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R-KKM THEOREMS ON L-CONVEX SPACES AND ITS APPLICATIONS

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Abstract. In this paper, we introduce the generalized $R-KKM$ mapping and the new class $R_{\psi_N}-KKM(X,Y)$, and we get some fixed point theorems, matching theorems, coincidence theorems, and minimax inequalities on L-convex spaces.

1 Introduction and Preliminaries In 1937, Neumann[9] established the well-knowm coincidence theorem, and then there had a lot of generalizations and applications. Park and Kim[11] introduce the conception of generalized convex (G-convex) spaces which is adequate establish theories on fixed points, coincidence points and variational inequalities. In [1], Ben-El-Mechaiekh defined a L-convex space by dropping some assumption on G-convex space and got some fixed point theorems. In this paper, we establish some coincidence theorems for the maps $R-KKM$ on L-convex spaces, and we use these coincidence theorems to establish the existence theorems concerning minimax inequalities.

We now introduce the notations used in this paper and recall some basic facts.

For a nonempty set X, 2^X denotes the class of all nonempty subsets of X, and $\langle X\rangle$ denotes the class of all nonempty finite subsets of X.

Throughout this paper, for a set-valued function $T: X \to 2^Y$, the following notations are used:

(i) $T(x) = \{y \in Y | y \in T(x)\}$,

(ii) $T(A) = \cup_{x\in A} T(x)$,

(iii) $T^{-1}(y) = \{x \in X | y \in T(x)\}$,

(iv) $T^{-1}(B) = \{x \in X | T(x) \cap B \neq \phi\}$,and

(v) $T^*y = X \setminus T^{-1}y$. Clearly, $x \in T^*y$ iff $y \notin Tx$, and

(vi) T is said lower semicontinuous ($l.s.c$)if for each open set B in Y, $T^{-1} = \{x \in X | T(x) \cap B \neq \phi\}$ is open in X.

T is said to be closed if its graph $\mathcal{G}_T = \{(x,y) \in X \times Y | y \in Tx, \forall x \in X\}$ is a closed subset of $X \times Y$. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y.

A subset D of a topological space X is said to be compactly closed (resp. open) in X if for any compact subset K of X the set $D \cap K$ is closed (resp. open) in K.Obviosly, D is compactly closed in X if and only if its complement $D^c = X\backslash D$ is compactly open in X.

Let Δ_n denote the standard n-simplex, that is; $\Delta_n = \{u \in R^{n+1} : u = \sum_{i=0}^n \lambda_i(u)e_i, \lambda_i(u) \geq 0, \sum_{i=0}^n \lambda_i(u) = 1\}$, where $e_i (i = 0,1,2,...,n)$ is the $(i+1)$-th the unit vector in R^{n+1}.

A L-convex space (X, D, Γ) consists of a topological space X,a nonempty subset D of X and a function $\Gamma : \langle D\rangle \to 2^X$ with nonempty values (in the sequal, we write $\Gamma(A)$ by ΓA for each $A \in \langle X\rangle$) such that for each $A \in \langle D\rangle$ with $|A| = n+1$, there exists a continuous

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function $\varphi_A : \Delta_n \to \Gamma A$ such that $J \in \langle A \rangle$ implies $\varphi_A(\Delta_{|J|-1}) \subset \Gamma(J)$, where $\Delta_{|J|-1}$ denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

If an L-convex space (X, D, Γ) satisfies the addition condition

$$\textit{for} \text{ each } \quad A, B \in \langle D \rangle, A \subset B \quad \text{implies } \Gamma(A) \subset \Gamma(B),$$

then (X, D, Γ) is called a generalized G-convex space.

A subset K of X is said to be L-convex if for each $B \in \langle D \rangle$, $B \subset K$ implies $\Gamma(B) \subset K$. The L-convex hull of K, denoted by $L - co(K)$ is the set
$\cap\{B \subset X | B$ is a L-convex subset of X containing $K\}$.

DEFINITION. Let (X, D, Γ) be a L-convex space, D is a nonempty subset of X, and $T : D \to 2^X$ is said to be a generalized relatively KKM $(R - KKM)$ mapping if for any $N \in \langle D \rangle$ with $|N| = n + 1$, there exists a continuous mapping $\psi_N : \Delta_n \to X$ such that $\psi_N(\Delta_n) \subset T(N)$.

(*This ψ_N may be different from the φ_N of the definition for the L-convex space.)

Now, we prove the intersection theorems involving $R - KKM$ mapping.

THEOREM 1. *Let X be a L-convex space and $T : X \to 2^X$ a multimap with closed values. Suppose that T is a $R-KKM$ mapping. Then $\{T(z)\}_{z\in X}$ has the finite intersection property.*

Proof. Since T is a $R - KKM$ mapping,we have that for any $N = \{x_0, x_1, ..., x_n\} \in \langle X \rangle$ with $|N| = n + 1$, there exists a continuous mapping $\psi_N : \Delta_n \to X$ such that $\psi_N(\Delta_n) \subset \cup_{i=0}^n T(x_i)$. If we set $B_i = \psi_N^{-1}(Tx_i)$ for $i = 0, 1, 2, ..., n$, then B_i is closed in Δ_n by the closedness of T and the continuity of ψ. From above arguments, it follow that $co\{e_0, e_1, ..., e_n\} \subset \psi_N^{-1}(\cup_{i=0}^n Tx_i) = \cup_{i=0}^n \psi_N^{-1} Tx_i = \cup_{i=0}^n B_i$. Apply the KKM theorem[7], we have $\cap_{i=0}^n B_i \neq \phi$. Let $b_0 \in \cap_{i=0}^n B_i$, then $b_0 \in B_i = \psi_N^{-1}(Tx_i)$ for all $i = 0, 1, ..., n$. Hence $\psi_N(b_0) \in \cap_{i=0}^n T(x_i)$. □

DEFINITION. Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N| = n + 1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. If $T : X \to 2^Y$ and $F : X \to 2^Y$ are two set-valued functions satisfying that $T(\psi_N(\Delta_n)) \subset F(N)$, for any $N \in \langle X \rangle$ with $|N| = n + 1$, then F is said to be a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N. Moreover, if the set-valued function $T : X \to 2^Y$ satisfies the requirement that for any generalized $R_{\psi_N} - KKM$ mapping F with respect to T and ψ_N the family $\{\overline{Fx} | x \in X\}$ has the finite intersection property, then T is said to have the $R_{\psi_N} - KKM$ property. The class $R_{\psi_N} - KKM(X, Y)$ is defined to be the set $\{T : X \to 2^Y | T$ has the $R_{\psi_N} - KKM$ property$\}$.

(*This ψ_N may be different from the φ_N of the definition for the L-convex space.)

2 COINCIDENCE THEOREMS AND FIXED POINT THEOREMS

First, we deal with the intersection theorems involving $R - KKM$ mappings and obtain some fixed point theorems.

THEOREM 2. *Let X be a compact L-convex space and $S, T : X \to 2^X$ two multivalued mappings such that*

(i) Tx is closed and $Sx \subset Tx$ for all $x \in X$,

(ii) $x \in Sx$ for all $x \in X$, and

(iii) S^*y is L-convex for all $y \in X$.

Then $\cap_{x\in X}Tx \neq \phi$.

Proof. First, we claim that T is a $R-KKM$ mapping. If not, we assume that T is not $R-KKM$, then there exists $N=\{x_0,x_1,...,x_n\}\in\langle X\rangle$ with $|N|=n+1$ such that for any continuous mapping $\psi_N:\Delta_n\to X$, $\psi_N(\Delta_n)\nsubseteq \cup_{i=0}^{n}Tx_i$. Let $y\in\psi_N(\Delta_n)\nsubseteq\cup_{i=0}^{n}Tx_i$, then $y\notin Tx_i$ for all $i=0,1,2,...,n$. This means $x_i\in T^*y$, for all $i=0,1,2,...,n$. By (i), $Sx\subset Tx$ for all $x\in X$, we have $T^*y\subset S^*y$ for all $y\in Y$. Hence, $\{x_0,x_1,...,x_n\}\subset S^*y$. Since S^*y is L-convex, we have $\Gamma(\{x_0,x_1,...,x_n\})\subset S^*y$. Thus $y\in\psi_N(\Delta_n)\subset\Gamma(N)\subset S^*y$, $y\notin Sy$, a contraction to (ii). Next, by Theorem 1 and X is a compact L-convex space, we have $\cap_{x\in X}Tx\neq\phi$. □

The following theorem is a variant form of theorem 2.

THEOREM 3. *Let X be a compact L-convex space and $S,T:X\to 2^X$ two multivalued mappings such that*

(i) $Sx\subset Tx$ for all $x\in X$,

(ii) Tx is L-convex for all $x\in X$, and

(iii) $S^{-1}y$ is open for all $y\in X$.

Then there exists $x_0\in X$ such that $x_0\in Tx_0$.

Next, we state some coincidence theorems for the class $R_{\psi_N}-KKM(X,Y)$.

THEOREM 4. *Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N|=n+1$,there exists a continuous mapping $\psi_N:\Delta_n\to X$. Let $H:X\to 2^Y$, $T:X\to 2^Y$, and $F\in R_{\psi_N}-KKM(X,Y)$ be three set-valued maps satisfying the following:*

(i) for each $y\in F(X)$, $N\in\langle H^{-1}(y)\rangle$ with $|N|=n+1$, implies $\psi_N(\Delta_n)\subset T^{-1}(y)$, and

(ii) $\overline{FX}\subset\cup\{intH(x):x\in M\}$ for some $M\in\langle X\rangle$.

Then there exists $\overline{x}\in X$ such that $F(\overline{x})\cap T(\overline{x})\neq\phi$.

Proof. First, we assume that $\overline{FX}\subset intH(x_0)$ for some $x_0\in X$. Then $F(N)\subset intH(x_0)$ where $N=\psi_{x_0}(\Delta_0)$, since $F\in R_{\psi_N}-KKM(X,Y)$. Choose $\overline{x}\in N$ and $\overline{y}\in F(\overline{x})$, then $\overline{y}\in intH(x_0)$, $\{x_0\}\in\langle H^{-1}(\overline{y})\rangle$. From (i), we have $N=\psi_{x_0}(\Delta_0)\subset T^{-1}(\overline{y})$. Thus, $\overline{y}\in\cap_{z\in N}T(z)\subset T(\overline{x})$, and hence $\overline{y}\in T(\overline{x})\cap F(\overline{x})$.

Next, we assume that $\overline{FX}\nsubseteq intH(x)$ for all $x\in X$. We now define $P:X\to 2^Y$ by $P(x)=\overline{FX}\backslash intH(x)$. Then $P(x)$ is nonempty and closed, and we claim that P is not a generalized $R_{\psi_N}-KKM$ mapping with respect to F and ψ_N.

Suppose that P is a generalized $R_{\psi_N}-KKM$ mapping with respect to F and ψ_N. Since $F\in R_{\psi_N}-KKM(X,Y)$, we have $\{P(x):x\in X\}$ has the finite intersection property. Hence, for all $N\in\langle X\rangle$, $\cap_{x\in N}P(x)\neq\phi$. So $\phi\neq\cap_{x\in N}P(x)=\cap_{x\in N}(\overline{FX}\backslash intH(x))=\overline{FX}\backslash\cup_{x\in N}intH(x)$. Thus, $\overline{FX}\nsubseteq\cup_{x\in N}intH(x)$ for all $N\in\langle X\rangle$, which contradicts to (ii). Since P is not a generalized $R_{\psi_N}-KKM$ mapping with respect to F and ψ_N, there exists a finite subset $N=\{z_0,z_1,z_2,...,z_n\}$ of X with $|N|=n+1$ such that $F(\psi_N(\Delta_n))\nsubseteq P(N)$. Choose $\overline{x}\in\psi_N(\Delta_n)$ such that $F(\overline{x})\nsubseteq P(N)$, and since $P(N)=\overline{F(X)}\backslash(\cap_{i=0}^{n}intH(z_i))$, we have that there exists $\overline{y}\in F(\overline{x})$ such that $\overline{y}\notin P(N)$, $\overline{y}\in\cap_{i=0}^{n}intH(z_i)$, and hence $z_i\in H^{-1}(\overline{y})$ for all $i=0,1,2,...,n$, that is; $N\subset H^{-1}(\overline{y})$. By condition (i), $\overline{x}\in\psi_N(\Delta_n)\subset T^{-1}(\overline{y})$. Therefore, $\overline{y}\in T(\overline{x})\cap F(\overline{x})$. This completes the proof. □

THEOREM 5. *Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N|=n+1$,there exists a continuous mapping $\psi_N:\Delta_n\to X$. Let $T:X\to 2^Y$ and $H,F:Y\to 2^X$ be three set-valued maps satisfying the following:*

(i) $T \in R_{\psi_N} - KKM(X,Y)$ *is compact,*
(ii) *for each* $y \in T(X)$, $N \in \langle F(y) \rangle$ *with* $|N| = n+1$, *implies* $\psi_N(\Delta_n) \subset H(y)$, *and*
(iii) $Y = \cup\{intF^{-1}(x) : x \in X\}$.

Then there exists $\overline{x} \in X$ *and* $\overline{y} \in Y$ *such that* $\overline{x} \in H(\overline{y})$, $\overline{y} \in T(\overline{x})$

Proof. Define $P, F' : X \to 2^Y$ by $P(x) = F^{-1}(x)$, $F'(x) = H^{-1}(x)$ for all $x \in X$. From (ii),for each $y \in T(X)$, $N \in \langle P^{-1}(y) \rangle$ with $|N| = n+1$, implies $\psi_N(\Delta_n) \subset (F')^{-1}(y)$. From (iii), we have $\overline{T(X)} \subset \cup\{intP(x) : x \in X\}$. Since T is compact, applying the theorem 4, we have that $T(\overline{x}) \cap F'(\overline{x}) \neq \phi$ for some $\overline{x} \in X$. Choose $\overline{y} \in T(\overline{x}) \cap F'(\overline{x})$, then $\overline{y} \in H^{-1}(\overline{x})$, that is; $\overline{x} \in H(\overline{y})$. This completes the proof. □

THEOREM 6. *Let* X *be a L-convex space,* Y *a topological space such that for each finite subset* N *of* X *with* $|N| = n+1$,*there exists a continuous mapping* $\psi_N : \Delta_n \to X$. *Let* $H : X \to 2^Y$, $T : X \to 2^Y$, *and* $F \in R_{\psi_N} - KKM(X,Y)$ *be three set-valued maps satisfying the following:*

(i) H *has open values,*
(ii) *for each* $y \in F(X)$, $N \in \langle H^{-1}(y) \rangle$ *with* $|N| = n+1$, *implies* $\psi_N(\Delta_n) \subset T^{-1}(y)$, *and*
(iii) $\overline{FX} \subset H(A)$ *for some* $A \in \langle X \rangle$.

Then F *and* T *have a coincidence point* $\overline{x}$ *in* X, *that is; there is* $\overline{x} \in X$ *such that* $F(\overline{x}) \cap T(\overline{x}) \neq \phi$.

Proof. Let $W = \overline{F(X)}$. Define a map $P : X \to 2^W$ by $P(z) = W \backslash H(z)$ for each $z \in X$. Then P has closed values. By (iii), $\cap_{z \in A} P(z) = \cap_{z \in A} H^c(z) = (H(A))^c \subset (\overline{FX})^c = W^c = \phi$. Then $\{P(z)\}_{z \in X}$ does not have the finite intersection property. Since $F \in R_{\psi_N} - KKM(X,Y)$, we have $F(\psi_M(\Delta_m)) \nsubseteq P(M)$ for some $M \in \langle X \rangle$ with $|M| = m+1$. Hence, there exists a $\overline{y} \in F(\psi_M(\Delta_m))$ such that $\overline{y} \notin P(z)$ for all $z \in M$. Therefore, $\overline{y} \in F(\psi_M(\Delta_m))$ and $\overline{y} \in H(z)$ for all $z \in M$, that is; $z \in H^{-1}(\overline{y})$ for all $z \in M$ and hence $M \in \langle H^{-1}(\overline{y}) \rangle$. Since $\overline{y} \in F(\psi_M(\Delta_m))$, by (ii), we have $\psi_M(\Delta_m) \subset T^{-1}(\overline{y})$, and hence $\overline{y} \in F(\psi_M(\Delta_m)) \subset F(T^{-1}(\overline{y}))$, and hence exists a $\overline{x} \in T^{-1}(\overline{y})$ such that $\overline{y} \in F(\overline{x})$. So we have that $\overline{y} \in F(\overline{x}) \cap T(\overline{x})$. This completes the proof. □

By using Theorem 6, we have the fixed point theorem.

THEOREM 7. *Let* X *be a nonempty compact L-convex space such that for each finite subset* N *of* X *with* $|N| = n+1$,*there exists a continuous mapping* $\psi_N : \Delta_n \to X$, *and let* $S, P : X \to 2^X$ *be two set-valued functions. Assume that*

(i) *for each* $x \in X$ *and for each* $N \in \langle S(x) \rangle$ *with* $|N| = n+1$, $\psi_N(\Delta_n) \subset P(x)$, *and*
(ii) $X = \cup\{int_X S^{-1}(y) : y \in X\}$

Then P *has a fixed point in* X, *that is; there exists* $\overline{x} \in X$ *such that* $\overline{x} \in P(\overline{x})$.

Proof. Let $H : X \to 2^X$ be defined by $H(z) = intS^{-1}(z)$ for $z \in X$. Then $H(z)$ is open in X, and $H^{-1}(z) = (intS^{-1})^{-1}(z) \subset S(z)$, for $z \in X$. Hence, for $N \in \langle H^{-1}(z) \rangle$, $\psi_N(\Delta_n) \subset P(x)$. Now, by assumption (ii) and the compactness of X, there exists $N \in \langle X \rangle$ such that $X = \cup_{y \in N} intS^{-1}(y) = H(N)$. So all assumptions of Theorem 6 are satisfied by letting $F = 1_X$ and $T = P^{-1}$. Hence there exists $\overline{x} \in X$ such that $\overline{x} \in \{F(\overline{x})\} \cap T(\overline{x})$, so we have $\overline{x} \in P(\overline{x})$. □

Now we shall use the above Theorem 7 to get the following fixed point theorem.

THEOREM 8. *Let* X *be a nonempty L-convex space such that for each finite subset* N *of* X *with* $|N| = n+1$,*there exists a continuous mapping* $\psi_N : \Delta_n \to X$, *and let* $S, T : X \to 2^X$ *be two set-valued functions. Assume that* *(i)* *for each* $x \in X$, $S(x) \subset T(x)$ *and for each* $N \in \langle S(x) \rangle$ *with* $|N| = n+1$, *implies* $\psi_N(\Delta_n) \subset T(x)$,

(ii) $X = \cup\{int_X S^{-1}(y) : y \in X\}$,

(iii) *there exists a nonempty subset* B_0 *of* X *such that* B_0 *is contained in a compact* L-*convex subset* B_1 *of* X *and let the set* $D = \cap\{X \backslash int_X S^{-1}(y) : y \in B_0\}$ *is either empty or compact, and*

(iv) *for each* $M \in \langle X \rangle$, *there exists a nonempty compact* L-*convex set* A *such that* $A \supset L - co(B_1 \cup M)$.

Then T *has a fixed point in* X.

Proof. First, we assume that $D = \phi$. Define a multifunction $F : B_1 \to 2^{B_1}$ by $F(x) = S(x) \cap B_1$ for all $x \in B_1$. Then for each $x \in B_1$, $F(x)$ is nonempty. Indeed, suppose that $F(\overline{x})$ is empty for some $\overline{x} \in B_1$, then $S(\overline{x}) \cap B_1 = \phi$. This implies that for all $x \in B_1$, $x \notin S(\overline{x})$, and $\overline{x} \notin S^{-1}(x) \supset int_X S^{-1}(x)$. So $\overline{x} \in X \backslash int_X S^{-1}(x)$ for all $x \in B_1$, and we get a contraction, since D is empty. Moreover, we have

(1), for all $x \in B_1$, $N \in \langle F(x) \rangle$ with $|N| = n+1$, then $N \in \langle S(x) \rangle$, this implies $\psi_N(\Delta_n) \subset T(x)$.

(2), by the assumption (ii) and $D = \phi$, we have $X = \cup\{int_X S^{-1}(y) : y \in B_0\}$, and hence $X = \cup\{int_X S^{-1}(y) : y \in B_1\}$. Thus, for each $y \in B_1$, $F^{-1}(y) = S^{-1}(y) \cap B_1$ and $int_X S^{-1}(y) \cap B_1 \subset int_{B_1}(S^{-1}(y) \cap B_1)$, we have $\cup_{y \in B_1}\{int_{B_1} F^{-1}(y)\} = \cup_{y \in B_1}\{int_{B_1}(S^{-1}(y)) \cap B_1\} \supset \cup_{y \in B_1}\{int_X(S^{-1}(y)) \cap B_1\} = X \cap B_1 = B_1$.Therefore, $\cup_{y \in B_1}\{int_{B_1} F^{-1}(y)\} = B_1$.

Now,by Theorem 7 and (1),(2), there exists $x_0 \in B_1$ such that $x_0 \in T(x_0)$.

Next,for the case that $D \neq \phi$, on the contrary, we assume that T has no fixed point. We divide the remaining proof into four parts.

Part 1. Claim that $X \backslash int_X S^{-1}(y) \neq \phi$ for all $y \in X$.

Suppose that $X \backslash int_X S^{-1}(y) = \phi$ for some $y \in X$, then $y \notin X \backslash int_X S^{-1}(y)$, $y \in int_X S^{-1}(y) \subset S^{-1}(y)$, and hence $y \in S(y) \subset T(y)$. This implies that y is a fixed point of T, a contradiction.

Part 2. Claim that for each $N \in \langle X \rangle$ with $|N| = n+1$, $\psi_N(\Delta_n) \subset \cup_{y \in N}\{X \backslash int_X S^{-1}(y)\}$.

Suppose that there exists $M = \{y_0, y_1, ..., y_m\} \in \langle X \rangle$ with $|M| = m+1$ such that $\psi_M(\Delta_m) \nsubseteq \cup_{i=1}^{m}\{X \backslash int_X S^{-1}(y_i)\}$. Choose $y \in \psi_M(\Delta_m)$ and $y \notin X \backslash int_X S^{-1}(y_i)$ for all $i = 0, 1, 2, ...m$. Then $y \in int_X S^{-1}(y_i) \subset S^{-1}(y)$ for all $i = 0, 1, 2, ...m$, and hence $y_i \in S(y)$ for all $i = 0, 1, 2, ...m$. Therefore, $M \in \langle S(y) \rangle$, this implies $y \in \psi_M(\Delta_m) \subset T(y)$. So y is a fixed point of T, a contradiction.

Part 3. Claim that $\cap_{y \in A}\{X \backslash int_X S^{-1}(y)\} \neq \phi$, where $A \supset L - co(B_1 \cup \{y_0, y_1, ..., y_n\})$, $\{y_0, y_1, ..., y_n\}$ is a finite subset of X, A is compact L-convex set which satisfies the assumptiom (iv).

Suppose $\cap_{y \in A}\{X \backslash int_X S^{-1}(y)\} = \phi$. Define $R : A \to 2^A$ by $R(x) = \{y \in A : x \notin X \backslash int_X S^{-1}(y)\}$ for all $x \in A$. Then $R(x)$ is nonempty. For $y \in A$, $R^{-1}(y) = \{x \in A : y \in R(x)\} = \{x \in A : x \notin X \backslash int_X S^{-1}(y)\} = \{x \in A : x \in int_X S^{-1}(y)\} = int_X S^{-1}(y)\} \cap A$. We now define the other set-valued map $P : A \to 2^A$ by $P(x) = \cup\{\psi_M(\Delta_m) : M \in \langle R(x) \rangle, |M| = m+1\}$ for all $x \in A$. We claim that $A = \cup_{y \in A}\{int_A R^{-1}(y)\}$. Indeed, since $\cap_{y \in A}\{X \backslash int_X S^{-1}(y)\} = \phi$, we have $\cup_{y \in A}\{int_A S^{-1}(y)\} = X$. Hence $A \supset \cup_{y \in A}\{int_A R^{-1}(y)\} \supset \cup_{y \in A}\{int_X S^{-1}(y) \cap A\} = X \cap A = A$. By Theorem 7 , there exists $x_0 \in A$ such that $x_0 \in P(x_0) = \cup\{\psi_M(\Delta_m) : M \in \langle R(x_0) \rangle, |M| = m+1\}$. This implies that there exists $M = \{y_0, y_1, ..., y_m\} \in \langle R(x_0) \rangle$ such that $x_0 \in \psi_M(\Delta_m)$. Hence $x_0 \in R^{-1}(y_i)$ for all $i = 0, 1, 2, ..., m$, $x_0 \notin X \backslash int_X S^{-1}(y_i)$ for all $i = 0, 1, 2, ..., m$, that is; $x_0 \in int_X S^{-1}(y_i) \subset S^{-1}(y_i)$ for all $i = 0, 1, 2, ..., m$. Thus, $y_i \in S(x_0)$ for all $i =$

$0, 1, 2, ..., m$, $M \in \langle S(x_0) \rangle$, this implies $\psi_M(\Delta_m) \subset T(x_0)$, and then $x_0 \in \psi_M(\Delta_m) \subset T(x_0)$. This shows that x_0 is a fixed point of T, a contradiction.

Part 4. By Claim(3), for each finite subset $\{y_0, y_1, ..., y_m\}$ of X, we have $B_1 \cup \{y_0, y_1, ..., y_m\} \subset A$ and $D \cap (\cap_{i=1}^m \{X \backslash int_X S^{-1}(y_i)\}) = (\cap_{y \in B_0} \{X \backslash int_X S^{-1}(y)\}) \cap (\cap_{i=1}^m \{X \backslash int_X S^{-1}(y_i)\}) \supset \cap_{y \in A} \{X \backslash int_X S^{-1}(y)\} \neq \phi$. Since D is compact and $\{X \backslash int_X S^{-1}(y)\}$ is closed, $\{X \backslash int_X S^{-1}(y) \cap D\}$ is compact for each $y \in X$. Hence $\cap_{y \in X} \{X \backslash int_X S^{-1}(y)\} \neq \phi$. This contradicts to our condition (ii). Therefore, T has a fixed point in X. □

We obtain the following generalization of the coincidence theorems for Park and Kim[10].

THEOREM 9. *Let X be a nonempty L-convex space, Y a Hausdorff space such that for each finite subset N of X with $|N| = n+1$, there exists a continuous mapping $\psi_N : \Delta_n \to X$, which has $\psi_N(\Delta_{|J|-1}) = \psi_J(\Delta_{|J|-1})$ for any $J \in \langle N \rangle$, and let $S, T : X \to 2^Y$, $F \in R_{\psi_N} - KKM(X, Y)$ be three set-valued functions. Assume that*

(i) $\overline{F(D)}$ is compact and $F|_D$ is closed for any nonempty compact subset D of X,

(ii) for each $x \in X$, $S(x)$ is compactly open in Y,

(iii) for each $y \in F(X)$, $N \in \langle S^{-1}(y) \rangle$ with $|N| = n+1$, implies $\psi_N(\Delta_n) \subset T^{-1}(y)$,

(iv) there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(X)$, and

(v) either

(1)$Y \backslash K \subset S(M)$ for some $M \in \langle X \rangle$, or

(2) for each $N \in \langle X \rangle$, there exists a compact L-convex subset B_N of X containing N such that $F(B_N) \backslash K \subset S(B_N)$.

Then there exists $\overline{x} \in X$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \phi$.

Proof. From (iv), we have $\overline{F(X)} \cap K = \cup_{x \in X} (S(x) \cap \overline{F(X)} \cap K)$, and by (ii), $S(x) \cap \overline{F(X)} \cap K$ is open in $\overline{F(X)} \cap K$ for each $x \in D$. Since $\overline{F(X)} \cap K$ is compact, there exists $A \in \langle X \rangle$ such that

$$(*) \qquad \overline{F(X)} \cap K \subset S(A).$$

First, we consider the case (1) of (v), $Y \backslash K \subset S(M)$ for some $M \in \langle X \rangle$, then there exists $N = M \cup A = \{x_0, x_1, ..., x_n\} \in \langle X \rangle$ with $|N| = n+1$ such that $\overline{F(X)} \subset S(N)$.

Let $\{\beta_i\}_{i=0}^n$ be a partition of unity subordinate to the open covering $\{Sx_i \cap \overline{F(\psi_N(\Delta_n))}\}_{i=1}^n$ of $\overline{F(\psi_N(\Delta_n))}$. Define $f : \overline{F(\psi_N(\Delta_n))} \to \Delta_n$ by $f(y) = \sum_{i=0}^n \beta_i(y) e_i = \sum_{i \in I(y)} \beta_i(y) e_i$ where $I(y) = \{i \in \{0, 1, 2, ..., n\} : \beta_i(y) \neq 0\}$. And, we let $J = \{x_i : i \in I(y)\} \subset N$. Then $J \in \langle S^{-1}(y) \rangle$. (since $i \in I(y)$, we have $\beta_i(y) \neq 0$, and hence $y \in Sx_i$, $x_i \in S^{-1}(y)$. Thus, by (iii), $\psi_N(\Delta_{|J|-1}) = \psi_J(\Delta_{|J|-1}) \subset T^{-1}(y)$ for all $y \in F(\psi_N(\Delta_n))$. This implies that $\psi_N(f(y)) \in \psi_N(\Delta_{|J|-1}) \subset T^{-1}(y)$ for all $y \in F(\psi_N(\Delta_n))$, that is; $y \in T(\psi_N(f(y)))$ for all $y \in F(\psi_N(\Delta_n))$. Now, since $F \in R_{\psi_N} - KKM(X, Y)$ and for this ψ_N, it is clear that $F\psi_N \in KKM(\Delta_n, Y)$. From theorem of Chang and Yen[2], we have $fF\psi_N \in KKM(\Delta_n, \Delta_n)$. It is obvious that $fF\psi_N$ is compact. And by (i), $F|_{\psi_N(\Delta_n)}$ is closed, $fF|_{\psi_N(\Delta_n)}$ is closed, and so $fF\psi_N$ is closed on Δ_n. Apply Theorem 2[3], we conclude that $fF\psi_N$ has a fixed point ξ in Δ_n, that is $\xi \in fF\psi_N(\xi)$. Put $\overline{x} = \psi_N(\xi)$. Then $f^{-1}(\xi) \cap F(\overline{x}) \neq \phi$. For $y \in f^{-1}(\xi) \cap F(\overline{x})$, we have $y \in F(\psi_N(\xi)) \subset F(\psi_N(\Delta_n))$ and $\psi_N(f(y)) = \psi_N(\xi) = \overline{x}$. So, $y \in T\psi_N f(y) = T(\overline{x})$. This implies that $\phi \neq f^{-1}(\xi) \cap F(\overline{x}) \subset T(\overline{x})$.Therefore, $F(\overline{x}) \cap T(\overline{x}) \neq \phi$.

Next, consider the case (2) of (v) and for this A of $(*)$, there exists a compact L-convex subset B_A of X containing A such that $F(B_A) \backslash K \subset S(B_A)$. By $(*)$, we have $\overline{F(B_A)} \cap K \subset \overline{F(X)} \cap K \subset S(A) \subset S(B_A)$. Thus $\overline{F(B_A)} \subset S(B_A)$. Since B_A is compact, by (i), $\overline{F(B_A)}$ is compact . Therefore, $F(B_A) \subset \overline{F(B_A)} \subset S(N)$ for some $N = \{x_0, x_1, ..., x_n\} \in \langle B_A \rangle$. With B_A in place of X, we can just follow the procedure of the previous paragraph to obtain the conclusion that $F(\overline{x}) \cap T(\overline{x}) \neq \phi$. This completes the proof. □

We deduce the following $R_{\psi_N} - KKM$-type theorem for L-convex spaces.

THEOREM 10. *Let X be a nonempty L-convex space, Y a Hausdorff space such that for each finite subset N of X with $|N| = n+1$, there exists a continuous mapping $\psi_N : \Delta_n \to X$, which has $\psi_N(\Delta_{|J|-1}) = \psi_J(\Delta_{|J|-1})$ for any $J \in \langle N \rangle$ and let $P : X \to 2^Y$, $F \in R_{\psi_N} - KKM(X, Y)$ be two set-valued functions. Assume that*

(i) $\overline{F(D)}$ is compact and $F|_D$ is closed for any nonempty compact subset D of X,

(ii) for each $x \in X$, $P(x)$ is compactly closed in Y,

(iii) there exists a nonempty compact subset K of Y such that either

(1) $\cap\{P(x) : x \in M\} \subset K$ for some $M \in \langle X \rangle$, or

(2) for each $N \in \langle X \rangle$, there exists a compact L-convex subset B_N of X containing N such that $F(B_N) \cap (\cap\{P(x) : x \in B_N\}) \subset K$,

(iv) for any $N \in \langle X \rangle$ with $|N| = n+1$, $F(\psi_N(\Delta_n)) \subset P(N)$.

Then $\overline{F(X)} \cap K \cap (\cap\{P(x) : x \in X\}) \neq \phi$.

Proof. Define $S : X \to 2^Y$ by $S(x) = Y \backslash P(x)$ for all $x \in X$. By (ii), $S(x)$ is compactly open for each $x \in X$. On the contrary, we assume that $\overline{F(X)} \cap K \subset S(X)$. We next define $H : Y \to 2^X$ and $T : X \to 2^Y$ by $H(y) = \cup\{\psi_M(\Delta_m) : M \in \langle S^{-1}(y)\rangle, |M| = m+1\}$ for $y \in Y$, and $T(x) = H^{-1}(x)$for all $x \in X$. Then for any $y \in F(X)$ and $M \in \langle S^{-1}(y) \rangle$, we have that $\psi_M(\Delta_m) \subset T^{-1}(y)$ for any $M \in \langle S^{-1}(y) \rangle$. (since, if $x \in \psi_M(\Delta_m)$ where $M \in \langle S^{-1}(y)\rangle$, then $x \in H(y)$, $y \in H^{-1}(x) = T(x)$, and hence $x \in T^{-1}(y)$). From (1) of (iii), $Y \backslash K \subset \cup_{x \in M}(Y \backslash P(x)) = \cup_{x\in M} S(x) = S(M)$. And, from (2) of (iii), $F(B_N)\backslash K \subset (F(B_N)\backslash(\cap\{P(x) : x \in B_N\}) \subset (\cup\{Y \backslash P(x) : x \in B_N\}) = S(B_N)$. So all of the requirements of Theorem 9 are satisfied, and hence there exists an $\overline{x} \in X$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \phi$. Choose $\overline{y} \in F(\overline{x}) \cap T(\overline{x})$, then $\overline{x} \in T^{-1}(\overline{y}) = \cup_{M \in \langle S^{-1}(\overline{y})\rangle} \psi_M(\Delta_m)$. Hence, there exists $M \in \langle S^{-1}(\overline{y})\rangle$ such that $\overline{x} \in \psi_M(\Delta_m)$. Noting that if $M \in \langle S^{-1}(\overline{y})\rangle$, then for any $x \in M$, $\overline{y} \in S(x)$, and hence $\overline{y} \in F(\overline{x}) \cap (\cap\{Sx : x \in M\} \subset F(\psi_M(\Delta_m)) \cap (\cap\{Sx : x \in M\}) = F(\psi_M(\Delta_m)) \cap S(M)$. This implies that $F(\psi_M(\Delta_m)) \nsubseteq P(M)$, a contradiction to (iv). Therefore, $\overline{F(X)} \cap K \cap (\cap\{P(x) : x \in X\}) \neq \phi$. We complete the proof. □

3 MATCHING THEOREMS

As a consequence of the Theorem 4, we obtain the following matching theorem.

THEOREM 11. *Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N| = n+1$, there exists a continuous mapping $\psi_N : \Delta_n \to X$. Let $T, F : X \to 2^Y$ be two set-valued maps satisfying the following:*

(i) $T \in R_{\psi_N} - KKM(X, Y)$, and

(ii) $\overline{T(X)} \subset \{intF(x) : x \in A\}$ for some $A \in \langle X \rangle$.

Then there exists an $M \in \langle X \rangle$ with $|M| = m+1$ such that $T(\psi_M(\Delta_m)) \cap (\cap_{x \in M} F(x)) \neq \phi$

Proof. For each $y \in Y$, we let

$$P^{-1}(y) = \cup\{\psi_N(\Delta_n) : N \in \langle F^{-1}(y)\rangle, |N| = n+1\}.$$

Then $P : X \to 2^Y$ and for each $y \in T(X)$, $N \in \langle F^{-1}(y)\rangle$ with $|N| = n+1$, implies $\psi_N(\Delta_n) \subset P^{-1}(y)$. By Theorem 4, there exists a coincidence point $\overline{x} \in X$ of T and P. Choose $\overline{y} \in T(\overline{x}) \cap P(\overline{x})$, and so $\overline{x} \in P^{-1}(\overline{y})$. Thus, there exists an $M = \{z_0, z_1, z_2, ..., z_m\} \in \langle F^{-1}(\overline{y})\rangle$ such that $\overline{x} \in \psi_M(\Delta_m)$. This implies that $\overline{y} \in F(z_i)$ for all $i = 0, 1, 2, ..., m$. Hence, $\overline{y} \in T(\overline{x}) \cap (\cap_{i=0}^m F(z_i)) \subset T(\psi_M(\Delta_m)) \cap (\cap_{x\in M} F(x)) \neq \phi$. This completes the proof. □

THEOREM 12. *Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N| = n+1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. Let $T, F : X \to 2^Y$ be two set-valued maps satisfying the following:*

(i) $T \in R - KKM(X,Y)$ is compact,
(ii) for each $x \in X$, $F(x)$ is open, and
(iii) $\overline{TX} \subset F(X)$.

Then there exists an $M \in \langle X\rangle$ with $|M| = m+1$ such that $T(\psi_M(\Delta_m)) \cap (\cap_{x\in M} F(x)) \neq \phi$.

Proof. Since $F(x)$ is open, we have $\overline{TX} \subset F(X) = \cup\{intF(x) : x \in X\}$. And, since T is compact, there exists $N \in \langle X\rangle$ such that $\overline{TX} \subset \cup\{intF(x) : x \in N\}$. Applying the Theorem 11, we complete the proof. □

THEOREM 13. *Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N| = n+1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. Let $T, F : X \to 2^Y$ be two set-valued maps satisfying the following:*

(i) $T \in R_{\psi_N} - KKM(X,Y)$ is compact,
(ii) for each $x \in X$, $F(x)$ is closed in Y, and
(iii) for any $N \in \langle X\rangle$ with $|N| = n+1$, $T(\psi_N(\Delta_n)) \subset F(N)$.

Then $\overline{T(X)} \cap (\cap_{x\in X} F(x)) \neq \phi$.

Proof. On the contrary, we assume that $\overline{T(X)} \cap (\cap_{x\in X} F(x)) = \phi$. Define $P : X \to 2^Y$ by $P(x) = Y \backslash F(x)$ for all $x \in X$. Then $\overline{T(X)} \subset P(X)$. By Theorem 12, there exists an $M \in \langle X\rangle$ with $|M| = m+1$, $T(\psi_M(\Delta_m)) \nsubseteq F(M)$, a contraction. This completes the proof □

We now establish the following $R_{\psi_N} - KKM$-type theorem, which is equivalent to the Ky Fan matching Theorem 15.

THEOREM 14. *Let X be a L-convex space and Y a Hausdroff space such that for each finite subset N of X with $|N| = n+1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. If $T : X \to 2^Y$,$F : X \to 2^Y$ are two set-valued functions satisfies the following:*

(i) $T \in R_{\psi_N} - KKM(X,Y)$ is compact,
(ii) for each $x \in X$, $F(x)$ is compactly closed in Y, and
(iii) F is a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N,

then $\overline{T(X)} \cap (\cap\{F(x)|x \in X\}) \neq \phi$.

Proof. Since T is compact, $\overline{T(X)}$ is a compact subset of Y. Define $H : X \to 2^Y$ by

$$H(x) = \overline{T(X)} \cap F(x) \qquad \text{for each} \quad x \in X.$$

Then we have

(1) $H(x)$ is compact closed for each $x \in X$, and

(2) H is a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N.

To prove (2), since F is a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N, hence for all $N \in \langle X \rangle$ with $|N| = n+1$, $T(\psi_N(\Delta_n)) \subset F(N)$, so we have

$$T(\psi_N(\Delta_n)) \subset \overline{T(X)} \cap T(\psi_N(\Delta_n)) \subset \overline{T(X)} \cap F(N) = H(N)$$

and so H is a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N. Since $T \in R_{\psi_N} - KKM(X,Y)$, hence the family $\{H(x)|x \in X\}$ has finite intersection property, since $H(x)$ is closed in a compact set for each x in X, so $\cap_{x \in X} H(x) \neq \phi$. That is, $\overline{T(X)} \cap (\cap\{F(x)|x \in X\}) \neq \phi$. □

As a consequence of the above theorem 14, we obtain the following generalization of the Ky Fan matching theorem.

THEOREM 15. *Let X be a L-convex space and Y a Hausdorff space such that for each finite subset N of X with $|N| = n+1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. If $T : X \to 2^Y$,$H : X \to 2^Y$ are two set-valued functions satisfies the following:*

(i) $T \in R_{\psi_N} - KKM(X,Y)$ *is compact,*
(ii) *For each $x \in X$, $H(x)$ is compactly open in Y, and*
(iii) $\overline{T(X)} \subset H(X)$,

then there exists $M \in \langle X \rangle$ with $|M| = m+1$ such that

$$T(\psi_M(\Delta_m)) \cap (\cap\{H(x)|x \in M\}) \neq \phi.$$

Proof. On the contrary, assume that $T(\psi_M(\Delta_m)) \cap (\cap\{H(x)|x \in M\}) = \phi$ for any $M \in \langle X \rangle$, then $T(\psi_M(\Delta_m)) \subset \cup_{x \in M} H^c(x)$. Define $F : X \to 2^Y$ by $F(x) = H^c(x)$ for each $x \in X$. Then F is a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N such that $F(x)$ is compactly closed for all $x \in X$. We see that all of the requirements in Theorem 14 are satisfied by T and H^c. Thus $\overline{T(X)} \cap (\cap\{F(x)|x \in X\}) \neq \phi$. This implies that $\overline{T(X)} \not\subset F^c(X) = H(X)$ a contradiction. So we complete the proof. □

In the above proof, we apply Theorem 14 to prove Theorem 15. In fact, these two theorems are equivalent. For this, suppose that the conclusion of theorem 14 is false. Then

$$\overline{T(X)} \cap (\cap\{F(x)|x \in X\}) = \phi,$$

which implies

$$\overline{T(X)} \subset (\cup\{F^c(x)|x \in X\}).$$

Define $H : X \to 2^Y$ by $H(x) = F^c(x)$ for all x in X, then $H(x)$ is compactly open in Z for each $x \in X$ and $\overline{T(X)} \subset H(X)$. By Theorem 15, there exists $M \in \langle X \rangle$ with $|M| = m+1$ such that

$$T(\psi_M(\Delta_m)) \cap (\cap\{H(x)|x \in M\}) \neq \phi,$$

and so

$$T(\psi_M(\Delta_m)) \not\subset \cup_{x \in M} H^c(x) = \cup_{x \in M} F(x),$$

that is, F is not a generalized $R_{\psi_N} - KKM$ mapping with respect to T and ψ_N. Which contradicts the condition (iii).

In order to establish the Corollary 17, we need to have the following lemma and applying the Theorem 1.

LEMMA 16. *Let X be a L-convex space, Y a topological space such that for each finite subset N of X with $|N| = n+1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. Suppose $f \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ is a single-valued function, then $f \in R_{\psi_N} - KKM(X, Y)$.*

Proof. Let $F : X \to 2^Y$ be any generalized $R_{\psi_N} - KKM$ mapping with respect to f such that $F(x)$ be closed for each $x \in X$. It suffices to show that the family $\{F(x)|x \in X\}$ has the finite intersection property. Since $f(\psi_N(\Delta_n)) \subset F(N)$ for all $N \in \langle X\rangle$ with $|N| = n+1$, which implies $\psi_N(\Delta_n) \subset f^{-1}F(N)$, and we have $f^{-1}F(x)$ is closed for each $x \in X$. By using Theorem 1 we have $\{f^{-1}F(x)|x \in X\}$ has the finite intersection property, and so does the family $\{F(x)|x \in X\}$. We have finished the proof. □

The following corollary is due to Park[12], which in turn is a generalization of Lemma 1 of Ky Fan[5].

COROLLARY 17. *Let X be a compact L-convex space, Y a Hausdorff space such that for each finite subset N of X with $|N| = n+1$,there exists a continuous mapping $\psi_N : \Delta_n \to X$. If $A : X \to 2^Y$ is a set-valued function satisfies the following:*

(i) For each $x \in X$, $A(x)$ is compactly open in Y, and

(ii) $A(X) = Y$,

then for all $f \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ there exists a $N = \{x_0, x_1, \dots, x_n\} \in X$ with $|N| = n+1$ and $\overline{x} \in \psi_N(\Delta_n)$ such that $f(\overline{x}) \in \cap_{i=0}^n A(x_i)$.

Proof. By Lemma 16 , we have $f \in R_{\psi_N} - KKM(X, Y)$. Since X is compact, $f(X)$ is also compact and $f(X) \subset Y$. Furthermore, by condition (ii), we have that

$$\overline{f(X)} \subset Y = A(X).$$

So all of the requirements of Theorem 15 are satisfied by f and A. Thus, there exists a finite subset $N = \{x_0, x_1, \dots, x_n\}$ of X with $|N| = n+1$ such that

$$f(\psi_N(\Delta_n)) \cap (\cap_{i=0}^n A(x_i)) \neq \phi.$$

That is, there exists $\overline{x} \in \psi_N(\Delta_n)$ such that $f(\overline{x}) \in \cap_{i=0}^n A(x_i)$. Thus we have completed the proof. □

4 MINIMAX INEQUATIES

Now, we apply the results of the theorem 2 to establish some minimax inequalities, which generalize some minimax results[8] in pseudoconvex space and H-space.

THEOREM 18. *Let X be a compact L-convex space and $f, g : X \times X \to \Re$ two functions such that the following assumptions hold:*

(i) $g(x, y) \le f(x, y)$ for all $(x, y) \in X \times X$,

(ii) $g(x, y)$ is lower semicontinuous in its second variable, that is; the set $\{y \in X : g(x, y) \le \alpha, \alpha \in \Re\}$ is closed, and

(iii) $f(x, y)$ is L-quasiconcave in first variable, that is; the set $\{x \in X : f(x, y) > \alpha\}$ is L-convex.

Then one of the following alternatives holds:

(I) there exists an elements $y_0 \in X$ *such that* $g(x, y_0) \leq \alpha$ *for all* $x \in X$

(II) there exists an elements $x_0 \in X$ *such that* $f(x_0, x_0) > \alpha$.

Proof. We define two set-valued mappings $F, T : X \to 2^X$ by $F(x) = \{y \in X : f(x, y) \leq \alpha, \alpha \in \Re\}$ and $T(x) = \{y \in X : g(x, y) \leq \alpha, \alpha \in \Re\}$, respectively. Then, by (i), we have $F(x) \subset T(x)$. First, we consider the case that if there is an $x_0 \in X$ such that $x_0 \notin Fx_0$, then $f(x_0, x_0) > \alpha$, which implies (II).

Next, we consider the other case that if $x \in F(x)$ for all $x \in X$. Claim that $F^*y = \{x \in X : f(x, y) > \alpha\}$ is L-convex for all $y \in X$. Let $N \in \langle X\rangle$ and $N \subset F^*y$. Then, by (iii), $\Gamma(N) \subset F^*y$. Now, by Theorem 2, we have $\cap_{x\in X} Tx \neq \phi$, which implies (I). □

From above Theorem 18, we have the following corollary.

COROLARY 19. *Under the assumptions of Theorem 18, we have*
$inf_{y\in X} sup_{x\in X} g(x, y) \leq sup_{x\in X} f(x, x)$.

From Theorem 4, we establish an abstract variational inequality.

THEOREM 20. *Let* X *be a nonempty* L*-convex space such that for each finite subset* N *of* X *with* $|N| = n+1$*, there exists a continuous mapping* $\psi_N : \Delta_n \to X$*, and let* $f : X \to \overline{\Re}$ *and* $g, h : X \times X \to \overline{\Re}$. *Assume that*

(i) $T \in R_{\psi_N} - KKM(X, X)$ *is compact,*

(ii) $h(x, y) \leqslant g(x, y)$ *for all* $(x, y) \in X \times X$,

(iii) for all $x \in X$ *and for all* $y \in T(x)$, $g(x, y) + f(y) \leqslant f(x)$,

(iv) for each $y \in T(X)$, $N \in \langle M_y\rangle$ *with* $|N| = n + 1$*, implies* $\psi_N(\Delta_n) \subset M_y$*, where* $M_y = \{x \in X : g(x, y) + f(y) > f(x)\}$*, and*

(v) for each $x \in X$*, the set* $\{y \in X : h(x, y) + f(y) > f(x)\}$ *is open.*

Then there exists a $\overline{y} \in \overline{T(X)}$ *such that*

$$h(x, \overline{y}) + f(\overline{y}) \leqslant f(x) \quad \textit{for all} \quad x \in X.$$

Proof. Define $F, G : X \to 2^X$ by $F(x) = \{y \in X : g(x, y) + f(y) > f(x)\}$ and $G(x) = \{y \in X : h(x, y) + f(y) > f(x)\}$. From (i), $G^{-1}(y) \subset F^{-1}(y)$ for all $y \in X$. By (iii), it is clear that for each $y \in T(X)$, $N \in \langle G^{-1}(y)\rangle$ with $|N| = n + 1$, implies $\psi_N(\Delta_n) \subset F^{-1}(y)$.

We now claim that $\overline{T(X)} \nsubseteq \cup\{intG(x) : x \in X\}$. On the contrary, we assume that $\overline{T(X)} \subset \cup\{intG(x) : x \in X\}$. Since T is compact, there exists a finite subset M of X such that $\overline{T(X)} \subset \cup\{intG(x) : x \in M\}$. Applying Theorem 4, there exists a $\overline{x} \in X$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \phi$. Choose $\overline{y} \in F(\overline{x}) \cap T(\overline{x})$. Then $\overline{y} \in T(\overline{x})$ and $g(\overline{x}, \overline{y}) + f(\overline{y}) > f(\overline{x})$, which contradicts (ii).

Therefore, there exists a $\overline{y} \in \overline{T(X)}$ such that $\overline{y} \notin \cup\{intG(x) : x \in X\}$. This implies that there exists a $\overline{y} \in \overline{T(X)}$ such that $h(x, \overline{y}) + f(\overline{y}) \leqslant f(x)$ for all $x \in X$. □

Next, applying Theorem 9 and we restrict the range of the continuous mapping ψ_N to be the set $\Gamma(N)$ for each $N \in \langle X\rangle$, then we obtain some results about minimax inequalities.

THEOREM 21. *Let* X *be a nonempty* L*-convex space,* Y *a Hausdorff space such that for each finite subset* N *of* X *with* $|N| = n + 1$*, there exists a continuous mapping* $\psi_N : \Delta_n \to \Gamma(N)$*,which has* $\psi_N(\Delta_{|J|-1}) = \psi_J(\Delta_{|J|-1})$ *for any* $J \in \langle N\rangle$ *and let* $F \in R_{\psi_N} - KKM(X, Y)$*. Let* A *and* B *be two nonempty subsets of a set* Z*,* K *a nonempty compact subset of* Y*, and let* $f, g : X \times Y \to Z$*. Assume that* *(i)* $\overline{F(D)}$ *is compact and* $F|_D$ *is closed for any nonempty compact subset* D *of* X*,*

(ii) for any $x \in X$*, the set* $\{y \in Y : g(x,y) \in A\}$ *is compactly open and contained in the set* $\{y \in Y : f(x,y) \in B\}$*,*

(iii) for any $y \in F(X)$*, the set* $\{x \in X : f(x,y) \in B\}$ *is L-convex, and*

(iv) either

(1) there exists an $M \in \langle X\rangle$ *such that for each* $y \in Y\backslash K$*,* $g(x,y) \in A$ *for some* $x \in M$*, or*

(2) for any $N \in \langle X\rangle$*, there exists a compact L-convex subset* B_N *of* X *containing* N *such that for any* $y \in F(B_N)\backslash K$*, there exists* $x \in B_N$ *such that* $g(x,y) \in A$*.*

Then either

(I) there exists $\overline{y} \in \overline{F(X)} \cap K$ *such that* $g(x,\overline{y}) \notin A$ *for all* $x \in X$*, or*

(II) there exists $\overline{x} \in X$ *and* $\overline{y} \in F(\overline{x})$ *such that* $f(\overline{x},\overline{y}) \in B$*.*

Proof. Define $S,T : X \to 2^Y$ by $S(x) = \{y \in Y : g(x,y) \in A\}$ for all $x \in X$ and $T(x) = \{y \in Y : f(x,y) \in B\}$ for all $x \in X$. Then, by (ii), $S(x)$ is compactly open for all $x \in X$. By (iii), $T^{-1}(y)$ is L-convex for each $y \in F(X)$. Moreover, for any $M \in \langle S^{-1}(y)\rangle$ with $|M| = m+1$, and if $x \in M$, then $x \in S^{-1}(y)$, by (ii), $y \in S(x) \subset T(x)$. Thus $M \subset T^{-1}(y)$. Since for each $M \in \langle X\rangle$ with $|M| = m+1$, there exists a continuous mapping $\psi_M : M \to \Gamma(M)$ such that $\psi_M(\Delta_m) \subset \Gamma(M)$, and by the L-convexity of $T^{-1}(y)$, we have $\psi_M(\Delta_m) \subset \Gamma(M) \subset T^{-1}(y)$. By (iv), either there exists an $M \in \langle X\rangle$ such that $Y\backslash K \subset S(M)$, or for any $N \in \langle X\rangle$, there exists a compact L-subset B_N of X containing N such that $F(B_N)\backslash K \subset S(B_N)$. Least, if we assume that (I) does not hold, then for any $y \in \overline{F(X)} \cap K$, there exists an $x_0 \in X$ such that $g(x_0,y) \in A$, which implies that $\overline{F(X)} \cap K \subset S(X)$. Now, applying the Theorem 9, we have that there exists a $\overline{x} \in X$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \phi$. Choose $\overline{y} \in F(\overline{x}) \cap T(\overline{x})$, then $\overline{y} \in F(\overline{x})$ and $f(\overline{x},\overline{y}) \in B$. We complete the proof. □

We now put $Z = \Re$, $A = (\alpha,\infty)$ and $B = (\beta,\infty)$ in above Theorem 21, where α,β are constants. Then we get the following theorem.

THEOREM 22. *Let* X *be a nonempty L-convex space,* Y *a Hausdorff space such that for each finite subset* N *of* X *with* $|N| = n+1$*, there exists a continuous mapping* $\psi_N : \Delta_n \to \Gamma(N)$*,which has* $\psi_N(\Delta_{|J|-1}) = \psi_J(\Delta_{|J|-1})$ *for any* $J \in \langle N\rangle$ *and let* $F \in R_{\psi_N} - KKM(X,Y)$*. Let* $\alpha,\beta \in \Re$ *and* $\beta \leqslant \alpha$*,* K *a nonempty compact subset of* Y*, and let* $f,g : X \times Y \to \overline{\Re}$*. Assume that*

(i) $\overline{F(D)}$ *is compact and* $F|_D$ *is closed for any nonempty compact subset* D *of* X*,*

(ii) $g(x,y) \leqslant f(x,y)$ *for all* $(x,y) \in X \times Y$*,*

(iii) for any $x \in X$*, the set* $\{y \in Y : g(x,y) > \alpha\}$ *is compactly open,*

(iv) for any $y \in F(X)$*, the set* $\{x \in X : f(x,y) > \beta\}$ *is L-convex, and*

(v) either

(1) there exists an $M \in \langle X\rangle$ *such that for each* $y \in Y\backslash K$*,* $g(x,y) > \alpha$ *for some* $x \in M$*, or*

(2) for any $N \in \langle X\rangle$*, there exists a compact L-convex subset* B_N *of* X *containing* N *such that for any* $y \in F(B_N)\backslash K$ *there exists a* $x \in B_N$ *such that* $g(x,y) > \alpha$*.*

Then either

(I) there exists $\overline{y} \in \overline{F(X)} \cap K$ *such that* $g(x,\overline{y}) \leqslant \alpha$ *for all* $x \in X$*, or*

(II) there exists $\overline{x} \in X$ *and* $\overline{y} \in F(\overline{x})$ *such that* $f(\overline{x}, \overline{y}) > \beta$.

Further, we put $\alpha = \beta = sup\{f(x,y) : (x,y) \in \mathcal{G}_{\mathcal{F}}\}$, and by Theorem 22, we get the following generalization of the Ky Fan minimax inequality.

THEOREM 23. *Let* X *be a nonempty L-convex space,* Y *a Hausdorff space such that for each finite subset* N *of* X *with* $|N| = n+1$*, there exists a continuous mapping* $\psi_N : \Delta_n \to \Gamma(N)$*,which has* $\psi_N(\Delta_{|J|-1}) = \psi_J(\Delta_{|J|-1})$ *for any* $J \in \langle N\rangle$ *and let* $F \in R_{\psi_N} - KKM(X,Y)$*. Let* $f, g : X \times Y \to \overline{\Re}$*,* K *a nonempty compact subset of* Y*. Assume that*

- *(i)* $\overline{F(D)}$ *is compact and* $F|_D$ *is closed for any nonempty compact subset* D *of* X*,*
- *(ii)* $g(x,y) \leqslant f(x,y)$ *for all* $(x,y) \in X \times Y$*,*
- *(iii) for any* $x \in X$*, the function* $y \mapsto g(x,y)$ *is lower semicontinuous on the compact subset of* Y*, and*
- *(iv) there exists an* $M \in \langle X\rangle$ *such that for each* $y \in Y\backslash K$*,* $g(x,y) > \alpha$ *for some* $x \in M$*, where* $\alpha = sup\{f(x,y) : (x,y) \in \mathcal{G}_{\mathcal{F}}\}$.

Then, there exists a $\overline{y} \in \overline{(F)} \cap K$ *such that*

$$g(\overline{x}, \overline{y}) \leqslant sup_{(x,y)\in\mathcal{G}_{\mathcal{F}}} f(x,y) \quad \text{for all } \overline{x} \in X, \quad \text{and}$$

Moreover, we also have the minimax inequality:

$$min_{y\in K} sup_{x\in X} \leqslant sup_{(x,y)\in\mathcal{G}_{\mathcal{F}}} f(x,y).$$

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