroups. (4) follows directly from (1), (2), and (3). We omit the details.

[Q.E.D.]

Now we arrive at the duality theorem.

Theorem 3. Under the above assumptions we have

(1) $K(\hat{L}, \hat{\varphi}, \hat{H}) \cong \hat{K}(H, \varphi, L)$,

(2) $K(H,\varphi,L)\cong\hat{K}(\hat{L},\hat{\varphi},\hat{H})$.

Proof. It is clear that $\hat{K}(H,\varphi,L)\supset K(\hat{L},\hat{\varphi},\hat{H})$. It remains to show that $\hat{K}(H,\varphi,L)\subset K(\hat{L},\hat{\varphi},\hat{H})$. Let χ be a character of $K(H,\varphi,L)$. Then there exists $\chi_j\in\hat{H}$ such that

$$\chi(he(\ell)\otimes\ell)=\chi_j(h)\chi_j(e(\ell))\chi(h_0\otimes\ell)=\chi_j(h)e(\chi_j)(\ell)\rho(\ell).$$

for some $\rho\in\hat{L}$. Hence we get $\chi=\chi_j e(\chi_j)\otimes\rho$, and this proves statement (1).

Statement (2) follows immediately from the isomorphisms

$$\hat{K}(\hat{L},\hat{\varphi},\hat{H})\cong\hat{\hat{K}}(H,\varphi,L)\cong K(H,\varphi,L).$$

[Q.E.D.]

Remark 1. We have established the exact sequence

$$1\longrightarrow\hat{L}\longrightarrow K(\hat{L},\hat{\varphi},\hat{H})\longrightarrow\hat{H}\longrightarrow 1$$

which is the dual of the exact sequence

$$1\longrightarrow H\longrightarrow K(H,\varphi,L)\longrightarrow L\longrightarrow 1.$$

Discussion Here we describe the relationship between hypergroups arising from fields and hypergroups associated with group actions studied in [H-J-K-K] and [K-I].

Let α be an action of a finite abelian group G on a finite commutative hypergroup $M=\{c_0,c_1,\cdots,c_n\}$. Then the action α induces an action of G on the $*$-algebra $A(M)$, which we also denote by α. Let E be a conditional expectation from $A(M)$ onto the fixed point algebra $A(M)^\alpha$ defined by

$$E(x):=\frac{1}{|G|}\sum_{g\in G}\alpha_g(x)\quad\text{for } x\in A(M).$$

Then the orbital hypergroup is given by

$$M^\alpha:=\{d\in A(M)^\alpha;\ d=E(c),\ c\in M\}.$$

The action α of G on M induces an action $\hat{\alpha}$ of G on the dual signed hypergroup $\hat{M}$ and also on $A(\hat{M})$ by

$$\hat{\alpha}_g(\chi)(c)=\chi(\alpha_g^{-1}(c))\quad\text{for }\chi\in\hat{M},c\in M.$$

In a similar way we define a conditional expectation F from $A(\hat{M})$ onto $A(\hat{M}^{\hat{\alpha}})$ and also the orbital signed hypergroup $\hat{M}^{\hat{\alpha}}$ defined by this action $\hat{\alpha}$ of G.

We denote by $K(G)$ the hypergroup associated with the group G, i.e.

$$K(G)=\{\ell_g;g\in G\}\quad\text{with } \ell_{g_1}\ell_{g_2}=\ell_{g_1g_2}.$$

For each $\ell_g\in K(G)$ we take the sets

$$\overline{X}(\ell_g)=\{\chi\in\hat{M};\hat{\alpha}_g(\chi)=\chi\}$$

and

$$X(\ell_g) = \{\rho \in \hat{M}^{\hat{\alpha}}; \rho = F(\chi), \chi \in \overline{X}(\ell_g)\}.$$

The regularity of the action α is required as the assumption which assures that the family $\{X(\ell_g); \ell_g \in K(G)\}$ satisfies the above conditions (i) and (ii). Let $H(\ell_g)$ denote the annihilator $A(M^\alpha, X(\ell_g))$ of $X(\ell_g)$ and $e(g)$ denote the Haar measure of $H(\ell_g)$.

In [H-J-K-K] we introduced the hypergroup associated with the regular action α of G on M by

$$K(M, G, \alpha) := \{he(g) \otimes \ell_g; h \in M^\alpha, g \in G\},$$

which coincides with the extension $K(M^\alpha, \varphi, K(G))$ arising from the field $\varphi; K(G) \ni \ell_g \longmapsto H(\ell_g) \subset M^\alpha$.

Next we review the other hypergroup $K(\hat{M}, \hat{G}, \hat{\alpha})$ associated with the regular action α which is studied in [K-I]. For each $\chi \in \hat{M}$ we put

$$\overline{Y}(\chi) = \{\ell_g \in K(G); \chi \in \overline{X}(\ell_g)\} = \{\ell_g \in K(G); \hat{\alpha}_g(\chi) = \chi\}.$$

It is easy to see that $\overline{Y}(\chi_p) = \overline{Y}(\chi_q)$ if $F(\chi_p) = F(\chi_q)$. We denote $\overline{Y}(\chi)$ by $Y(\rho)$ for each $\rho \in \hat{M}^{\hat{\alpha}}$ such that $\rho = F(\chi)$. Take the annihilator $Z(\rho) := A(K(\hat{G}), Y(\rho))$ of $Y(\rho)$ and denote the Haar measure of $Z(\rho)$ by $\tau(\rho)$.

Then we obtain the hypergroup $K(\hat{M}, \hat{G}, \hat{\alpha})$ investigated in [K-I] as

$$K(\hat{M}, \hat{G}, \hat{\alpha}) = \{\rho \otimes \tau(\rho)\tau; \rho \in \hat{M}^{\hat{\alpha}}, \tau \in K(\hat{G})\},$$

which coincides with the extension $K(K(\hat{G}), \hat{\varphi}, \hat{M}^{\hat{\alpha}})$ of $\hat{M}^{\hat{\alpha}}$ by $K(\hat{G})$ arising from the dual field $\hat{\varphi}; \hat{M}^{\hat{\alpha}} \ni \rho \longmapsto Z(\rho) \subset K(\hat{G})$.

5 Applications and examples

We apply our results for some concrete examples. Let $H = \{h_0, h_1, h_2, h_3\}$ be a finite commutative hypergroups whose structure equations are given by

$$h_1^2 = \frac{1}{2}h_0 + \frac{1}{2}h_1,$$

$$h_2^2 = \frac{1}{2}h_0 + \frac{1}{2}h_2,$$

$$h_3^2 = \frac{1}{4}h_0 + \frac{1}{4}h_1 + \frac{1}{4}h_2 + \frac{1}{4}h_3,$$

$$h_1 h_2 = h_3,$$

$$h_1 h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_3,$$

$$h_2 h_3 = \frac{1}{2}h_1 + \frac{1}{2}h_3.$$

Hence, the weight $w(h_i)$ of h_i and the total weight $w(H)$ of H are

$$w(h_0) = 1,\ w(h_1) = 2,\ w(h_2) = 2,\ w(h_3) = 4,\ \text{and}$$

$$w(H) = w(h_0) + w(h_1) + w(h_2) + w(h_3) = 1 + 2 + 2 + 4 = 9,$$

respectively. The subhypergroups of H are

$$H_0 = \{h_0\},$$

$$H_1 = \{h_0, h_1\},$$

$$H_2 = \{h_0, h_2\} \text{ and}$$

$$H_3 = H = \{h_0, h_1, h_2, h_3\},$$

and the Haar measures e_i of H_i ($i = 0, 1, 2, 3$) are given by

$$e_0 = h_0,$$

$$e_1 = \frac{1}{3}h_0 + \frac{2}{3}h_1,$$

$$e_2 = \frac{1}{3}h_0 + \frac{2}{3}h_2 \text{ and}$$

$$e_3 = e(H) = \frac{1}{9}h_0 + \frac{2}{9}h_1 + \frac{2}{9}h_2 + \frac{4}{9}h_3.$$

Let $\hat{H} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ be the dual of H which are determined by the following character table.

	h_0	h_1	h_2	h_3
χ_0	1	1	1	1
χ_1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$
χ_2	1	$-\frac{1}{2}$	1	$-\frac{1}{2}$
χ_3	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$

From this table we see that $\hat{H}$ is isomorphic to H by the correspondences $\chi_i \longleftrightarrow h_i$ $(i = 0, 1, 2, 3)$.

The subhypergroups of $\hat{H}$ are

$$X_0 = \{\chi_0\},$$

$$X_1 = \{\chi_0, \chi_1\},$$

$$X_2 = \{\chi_0, \chi_2\} \text{ and}$$

$$X_3 = \hat{H} = \{\chi_0, \chi_1, \chi_2, \chi_3\}.$$

We note that the annihilators of X_0, X_1, X_2, X_3 are H_3, H_1, H_2, H_0 respectively.

Let $L = \{\ell_0, \ell_1, \ell_2, \ell_3\}$ be another finite commutative hypergroup whose structure equations are

$$\ell_1^2 = \ell_2^2 = \ell_3^2 = \ell_0,$$

$$\ell_1\ell_2 = \ell_3,\ \ell_1\ell_3 = \ell_2,\ \ell_2\ell_3 = \ell_1.$$

Here, we present a list of all possible extensions $K_i := K(H, \varphi_i, L)$ of L by H arising from fields φ_i which satisfy both subhypergroup condition and regularity condition among all subhypergroups of H described in section 3. We obtain 16 kinds of fields φ_i as given in the following list.

	$H(\ell_0)$	$H(\ell_1)$	$H(\ell_2)$	$H(\ell_3)$
φ_1	H_0	H_0	H_0	H_0
φ_2	H_0	H_3	H_3	H_3
φ_3	H_0	H_1	H_1	H_1
φ_4	H_0	H_2	H_2	H_2
φ_5	H_0	H_3	H_1	H_2
φ_6	H_0	H_3	H_2	H_1
φ_7	H_0	H_1	H_3	H_2
φ_8	H_0	H_1	H_2	H_3
φ_9	H_0	H_2	H_3	H_1
φ_{10}	H_0	H_2	H_1	H_3
φ_{11}	H_0	H_0	H_1	H_1
φ_{12}	H_0	H_1	H_0	H_1
φ_{13}	H_0	H_1	H_1	H_0
φ_{14}	H_0	H_0	H_2	H_2
φ_{15}	H_0	H_2	H_0	H_2
φ_{16}	H_0	H_2	H_2	H_0

Example 1.

$$H(\ell_0) = H(\ell_1) = H(\ell_2) = H(\ell_3) = H_0$$

$$X(\ell_0) = X(\ell_1) = X(\ell_2) = X(\ell_3) = X_3$$

$$K_1 = K(H, \varphi_1, L) = \{h_i \otimes \ell_j\ ;\ i, j = 0, 1, 2, 3\} = H \times L$$

The number $\mid K_1 \mid$ of elements of K_1 is 16.

Example 2.

$$H(\ell_0) = H_0,\ H(\ell_1) = H(\ell_2) = H(\ell_3) = H$$

$$X(\ell_0) = X_3,\ X(\ell_1) = X(\ell_2) = X(\ell_3) = X_0$$

$$K_2 = K(H, \varphi_2, L) = H \vee L, \quad \mid K_2 \mid = 7$$

Example 3.

$$H(\ell_0) = H_0,\ H(\ell_1) = H(\ell_2) = H(\ell_3) = H_1$$

$$X(\ell_0) = X_3,\ X(\ell_1) = X(\ell_2) = X(\ell_3) = X_1$$

$$K_3 = K(H, \varphi_3, L) = S(Q_1 \times L; Q_1 \to H) \text{ for } Q_1 = H/H_1,\ \mid K_3 \mid = 10$$

Example 4.

$$H(\ell_0) = H_0,\ H(\ell_1) = H_3,\ H(\ell_2) = H_2,\ H(\ell_3) = H_1$$

$$X(\ell_0) = X_3,\ X(\ell_1) = X_0,\ X(\ell_2) = X_2,\ X(\ell_3) = X_1$$

$$K_6 = K(H, \varphi_6, L), \quad \mid K_6 \mid = 9$$

Example 5.

$$H(\ell_0) = H_0,\ H(\ell_1) = H_0,\ H(\ell_2) = H(\ell_3) = H_1$$

$$X(\ell_0) = X_3,\ X(\ell_1) = X_3,\ X(\ell_2) = X(\ell_3) = X_1$$

$$K_{11} = K(H, \varphi_{11}, L), \quad \mid K_{11} \mid = 12$$

Remark 1. Since the roles of h_1 and h_2 (χ_1 and χ_2) and also those of ℓ_1, ℓ_2, ℓ_3 can be exchanged we can see that mutually non-isomorphic hypergroups among the extensions $K_i = K(H, \varphi_i, L)$ $(i = 1, 2, \cdots, 16)$ of L by H are essentially the 5 kinds as shown in Examples 1, 2, 3, 4, and 5.

Remark 2. Let N and G be abelian groups with $N = \{(n_i, n_j);\ i, j = 0, 1, 2\} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $G = \{e, g, h, gh\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ so that $n_1^2 = n_2, n_2^2 = n_1, n_1 n_2 = n_2 n_1 = n_0$, and $g^2 = h^2 = e$. Let α be the action of G on N defined by

(i) $\alpha_g((n_i, n_j)) = (n_i^2, n_j^2) \quad (i, j = 0, 1, 2)$

(ii) $\alpha_h((n_i, n_j)) = (n_j, n_i) \quad (i, j = 0, 1, 2)$

Then by simple calculations we can show that

$$K_6 = K(H, \varphi_6, L) \cong K(N \rtimes_\alpha G),$$

where $L = K(G)$, $H = K(N)^\alpha$, and $K(N \rtimes_\alpha G)$ is the class hypergroup of the semi-direct product $N \rtimes_\alpha G$, referred to in Example 4 of the paper [H-J-K-K].

Indeed, the structure equations of

$$K_6 = K(H, \varphi_6, L) = \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$$

where

$$c_0 = h_0 \otimes \ell_0,\ c_1 = h_1 \otimes \ell_0,\ c_2 = h_2 \otimes \ell_0,\ c_3 = h_3 \otimes \ell_0,$$

$$c_4 = e_3 \otimes \ell_1,\ c_5 = e_2 \otimes \ell_2,\ c_6 = h_1 e_2 \otimes \ell_2,\ c_7 = e_1 \otimes \ell_3,\ c_8 = h_2 e_1 \otimes \ell_3,$$

are given as follows:

$$c_1^2 = \frac{1}{2}c_0 + \frac{1}{2}c_1 \ , \quad c_2^2 = \frac{1}{2}c_0 + \frac{1}{2}c_2 \ ,$$

$$c_3^2 = \frac{1}{4}c_0 + \frac{1}{4}c_1 + \frac{1}{4}c_2 + \frac{1}{4}c_3 \ ,$$

$$c_4^2 = \frac{1}{9}c_0 + \frac{2}{9}c_1 + \frac{2}{9}c_2 + \frac{4}{9}c_3 \ ,$$

$$c_5^2 = \frac{1}{3}c_0 + \frac{2}{3}c_2 \ ,$$

$$c_6^2 = \frac{1}{6}c_0 + \frac{1}{6}c_1 + \frac{1}{3}c_2 + \frac{1}{3}c_3 \ ,$$

$$c_7^2 = \frac{1}{3}c_0 + \frac{2}{3}c_1 \ ,$$

$$c_8^2 = \frac{1}{6}c_0 + \frac{1}{3}c_1 + \frac{1}{6}c_2 + \frac{1}{3}c_3 \ ,$$

$$c_1 c_2 = c_3 \ , \quad c_1 c_3 = \frac{1}{2}c_2 + \frac{1}{2}c_3 \ , \quad c_1 c_4 = c_4 \ , \quad c_1 c_5 = c_6 \ ,$$

$$c_1 c_6 = \frac{1}{2}c_5 + \frac{1}{2}c_6 \ , \quad c_1 c_7 = c_7 \ , \quad c_1 c_8 = c_8 \ ,$$

$$c_2c_3 = \frac{1}{2}c_1 + \frac{1}{2}c_3 \ , \quad c_2c_4 = c_4 \ , \quad c_2c_5 = c_5 \ ,$$

$$c_2c_6 = c_6 \ , \quad c_2c_7 = c_8 \ , \quad c_2c_8 = \frac{1}{2}c_7 + \frac{1}{2}c_8 \ ,$$

$$c_3c_4 = c_4 \ , \quad c_3c_5 = c_6 \ , \quad c_3c_6 = \frac{1}{2}c_5 + \frac{1}{2}c_6 \ ,$$

$$c_3c_7 = c_8 \ , \quad c_3c_8 = \frac{1}{2}c_7 + \frac{1}{2}c_8 \ ,$$

$$c_4c_5 = \frac{1}{3}c_7 + \frac{2}{3}c_8 \ , \quad c_4c_6 = \frac{1}{3}c_7 + \frac{2}{3}c_8 \ ,$$

$$c_4c_7 = \frac{1}{3}c_5 + \frac{2}{3}c_6 \ , \quad c_4c_8 = \frac{1}{3}c_5 + \frac{2}{3}c_6 \ ,$$

$$c_5c_6 = \frac{1}{3}c_0 + \frac{2}{3}c_3 \ , \quad c_5c_7 = c_4 \ , \quad c_5c_8 = c_4 \ ,$$

$$c_6c_7 = c_4 \ , \quad c_6c_8 = c_4 \ ,$$

$$c_7c_8 = \frac{1}{3}c_2 + \frac{2}{3}c_3 \ .$$

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