WHEN THE HEWITT REALCOMPACTIFICATION AND THE P-COREFLECTION COMMUTE

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Abstract. If X is a Tychonoff space then its P-coreflection X_δ is a Tychonoff space that is a dense subspace of the realcompact space $(\upsilon X)_\delta$, where υX denotes the Hewitt realcompactification of X. We investigate under what conditions X_δ is C-embedded in $(\upsilon X)_\delta$, i.e. under what conditions $\upsilon(X_\delta) = (\upsilon X)_\delta$. An example shows that this can fail for the product of a compact space and a P-space. It is possible for a von Neumann regular ring A to be isomorphic to a $C(Y)$ and lie between $C(X)$ and $C(X_\delta)$ without being isomorphic to $C(X_\delta)$. This cannot occur if X is realcompact or more generally if $\upsilon(X_\delta) = (\upsilon X)_\delta$. Applications are given to the epimorphic hull of $C(X)$.

1 Introduction Throughout the symbols $\sim$ and $\cong$ will signify, respectively, homeomorphism and isomorphism. Undefined notation and terminology can be found in [Gillman & Jerison (1960)] and [Porter & Woods (1988)]. Let X denote a Tychonoff topological space (henceforth abbreviated "space"). Then X is a dense, C-embedded subspace of the realcompact space υX, its Hewitt realcompactification, and these properties determine υX uniquely as an extension of X (see [Gillman & Jerison (1960), Chapter 8] or [Porter & Woods (1988), Chapter 5]. If X is re-topologized by using its G_δ-sets (equivalently its zero-sets) as a base for a new topology, the resulting space X_δ is a Tychonoff P-space. (Recall a space is a P-space if its zero sets (equivalently its G_δ sets) are open). If j denotes the identity map on the underlying set of X, then $j : X_\delta \to X$ is a continuous bijection and the pair (X_δ, j) is the coreflection of X in the category of P-spaces and continuous maps; thus if T is another P-space and $f : T \to X$ is continuous, there is a continuous function $f^* : T \to X_\delta$ such that $f = j \circ f^*$. See [Walker (1974), chapter 10] for details.

Since X is G_δ-dense in υX (see [Porter and Woods, 5.11(b)], and since the subspace topology that X inherits from $(\upsilon X)_\delta$ is just the topology of X_δ, it follows that X_δ is a dense subspace of the realcompact space $(\upsilon X)_\delta$ (see 1.1 (b) below). The theme of this article is an investigation of when X_δ is C-embedded in $(\upsilon X)_\delta$, ie. of when $\upsilon(X_\delta) = (\upsilon X)_\delta$.

The following known results show why the equality $\upsilon(X_\delta) = (\upsilon X)_\delta$ holds for P-spaces and for realcompact spaces.

Lemma 1.1. *(a) [Gillman & Jerison (1960), 8A(4)]. If X is a P-space, then υX is a P space.*

(b) If a space X is realcompact then X_δ is also realcompact.

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We thank Professor W. Comfort for referring us to [Comfort & Retta (1985), 4.8(a)] where the history of (b) is discussed. In particular this result appears in [Frolík (1971), Theorem 4] as pointed out by Comfort and Retta. It is also implicit in [Hewitt (1950)].

Remarks 1.2.

(a) In [Levy & Rice (1981), 5.3 R(iii)] the assertion of Lemma 1.1 (b) is justified by the claim that if (Y,τ) is realcompact and if σ is a Tychonoff topology on Y containing τ, then (Y,σ) is realcompact. In fact this claim is false in general, as an examination of the proof of [Gillman & Jerison (1960), 8.17] reveals. A proof of 1.1 (such as that given in [Frolík (1971)]) is needed.

(b) "δ commutes with subspaces and finite products" in the following sense. Let X and Y be spaces. First, if X is a subspace of Y, then X viewed as a subspace of Y_δ is just X_δ. Second, the identity map on the underlying set is a homeomorphism from $X_\delta \times Y_\delta$ onto $(X\times Y)_\delta$, and we will identify $(X\times Y)_\delta$ with $X_\delta \times Y_\delta$. Each of these assertions is easily verified.

It follows from Lemma 1.1 that if X is realcompact then $(\upsilon X)_\delta = X_\delta = \upsilon(X_\delta)$, the latter equality holding because X_δ is realcompact. More generally, because every non-empty G_δ-set of υX has non-empty intersection with X (see above), it follows that X_δ is a dense subspace of the realcompact space $(\upsilon X)_\delta$. Hence $(\upsilon X)_\delta = \upsilon(X_\delta)$ (up to equivalence of extensions; see [Porter & Woods (1988), Chapter 4]) if and only if X_δ is C-embedded in $(\upsilon X)_\delta$. We formalize this.

Definition 1.3. *Let X be a Tychonoff space. We say that X is an $\upsilon\delta$-commuting space if X_δ is C-embedded in $(\upsilon X)_\delta$, and we write $\upsilon(X_\delta) = (\upsilon X)_\delta$.*

Note that since X_δ is G_δ-dense in $(\upsilon X)_\delta$ we have C-embedding if and only if X_δ is z-embedded in $(\upsilon X)_\delta$ (see [Blair & Hager (1974), 4.4].

We have seen that realcompact spaces and P-spaces are $\upsilon\delta$-commuting spaces. In [Porter & Woods (1988), 5F(7)] it is erroneously claimed that all Tychonoff spaces are $\upsilon\delta$-commuting spaces. (This error is the responsibility of Woods, not Porter). Here is a class of counterexamples; more follow later.

Proposition 1.4. *Let X be a space. Suppose that $X \subseteq T \subsetneq \upsilon X$ and that T_δ is realcompact. Then X is not a $\upsilon\delta$-commuting space.*

Proof. Clearly $X_\delta \subset T_\delta \subsetneq (\upsilon X)_\delta$. If X were a $\upsilon\delta$-commuting space then X_δ would be C-embedded in $(\upsilon X)_\delta$. It follows that T_δ would be C-embedded in $(\upsilon X)_\delta$, since if $f \in C(T_\delta)$ then the extension of $f|X_\delta$ to $(\upsilon X)_\delta$ would extend f. But T_δ is dense in the realcompact space $(\upsilon X)_\delta$, and a C-embedded realcompact subspace of $(\upsilon X)_\delta$ (such as T_δ) must be closed; see [Gillman & Jerison (1960), 8A (1)]. This is a contradiction, and the result follows. $\square$

Corollary 1.5. *(a) If X is a non-realcompact space for which X_δ is realcompact, then X is not a $\upsilon\delta$-commuting space.*

(b) A non-realcompact space X of countable pseudocharacter and non-measurable cardinality is not a $\upsilon\delta$-commuting space. Consequently a space of countable pseudocharacter and non-measurable cardinality is $\upsilon\delta$-commuting if and only if it is realcompact.

Proof. (a) Let T be X in Proposition 1.4.

(b) As X has countable pseudocharacter, X_δ is discrete and of non-measurable cardinality, and hence realcompact (see [Gillman & Jerison (1960), 12.2] (a). $\square$

Thus, for example, the space of countable ordinals is not an $\upsilon\delta$-commuting space. Neither is the space Ψ of [Gillman & Jerison (1960), 5I] although each of its points has a $\upsilon\delta$-commuting neighbourhood. We will show below that there are pseudocompact, noncompact spaces that are $\upsilon\delta$-commuting. Note that some results concerning $\upsilon\delta$-commuting spaces appear in [Levy & Rice (1981), 5.8, 5.9].

Note also (see [Hernandez-Hernandez, Ishiu (2004)]) that there are perfectly normal spaces of cardinality $\aleph_1$ that are not realcompact and hence not $\upsilon\delta$-commuting by 1.5. Thus perfect normality does not imply $\upsilon\delta$-commutativity.

Baire Sets 1.6.

(a) Recall that a *Baire set* of a space X is a member of the σ-algebra $S(X)$ of subsets of X generated by the family $Z(X)$ of zero-sets of X.

Since countable unions of clopen sets of a P-space are clopen, and since $Z(X) \subseteq Z(X_\delta)$, it follows that $S(X) \subseteq S(X_\delta) = Z(X_\delta)$, which is the family of clopen subsets of X_δ.

(b) The map $Z \to Z \cap X$ is a lattice isomorphism from $Z(\upsilon X)$ onto $Z(X)$ (see [Porter & Woods (1988), 5.11(g)]). As $S(X)$ is determined by $Z(X)$ and its order structure (under inclusion), it follows that the map $A \to A \cap X$ is an order-isomorphism from $S(\upsilon X)$ onto $S(X)$. We will denote the unique member of $S(\upsilon X)$ whose intersection with X is $A \in S(X)$ by A^*.

The following theorem appears in [Negrepontis (1967)].

Theorem 1.7. *A Baire set of a realcompact space is realcompact.*

2 Subspaces of $\upsilon\delta$-commuting spaces and subspaces which are $\upsilon\delta$-commuting

When one introduces a new topological property (such as that of being $\upsilon\delta$-commuting) it is traditional to consider whether it is preserved by certain sorts of subspaces, by products, and by direct and inverse images under certain sorts of continuous functions. We begin by investigating subspaces. Our first theorem is an analogue to 1.7.

Theorem 2.1. *A Baire set of a $\upsilon\delta$-commuting space is $\upsilon\delta$-commuting.*

Proof. Let X be a $\upsilon\delta$-commuting space and let $A \in S(X)$ (see 1.6(a)). By Lemma 1.1 and Theorem 1.7 it follows that $(A^*)_\delta$ is realcompact and contains A_δ (see 1.6(b) for notation).

We claim that A_δ is dense in $(A^*)_\delta$. To see this, note that $\{Z \cap A^* : Z \in Z(\upsilon X)\}$ is an open base for $(A^*)_\delta$. But if $Z \in Z(\upsilon X)$ and $Z \cap A^* \neq \emptyset$, then $Z \cap A^*$ is a non-empty Baire set of υX. As the map $B \to B \cap X$ is an order-isomorphism (and, in particular, one-to-one) from $S(\upsilon X)$ onto $S(X)$, it follows that $(Z \cap A^*) \cap X \neq \emptyset$, i.e. $Z \cap A \neq \emptyset$. Thus A_δ is dense in $(A^*)_\delta$ as claimed. Thus $(A^*)_\delta$ is a realcompact extension of A_δ.

Let $f \in C(A_\delta)$. As A_δ is clopen in X_δ (see 1.6(a)), f extends continuously to $f^* \in C(X_\delta)$. As X is a $\upsilon\delta$-commuting space, X_δ is C-embedded in $(\upsilon X)_\delta$, so f^* extends to $\upsilon f^* \in C((\upsilon X)_\delta)$. Then $\upsilon f^*|A^* \in C((A^*)_\delta)$, and so A_δ is C-embedded in $(A^*)_\delta$. Hence $\upsilon(A_\delta) = (A^*)_\delta$.

As A^* is realcompact, the embedding map $j : A \to A^*$ can be continuously extended to $\upsilon j : \upsilon A \to A^*$ (see [Gillman & Jerison (1960), 8.7 (I)]). It is easy to see that when regarded as a function from $(\upsilon A)_\delta$ to $(A^*)_\delta$, υj is continuous and fixes A_δ pointwise.

Now by Lemma 1.1, $(\upsilon A)_\delta$ is a realcompact extension of A_δ so there is a continuous function $g : \upsilon(A_\delta) \to (\upsilon A)_\delta$ that fixes A_δ pointwise. By the second last paragraph, g is a function from $(A^*)_\delta$ that fixes A_δ pointwise. Clearly $\upsilon j \circ g : (A^*)_\delta \to (A^*)_\delta$ fixes the dense set A_δ pointwise and hence is the identity. Similarly $g \circ \upsilon j : (\upsilon A)_\delta \to (\upsilon A)_\delta$ fixes A_δ pointwise. It follows that $g : (A^*)_\delta \to (\upsilon A)_\delta$ is a homeomorphism fixing A_δ pointwise; i.e. $g : \upsilon(A_\delta) \to (\upsilon A)_\delta$ is a homeomorphism fixing A_δ pointwise. Thus $\upsilon(A_\delta) = (\upsilon A)_\delta$ and A_δ is a $\upsilon\delta$-commuting space. □

However, any Tychonoff space of non-measurable cardinality is homeomorphic to a C-embedded closed subspace of a $\upsilon\delta$-commuting space, and so being a $\upsilon\delta$-commuting space is not, in general, a property inherited by C-embedded closed subspaces. To establish this we use a well known technique. We begin with some lemmas.

The following result is due to [Noble(1969)] and to Hager-Mrowka [Hager (1969)]. (Recall that a map is z-closed if images of zero sets are closed sets).

Lemma 2.2. *Let X and Y be spaces. Then the space $X \times Y$ is C^*-embedded in $X \times \beta Y$ if and only if the projection map $\pi_X : X \times Y \to X$ is z-closed.*

The following lemma is part of [Husek (1972), Corollary 4].

Lemma 2.3. *Let X be a space and Y a pseudocompact space, each of non-measurable cardinality. If $\pi_X : X \times Y \to X$ is z-closed, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.*

Lemma 2.4. *Let S be a space of non-measurable cardinality and let λ be an ordinal of non-measurable cardinality whose cofinality is greater than the cardinality $|S|$ of S. Then $\upsilon(S \times [0, \lambda)) = \upsilon S \times [0, \lambda]$. ($[0, \lambda)$ denotes the space of ordinals less than λ).*

Proof. Since $[0, \lambda)$ is pseudocompact, and since $\upsilon[0, \lambda) = \beta[0, \lambda) = [0, \lambda]$ by Lemmas 2.2 and 2.3 it suffices to show that $S \times [0, \lambda)$ is C^*-embedded in $S \times [0, \lambda]$. Suppose that $f \in C^*(S \times [0, \lambda))$. As λ evidently has uncountable cofinality if S is infinite, for each $s \in S$ there exists an ordinal $\delta(s) < \lambda$ and a $c(s) \in \mathbf{R}$ such that $f[\{s\} \times [\delta(s), \lambda)] = \{c(s)\}$. As the cofinality of λ is greater than $|S|$, the ordinal $\sup\{\delta(s) : s \in S\}$, which we denote by δ, is less than λ. Hence if $s \in S$ and $\alpha \in [\delta, \lambda)$, it follows that $f(s, \alpha) = c(s)$. It is now clear that if we define $F : S \times [0, \lambda] \to \mathbf{R}$ by $F|S \times [0, \lambda) = f$, and $F(s, \lambda) = c(s)$ for each $s \in S$, then F is a continuous extension of f to $S \times [0, \lambda]$. The lemma follows. □

Proposition 2.5. *Let X and T be spaces for which $X \subset T \subset \upsilon X$. If X is a $\upsilon\delta$-commuting space then T is a $\upsilon\delta$-commuting space.*

Proof. Clearly $\upsilon T = \upsilon X$ so $(\upsilon T)_\delta = (\upsilon X)_\delta = \upsilon(X_\delta)$ by hypothesis. But $X_\delta \subseteq T_\delta \subseteq (\upsilon X)_\delta = \upsilon(X_\delta)$, so $\upsilon(T_\delta) = \upsilon(X_\delta)$. Combining these equations gives $\upsilon(T_\delta) = (\upsilon T)_\delta$. □

Lemma 2.6. *Let λ be an ordinal of cofinality at least ω_2. If $g \in C([0, \lambda)_\delta)$ then there exists $\alpha(g) < \lambda$ such that g is constant on $[\alpha(g), \lambda)$.*

Proof. A straightforward generalization of the proof of [Gillman & Jerison (1960), 9L(4)] , with λ in place of ω_2, gives the result. □

Theorem 2.7. *Each Tychonoff space of non-measurable cardinality is homeomorphic to a closed C-embedded subspace of a $\upsilon\delta$-commuting space.*

Proof. Let S be Tychonoff and non-measurable, let λ be an ordinal satisfying $cof(\lambda) > \max\{\omega_2, |\upsilon S|\}$ (but non-measurable) and let $X = (\upsilon S \times [0,\lambda])\backslash((\upsilon S\backslash S) \times \{\lambda\})$. By Lemma 2.4 $\upsilon(S \times [0,\lambda)) \subseteq X \subseteq \upsilon S \times [0,\lambda]$, so by Proposition 2.5 if $\upsilon S \times [0,\lambda)$ is an $\upsilon\delta$-commuting space it will follow that X is a $\upsilon\delta$-commuting space. But obviously $S \times \{\lambda\}$ is closed in X and homeomorphic to S. Furthermore, if $f \in C(S \times \{\lambda\})$ one can clearly extend f continuously to $\upsilon S \times \{\lambda\}$ and from there to $\upsilon S \times [0,\lambda]$; the restriction of this extension to X then shows that $S \times \{\lambda\}$ is C-embedded in X.

It remains to show that $\upsilon S \times [0,\lambda)$ is a $\upsilon\delta$-commuting space. By Lemma 2.4 and 1.2 (b) we see that

$$(\upsilon(\upsilon S \times [0,\lambda)))_\delta = (\upsilon S)_\delta \times [0,\lambda]_\delta$$

while $(\upsilon S \times [0,\lambda))_\delta = (\upsilon S)_\delta \times [0,\lambda)_\delta$. So it suffices to show that $(\upsilon S)_\delta \times [0,\lambda)_\delta$ is C-embedded in $(\upsilon S)_\delta \times [0,\lambda]_\delta$.

If $f \in C((\upsilon S)_\delta \times [0,\lambda)_\delta)$ and $x \in \upsilon S$, then by Lemma 2.6 and the fact that $\lambda \geq \omega_2$, there is an ordinal $\delta(x) < \lambda$ such that $f|\{x\} \times [0,\lambda)$ is constant on $[\delta(x),\lambda)$. Let $\delta = \max\{\delta(x) : x \in \upsilon S\}$. As $\mathrm{cof}(\lambda) > |\upsilon S|$ it follows that $\delta < \lambda$. Hence for each $x \in \upsilon S$ there exists $c(x) \in \mathbf{R}$ such that if $\delta \leq \alpha < \lambda$ and $x \in \upsilon S$ then $f(\alpha, x) = c(x)$. It is now clear that if we extend f to $(\upsilon S)_\delta \times [0,\lambda]_\delta$ by defining $f(x,\lambda) = c(x)$ for each $x \in \upsilon S$, then this extension is continuous and the proof is complete. □

We cannot (as far as we know) replace "closed" by "open" in Theorem 2.7, but it is easy to see that open subspaces of $\upsilon\delta$-commuting spaces need not be $\upsilon\delta$-commuting. Each locally compact space is an open subspace of each of its compactifications, which (being realcompact) are $\upsilon\delta$-commuting spaces. But there are locally compact spaces (for example the countable ordinals) that are not $\upsilon\delta$-commuting spaces.

We now consider whether unions of $\upsilon\delta$-commuting spaces are $\upsilon\delta$-commuting.

Theorem 2.8. *Let A be a Baire set of the space X. Assume that A and $X\backslash A$ are C-embedded in A^* and $(X\backslash A)^*$ respectively. The following are equivalent.*

(a) X is a $\upsilon\delta$-commuting space.

(b) A and $X\backslash A$ are $\upsilon\delta$-commuting spaces.

Proof. (a) ⇒ (b): This is a special case of Theorem 2.1; here the "C-embedded" hypothesis is unnecessary.

(b) ⇒ (a): By hypothesis A is C-embedded in A^* so $A \subseteq A^* \subseteq \upsilon A$. Thus by Remark (b) of 1.2, $A_\delta \subseteq (A^*)_\delta \subseteq (\upsilon A)_\delta$ and $(\upsilon A)_\delta = \upsilon(A_\delta)$ by hypothesis. Thus as A_δ is C-embedded in $\upsilon(A_\delta)$, it is C-embedded in its subspace $(A^*)_\delta$. Similarly $(X\backslash A)_\delta$ is C-embedded in $((X\backslash A)^*)_\delta$. Now let $f \in C(X_\delta)$. Then $f|A_\delta$ extends continuously to $f_A \in C((A^*)_\delta)$ and $f|(X\backslash A)_\delta$ extends to $f_{X\backslash A} \in C(((X\backslash A)^*)_\delta)$. As $(\upsilon X)_\delta = (A^*)_\delta \oplus ((X\backslash A)^*)_\delta$, it follows that $f_A \cup f_{X\backslash A}$ is a continuous extension of f to $(\upsilon X)_\delta$. Thus X_δ is C-embedded in $(\upsilon X)_\delta$ and so X is an $\upsilon\delta$-commuting space. □

Corollary 2.9. *Let Z be a C-embedded zero-set of the space X. The following are equivalent.*

(a) X is an $\upsilon\delta$-commuting space.

(b) Z and $X\backslash Z$ are $\upsilon\delta$-commuting spaces.

Proof. By Theorem 2.8 it suffices to show that the cozero-set $X\backslash Z$ is C-embedded in $(X\backslash Z)^*$. By [Gillman & Jerison (1960), 8.8(b)] $cl_{\upsilon X}Z \in Z(\upsilon X)$ so $\upsilon X\backslash cl_{\upsilon X}Z$ is a cozero-set of υX that intersects X in $X\backslash Z$. Thus $(X\backslash Z)^* = \upsilon X\backslash cl_{\upsilon X}Z$. By [Gillman & Jerison (1960), 8G (1)], $X\backslash Z$ is C-embedded in $(X\backslash Z)^*$. Our result follows. □

Remark 2.10.

"C-embedded" cannot be dropped from the hypotheses of "(b) ⇒ (a)" in Corollary 2.9, as let T be a "Ψ-space" (see [Gillman & Jerison (1960), 5I]) and let $Z = T\backslash N$. (Note that by (b) of Corollary 1.5, T is not a $\upsilon\delta$-commuting space, while its discrete subspaces Z and $T\backslash Z$ are.)

Corollary 2.11. *Let X be a normal space. The following are equivalent:*

(a) X is an $\upsilon\delta$-commuting space.

(b) There exists a zero-set Z of X such that $X\backslash Z$ and Z are $\upsilon\delta$-commuting.

(c) For each zero-set Z of X, $X\backslash Z$ and Z are $\upsilon\delta$-commuting.

The following is an easy consequence of work by Blair and Hager.

Lemma 2.12. *Let X be dense and z-embedded in Y. Let $g(Y)$ be the intersection of the cozero sets of Y that contain X. Then every function in $C(X)$ extends to $g(Y)$. If Y is realcompact, then $g(Y)$ is a copy of υX.*

Proof. By [Blair & Hager (1974), 2.4] each function on X extends to a countable intersection of cozero sets of Y containing X and hence to the smaller set $g(Y)$. When Y is realcompact, so is $g(Y)$. □

Theorem 2.13. *Suppose that X is the union of z-embedded $\upsilon\delta$-commuting subspaces X_i. Suppose furthermore that the family $\{X_i\}$ is locally finite in $(\upsilon X)_\delta$ and that for each i, Y_i, the G_δ-closure of X_i in υX, is G_δ-open. Then X is $\upsilon\delta$-commuting. In particular, the free union of non-measurably many $\upsilon\delta$-commuting spaces is $\upsilon\delta$-commuting.*

Proof. We must show that X_δ is z-embedded in $(\upsilon X)_\delta$. (See the remarks following 1.3). Since X_i is z-embedded in X, and hence in υX, each Y_i is a copy of υX_i by 2.12. Since the family $\{X_i\}$ is locally finite in $(\upsilon X)_\delta$ so is the family $\{Y_i\}$. Furthermore since X_δ is dense in $(\upsilon X)_\delta$, the Y_i cover υX. Let Z be a zero set in $(\upsilon X)_\delta$ with trace A_i and complement B_i in X_i. Since X_i is $\upsilon\delta$-commuting, there are clopen sets C_i and D_i in $(Y_i)_\delta$ so that C_i has trace A_i on X_i and D_i has trace B_i on X_i. The families $\{C_i\}$ and $\{D_i\}$ are locally finite families of clopen sets of $(\upsilon X)_\delta$ so their respective unions C and D are clopen. Thus $C\backslash D$ is also clopen and its trace on X equals Z.

The assertion about the free union of non-measurably many spaces holds because the free union of non-measurably many realcompact spaces is realcompact, and υX is the free union of the υX_i if X is the free union of the X_i. □

Recall that a space X is called *almost Lindelöf* , if of any two disjoint zero-sets, at least one must be Lindelöf.

Proposition 2.14. *(a) If X is a Tychonoff space such that X_δ is almost Lindelöf , then X is $\upsilon\delta$-commuting.*

(b) Suppose that X is the union of z-embedded subspaces X_i. Suppose also that the family $\{X_i\}$ is locally finite in $(\upsilon X)_\delta$ and that for each i, Y_i, the G_δ-closure of X_i in υX is G_δ-open. Then X is $\upsilon\delta$-commuting. The free union of non-measurably many spaces, each almost Lindelöf in the δ-topology, is $\upsilon\delta$-commuting.

Proof. (a) It suffices to show that X_δ is z-embedded in $(\upsilon X)_\delta$. Suppose that A is a zero-set in X_δ and let B denote its complement, also a zero-set. One of A and B is Lindelöf. Assume without loss of generality that it is A that is Lindelöf. Since Lindelöf subspaces of P-spaces are C-embedded ([Blair & Hager (1974), 4.6]), A is C-embedded in $(\upsilon X)_\delta$. Since A is also realcompact, it is closed in $(\upsilon X)_\delta$ by [Gillman & Jerison (1960), 8A(1)]. If C is the closure of B in $(\upsilon X)_\delta$ then C is disjoint from A and so $A = (\upsilon X)_\delta \backslash C$. Thus A and C are clopen in, and hence zero-sets of, $(\upsilon X)_\delta$, and $B = C \cap X_\delta$.

(b) follows from (a) and 2.13. ☐

Remark 2.15.

The spaces of [Hrusak, Raphael, & Woods (2005), Theorem 6] satisfy the hypothesis of Proposition 2.14. They are almost Lindelöf, but not Lindelöf in the δ-topology. Another example with additional properties is given in [Levy & Rice (1981), Example 4].

Note that if X_δ is almost Lindelof then X is also, but the converse fails. In fact " X_δ is almost Lindelof" cannot be replaced by "X is almost Lindelof" in 2.14; ω_1 is a counterexample.

We now consider other ways of generating $\upsilon\delta$-commuting spaces.

Theorem 2.16. *Let X be a $\upsilon\delta$-commuting dense z-embedded subspace of the realcompact space Y. Let $g(Y)$ be the copy of the Hewitt realcompactification υX in Y (See 2.12.) Suppose that $\{K_n\}$ is a countable family of subsets of Y that are closed in Y_δ and form a locally finite family in Y_δ. Then the space $W = X \cup (\bigcup K_n)$ is $\upsilon\delta$-commuting. In particular, this holds if the K_n are closed and pairwise disjoint in Y, or more generally if no point of Y lies in infinitely many K_n.*

Proof. The last claim follows from the theorem because G_δ-sets of Y are open in Y_δ.

To prove the theorem one must show that W_δ is C-embedded in $(\upsilon W)_\delta$. First we claim that $T = g(Y) \cup (\bigcup_n K_n)$ is realcompact and is a copy of υW. As K_n is closed in Y_δ, each K_n is an intersection of cozero-sets of Y (see [Blair & Hager (1974)] before 2.6). As well $g(Y)$ is an intersection of cozero-sets. It now follows from infinite distributivity that $T = g(Y) \cup (\bigcup_n K_n) = \bigcap \{V \cup (\bigcup_n U_n)\}$ where V denotes an arbitrary cozero-set of Y containing $g(Y)$ and U_n an arbitrary cozero-set containing K_n. As countable unions of cozero-sets are cozero-sets, T is the intersection of cozero-sets of the realcompact space Y and hence is realcompact.

We claim that W is C-embedded in T. This follows as X is dense in T, and given any function $f \in C(W)$, by lemma 2.12, $f|X$ extends to $g(Y)$ and also extends continuously to each point of each K_n as $f|X$ extends to W. Thus f extends to T and thus T is a copy of υW.

Now one must show that W_δ is C-embedded in T_δ. Given a function $h \in C(W_\delta)$, $h|X$ extends continuously to $(g(Y))_\delta$ because X is $\upsilon\delta$-commuting. So we have a set-theoretic extension H of h to all of T. We need to show that H is continuous. Now the space $(g(Y))_\delta$ is closed in Y_δ and if one adjoins it to the family $\{K_n\}$ the result is a locally finite family of closed sets of Y_δ. So T_δ is the union of the closed sets in the new family, and the new family is locally finite in T_δ. Now the continuity of H follows from [Gillman & Jerison (1960), 1A.3]. □

Examples 2.17.

The conditions hold if $\{K_n\}$ forms a locally finite family of closed sets of Y. They also hold if the family $\{K_n\}$ is hypothesized to be a locally finite (in Y_δ) family of Lindelöf subspaces of Y, since a Lindelöf subspace of Y is an intersection of cozero-sets of Y and hence is closed in Y_δ.

There are two other applications of the theorem.

Corollary 2.18. *Let X be dense and z-embedded in Y. Suppose that C is a countable subspace of Y. If X is $\upsilon\delta$-commuting then so is $X \cup C$.*

Proof. By replacing Y by υY if necessary, we can assume that Y is realcompact. Now invoke theorem 2.16 taking the points of C as the closed sets. □

Corollary 2.19. *Let X be dense and z-embedded in a realcompact space Y. Suppose that C is a cozero set of Y. If X is $\upsilon\delta$-commuting then so is $X \cup C$.*

Proof. Let C be the cozero-set of a function f that maps Y into the interval $[0,1]$. Then C is the union of the disjoint sets $f^{-1}(1/(n+1), 1/n]$ each of which is clopen in Y_δ. They form a locally finite family in Y_δ so theorem 2.16 applies. □

For brevity's sake we omit the proofs of the results in the remainder of this section. They follow from the properties of z-embeddings, locally finite families, P-spaces, and C-embedded subspaces. The flavour of the proofs is similar to that of 2.16.

Proposition 2.20. *Let $Y = X \cup K$, where X is an $\upsilon\delta$-commuting space and K is realcompact. Suppose further that X is z-embedded in Y and that K is C-embedded in Y. Then Y is $\upsilon\delta$-commuting.*

The third part of the next result relies on [Barr, Raphael & Woods (2005), 5.1] where it is shown that P-subspaces that induce epimorphisms in the category of rings must be z-embedded.

Corollary 2.21. *Let $Y = X \cup K$, where X is an $\upsilon\delta$-commuting space and K is realcompact and C-embedded in Y. Then Y is $\upsilon\delta$-commuting if X satisfies any of the following conditions: (i) X is almost compact, (ii) X is Lindelöf, (iii) X is a P-space whose inclusion in Y induces an epimorphism in the category of rings.*

Theorem 2.22. *Let Y be a topological space with a subspace $K = \bigcup K_i$ where each subspace K_i is $\upsilon\delta$-commuting and closed and C-embedded in Y. Suppose further that the $\{K_i\}$ form a locally finite family in υY. Let $X = Y \backslash K$ and assume that for all $f \in C(Y_\delta)$, $f|X_\delta$ extends continuously to the closure of X_δ in $(\upsilon Y)_\delta$. Then Y is $\upsilon\delta$-commuting.*

The corollary which follows will apply to the spaces constructed below in 4.12 in the case that one begins with a space which is $\upsilon\delta$-commuting (the copy of X will be closed and C-embedded and its complement will be discrete).

Corollary 2.23. *Let Y be a topological space with a family of realcompact C-embedded subspaces $\{K_i\}$ that is locally finite in υY. Assume $X = Y \setminus \bigcup K_i$ is $\upsilon\delta$-commuting. Then Y is $\upsilon\delta$-commuting. In particular, if Y has an $\upsilon\delta$-commuting subspace X such that $Y \setminus X$ is the union of a finite family $\{K_i\}$ of realcompact C-embedded subspaces of Y, then Y is $\upsilon\delta$-commuting. Thus if K is a compact subspace of a space Y and $Y \setminus K$ is $\upsilon\delta$-commuting, then Y is $\upsilon\delta$-commuting.*

Remark 2.24. *Note the importance of the demand for C-embedding in the previous results. The space Ψ is not $\upsilon\delta$-commuting, but it is the union of two spaces that are disjoint and $\upsilon\delta$-commuting, one z-embedded, and the other a zero-set.*

We get a more general version (without disjointness) of Corollary 2.23 when X is a P-space.

Corollary 2.25. *Let Y be a topological space with a family of realcompact C-embedded subspaces $\{K_i\}$ that is locally finite in υY. Assume X is a subspace of Y that is a P-space and that furthermore $Y = X \cup \bigcup K_i$. Then Y is $\upsilon\delta$-commuting. In particular, if Y is the union of a P-subspace X and a subspace K that is a finite union of realcompact C-embedded subspaces of Y, then Y is $\upsilon\delta$-commuting. The union of a subspace that is a P-space and a compact subspace is $\upsilon\delta$-commuting.*

Nothwithstanding the results above we have not been able to settle the following question in general.

Question 2.26.

Let $Y = X \cup K$, where X is an $\upsilon\delta$-commuting space and K is compact. Must Y be $\upsilon\delta$-commuting? (see the open questions in section 5).

There is a variant of Theorem 2.16, that allows for an arbitrary locally finite family but it makes demands on the original topology rather than the δ-topology.

Theorem 2.27. *Let X be $\upsilon\delta$-commuting subspace of the realcompact space Y. Let X be C-embedded in $cl_Y X$ (a copy of υX). Suppose that $\{K_i\}$ is a locally finite family of $\upsilon\delta$-commuting subsets of Y that satisfy the condition $cl_Y K_i = \upsilon K_i$ (for example, the spaces K_i are compact, or C-embedded in Y). Then the space $W = X \cup (\bigcup K_i)$ is $\upsilon\delta$-commuting.*

Several of the previous results postulate a realcompact ambient space, or that families be locally finite in a Hewitt realcompactification. It is possible to drop these demands if one strengthens the demand for local finiteness, to a demand for *strong discreteness* (see the remarks preceding [Henriksen, Raphael, & Woods (2002), 2.9].

Proposition 2.28. *Suppose that Y is the union of a P-space X and a family of compact subsets $\{K_i\}$. Suppose furthermore that there exists a discrete family of open subsets $\{U_i\}$ of Y with the property that $K_i \subset U_i$ for each i. Then Y is $\upsilon\delta$-commuting.*

Remark 2.29. *The result also holds, by invoking 2.23, if Y is normal, and each K_i is a finite union of closed realcompact subspaces because K_i and $Y \backslash U_i$ are completely separated. It also works if one only assumes that X is $\upsilon\delta$-commuting but postulates the disjointness of X and $\bigcup K_i$, and that Y has a base of clopen sets.*

3 Products of $\upsilon\delta$-commuting spaces In this section we investigate when the product of two $\upsilon\delta$-commuting spaces is $\upsilon\delta$-commuting. Although "δ commutes with finite products" (see remark (b) of 1.2), υ does not in general. We will use the following results of H. Ohta that give criteria for when $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ (i.e. when $X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$).

Theorem 3.1. *[Ohta (1982), 1.1]. Let X be a space of non-measurable cardinality. The following are equivalent.*

(a) X is locally compact and realcompact

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any space Y.

The following result is a special case of [Ohta (1982), 1.3], where the cardinal n mentioned there is $\aleph_1$.

Theorem 3.2. *The following conditions on a Tychonoff space X of non-measurable cardinality are equivalent:*

(a) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any P-space Y.

(b) Each point of υX has an υX-neighbourhood G such that $G \cap X$ is weakly Lindelöf.

(Recall that a space S is called *weakly Lindelöf* if each open cover of S has a countable subfamily whose union is dense in S. It is well known and easily proved that if S is weakly Lindelöf and V is open in S, then $cl_S V$ is weakly Lindelöf).

If X is realcompact then Theorem 3.2 becomes:

Theorem 3.3. *Let X be a realcompact space of non-measurable cardinality. The following are equivalent.*

(a) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any P-space Y.

(b) X is locally weakly Lindelöf (i.e. each point of X has a neighbourhood base of weakly Lindelöf neighbourhoods).

We now investigate the relation among X, Y, and $X \times Y$ being $\upsilon\delta$-commuting.

Theorem 3.4. *Let X and Y be spaces for which $X \times Y$ is $\upsilon\delta$-commuting. Then X and Y are $\upsilon\delta$-commuting.*

Proof. We show that X is $\upsilon\delta$-commuting. Let $y_0 \in Y$. Then $X \times \{y_0\}$ is C-embedded in $X \times Y$ and hence in $\upsilon(X \times Y)$. Thus $X \times \{y_0\}$ is dense and C-embedded in the realcompact space $\mathrm{cl}_{\upsilon(X \times Y)}(X \times \{y_0\})$. Hence $\mathrm{cl}_{\upsilon(X \times Y)}(X \times \{y_0\}) = \upsilon X \times \{y_0\} \simeq \upsilon X$, and so $(\mathrm{cl}_{\upsilon(X \times Y)}(X \times \{y_0\}))_\delta \simeq (\upsilon(X \times \{y_0\}))_\delta \simeq (\upsilon X)_\delta$. Thus if $(X \times \{y_0\})_\delta$ is C-embedded in $(\mathrm{cl}_{\upsilon(X \times Y)}(X \times \{y_0\}))_\delta$ we will have, in effect, that X_δ is C-embedded in $(\mathrm{cl}_{\upsilon(X \times Y)}(X \times \{y_0\}))_\delta$ and hence X will be an $\upsilon\delta$-commuting space.

Now $(X \times \{y_0\})_\delta = X_\delta \times \{y_0\}$, which is C-embedded in $X_\delta \times Y_\delta$ and hence in $\upsilon(X_\delta \times Y_\delta)$ which is just $\upsilon((X \times Y)_\delta)$ by Remark (b) of 1.2. But $X \times Y$ is a $\upsilon\delta$-commuting space so $\upsilon((X \times Y)_\delta) = (\upsilon(X \times Y))_\delta$. Thus $(X \times \{y_0\})_\delta$ is C-embedded in $\upsilon(X \times Y)_\delta$ and hence is dense and C-embedded in its subspace $(\mathrm{cl}_{\upsilon(X \times Y)}(X \times \{y_0\}))_\delta$. Our result follows. $\square$

Next we prove a partial converse to Theorem 3.3.

Theorem 3.5. *Let X and Y be two $\upsilon\delta$-commuting spaces for which $\upsilon(X_\delta \times Y_\delta) = \upsilon(X_\delta) \times \upsilon(Y_\delta)$. Then $X \times Y$ is an $\upsilon\delta$-commuting space.*

Proof. As $\upsilon X \times \upsilon Y$ is a realcompact extension of $X \times Y$ there is a continuous function $j : \upsilon(X \times Y) \to \upsilon X \times \upsilon Y$ that fixes $X \times Y$ pointwise. If we apply the P-coreflection, then the underlying map j (now denoted j_δ) remains continuous and we have $j_\delta : (\upsilon(X \times Y))_\delta \to (\upsilon X \times \upsilon Y)_\delta$.

But $(\upsilon X \times \upsilon Y)_\delta = (\upsilon X)_\delta \times (\upsilon Y)_\delta$ (by Remark (b) of 1.2)$=\upsilon(X_\delta) \times \upsilon(Y_\delta)$ (as X and Y are $\upsilon\delta$-commuting)$= \upsilon(X_\delta \times Y_\delta)$ (by hypothesis) $= \upsilon((X \times Y)_\delta)$ (by remark (b) of 1.2) so $j_\delta : (\upsilon(X \times Y))_\delta \to \upsilon((X \times Y)_\delta)$ is continuous and fixes $(X \times Y)_\delta$ pointwise. However, $(\upsilon(X \times Y))_\delta$ is a realcompact extension of $(X \times Y)_\delta$, by remark (a) of 1.2 , so there is a continuous function $k : \upsilon((X \times Y)_\delta) \to (\upsilon(X \times Y))_\delta$ that fixes $(X \times Y)_\delta$ pointwise. But then $k \circ j_\delta|(X \times Y)_\delta$ and $j_\delta \circ k|(X \times Y)_\delta$ both fix $(X \times Y)_\delta$ pointwise, so by a standard argument k is a homeomorphism from $\upsilon((X \times Y)_\delta)$ onto $(\upsilon(X \times Y))_\delta$ that fixes $(X \times Y)_\delta$ pointwise. Thus $\upsilon((X \times Y)_\delta) = (\upsilon(X \times Y))_\delta$ and so $X \times Y$ is $\upsilon\delta$-commuting. □

We now show that the assumption that $\upsilon(X_\delta \times Y_\delta) = \upsilon(X_\delta) \times \upsilon(Y_\delta)$ cannot be dropped in the preceding theorem. We begin with a lemma. Its proof is straightforward but we include it for convenience.

Lemma 3.6. *$(\beta N \backslash N)_\delta$ is not locally weakly Lindelöf.*

Proof. It is well-known and easily verified that a weakly Lindelöf P-space is Lindelöf. Furthermore, each zero-set of $\beta N \backslash N$ contains a clopen copy of $\beta N \backslash N$. Consequently if $(\beta N \backslash N)_\delta$ were locally weakly Lindelöf it would be a Lindelöf P-space and hence functionally countable, ie if $f \in C((\beta N \backslash N)_\delta)$, then $|f[(\beta N \backslash N)_\delta]| \leq \aleph_0$. Consequently $\beta N \backslash N$ would be functionally countable, which is a contradiction as it contains no isolated points (see, for example, [Levy & Rice (1981), 3.1]). □

Lemma 3.7. *Let X be a realcompact locally weakly Lindelöf space of non-measurable cardinality such that X_δ is not locally weakly Lindelöf. Then there is a P-space T such that $X \times T$ is not $\upsilon\delta$-commuting.*

Proof. By Theorem 3.3, since X_δ is realcompact by Lemma 1.1 there exists a P-space T for which $\upsilon(X_\delta \times T) \neq \upsilon(X_\delta) \times \upsilon T = X_\delta \times \upsilon T$. It follows that

$$\upsilon((X \times T)_\delta) = \upsilon(X_\delta \times T_\delta) = \upsilon(X_\delta \times T) \neq X_\delta \times \upsilon T \tag{1}$$

However, $\upsilon(X \times T) = \upsilon X \times \upsilon T$ by 3.3. Thus

$$(\upsilon(X \times T))_\delta = (\upsilon X \times \upsilon T)_\delta = (X \times \upsilon T)_\delta = X_\delta \times (\upsilon T)_\delta = X_\delta \times \upsilon T \tag{2}$$

see [Gillman & Jerison (1960), 8A (4)].

By combining (1) and (2) we see that $\upsilon((X \times T)_\delta) \neq ((\upsilon X) \times T)_\delta$, so $X \times T$ is not $\upsilon\delta$-commuting. □

Example 3.8. *The product of a compact space and a P-space need not be $\upsilon\delta$-commuting.*

Proof. If X is a space satisfying the hypotheses of Lemma 3.7 then its product with some P-space is not $\upsilon\delta$-commuting (both spaces are $\upsilon\delta$-commuting, of course). By Lemma 3.6, $\beta N\backslash N$ satisfies the conditions on X. Thus there is a P-space T such that $(\beta N\backslash N) \times T$ is not $\upsilon\delta$-commuting. □

We can use Theorem 3.1 to derive a positive result.

Theorem 3.9. *Let X and Y be spaces, one of which is of non-measurable cardinality and countable pseudocharacter. The following are equivalent*
(a) X and Y are $\upsilon\delta$-commuting
(b) $X \times Y$ is $\upsilon\delta$-commuting

Proof. $(b) \Rightarrow (a)$. This is Theorem 3.3 (with an unnecesary extra hypothesis). $(a) \Rightarrow (b)$. Assume X has countable pseudocharacter. Then X_δ is discrete and of non-measurable cardinality. Then by Theorem 3.1 $\upsilon(X_\delta \times Y_\delta) = \upsilon(X_\delta) \times \upsilon(Y_\delta)$ so (b) follows from Theorem 3.4. □

Remark 3.10.

The statements "$\upsilon(X \times Y) = \upsilon X \times \upsilon Y$" and "$\upsilon(X_\delta \times Y_\delta) = \upsilon(X_\delta) \times \upsilon(Y_\delta)$" are independent of each other. To verify this note that by the paragraph preceding Theorem 3.9 , there exists a P-space T such that $\upsilon((\beta N\backslash N)_\delta \times T_\delta) \neq (\upsilon((\beta N\backslash N) \times T))_\delta$. However, $\beta N\backslash N$ is locally compact, realcompact and of non-measurable cardinality, so by Theorem 3.1 it follows that $\upsilon((\beta N\backslash N) \times T) = \upsilon(\beta N\backslash N) \times \upsilon T$.

By contrast, the space Q of rational numbers is not locally compact so by Theorem 3.1 there exists a space T such that $\upsilon(Q \times T) \neq \upsilon Q \times \upsilon T$. However, Q_δ is a countable discrete space, so again by Theorem 3.1 $\upsilon(Q_\delta \times T_\delta) = \upsilon(Q_\delta) \times \upsilon(T_\delta)$.

We close this section by showing that $\upsilon\delta$-commutativity is not preserved either directly or inversely by perfect continuous surjections.

Examples 3.11.

(a) Let A denote the so-called "Dieudonné plank" constructed as follows. Let $N^* = N \cup \{p\}$ denote the one-point compactification of N and let $L = D \cup \{q\}$ denote the one-point Lindelöfication of the discrete space D of cardinality $\aleph_1$ (neighbourhoods of q are co-countable sets). The space A is defined to be $(N^* \times L)\backslash\{(p, q)\}$. It is shown in [Kato (1979), Example A, Page 1256] that A is almost realcompact but not realcompact and that $\upsilon A = N^* \times L$, and in [Dykes (1969), 1.7] that the absolute EX of any almost realcompact space X is realcompact. (See [Porter & Woods (1988), chapter 6] for a discussion of absolutes, and [Porter & Woods (1988), Problems 6U, 6V, and 6W] for a discussion of almost realcompactness and the space A). Note that there is a perfect irreducible continuous surjection from the absolute EX of a Tychonoff space X onto X). The space A_δ is easily seen to be the free union of the Lindelöf space $N \times L$ and the discrete space $\{p\} \times D$ of cardinality $\aleph_1$; hence A_δ is realcompact. It follows from Proposition 1.4 that A is not a $\upsilon\delta$-commuting space. Hence the perfect irreducible continuous image of a realcompact (hence $\upsilon\delta$-commuting) space need not be $\upsilon\delta$-commuting.

(b) As noted before Theorem 3.9, there is a P-space T (which thus is $\upsilon\delta$-commuting) such that the product $(\beta N\backslash N) \times T$ is not $\upsilon\delta$-commuting. As $\beta N\backslash N$ is compact, the projection map from $(\beta N \backslash N) \times T$ onto T is a perfect continuous surjection. Hence the perfect continuous pre-image of a P-space need not be $\upsilon\delta$-commuting.

We do not have an example of a perfect irreducible pre-image of a $\upsilon\delta$-commuting space that fails to be $\upsilon\delta$-commuting.

4 Regular ring extensions of $C(X)$ Let X be a Tychonoff space, and let $F(X)$ denote the ring of all real-valued functions with domain X. The smallest (von Neumann) regular ring A for which $C(X) \subseteq A \subseteq F(X)$ is denoted $G(X)$. In [Henriksen, Raphael, & Woods (2002)] and [Raphael & Woods (2000)] conditions on X were investigated that are equivalent to $G(X)$ being ring-isomorphic to a ring of the form $C(Y)$, where Y is a Tychonoff space. If X is realcompact, this is equivalent to $G(X)$ being equal to $C(X_\delta)$. Such spaces are called *RG*-spaces; in [Henriksen, Raphael, & Woods (2002)] compact *RG*-spaces and metric *RG*-spaces are characterized, although no characterization of realcompact *RG*-spaces is yet available.

In this section we generalize this result by showing that if X and Y are Tychonoff spaces for which $G(X)$ is ring isomorphic to $C(Y)$, then $C(Y) \cong C((\upsilon X)_\delta)$. Thus X is an *RG*-space if and only if $G(X)$ is isomorphic to some $C(Y)$ and X is an $\upsilon\delta$-commuting space.

We begin with a theorem on regular rings between $C(X)$ and $F(X)$.

Theorem 4.1. *If X is realcompact and if A is a regular ring that is isomorphic to a $C(Y)$ such that $C(X) \subseteq A \subseteq C(X_\delta)$ then $A = C(X_\delta)$.*

Proof. We can assume without loss of generality that Y is realcompact. Let m be the isomorphism from A to $C(Y)$, with n being its inverse. Let i embed $C(X)$ in A and let j embed A in $C(X_\delta)$. Then $mi : C(X) \to C(Y)$ is a ring embedding. As X is realcompact, by the proof of [Gillman & Jerison (1960), 10.6] (with X and Y interchanged) there is a continuous map $t : Y \to X$ such that for all $g \in C(X)$ and all $y \in Y$, $(mi)(g)(y) = g(t(y))$. As i embeds $C(X)$ in A this becomes, for all g and y as above,

$$m(g)(y) = g(t(y)) \tag{1}$$

Similarly the map $jn : C(Y) \to C(X_\delta)$ is a ring homomorphism fixing 1 so again using [Gillman & Jerison (1960), 10.6], we similarly get a continuous function $s : X_\delta \to Y$ such that for all $k \in C(Y)$ and all $z \in X_\delta$, we have $(jn)(k)(z) = k(s(z))$, which becomes, for all k and z as above, $(n)(k)(z) = k(s(z))$. Hence as X_δ is Tychonoff it follows that for each $k \in C(Y)$,

$$n(k) = ks \tag{2}$$

If $g \in C(X)$ then $m(g) \in C(Y)$, so replacing k by $m(g)$ in (2) and using $nm = 1$, we get that for all $g \in C(X)$ and all $z \in X_\delta$,

$$g(z) = m(g)(s(z)) \tag{3}$$

Replacing y by $s(z)$ in (1) we have, for all $z \in X_\delta$ and $g \in C(X)$,

$$m(g)(s(z)) = g(t(s(z))) \tag{4}$$

Combining (3) and (4) gives, for all $g \in C(X)$ and $z \in X_\delta, g(z) = g(t(s(z))$ (note z is a point of X and of X_δ). As X is Tychonoff this implies that for all $z \in X_\delta, z = t(s(z))$. Thus $ts = 1$, and as 1 is the canonical map from X_δ to X, and as Y is a P-space since A and hence $C(Y)$ is regular, it follows that there is a continuous map $a : Y \to X_\delta$ such that $1a = t$ (see [Porter & Woods (1988), 1W(7)]). Thus as is the identity on X_δ.

We complete the proof by establishing three claims.

Claim 1. $s[X_\delta]$ is dense in Y

Proof of the claim. If not, let $p \in Y \backslash cl(s[X_\delta])$. As Y is Tychonoff there exists $h \in C(Y)$ such that $h(p) = 1$ and $h[s[X_\delta]] = 0$. Apply (2) to h and get: for all $z \in X_\delta, (n(h))(z) = h(s(z)) = 0$. Thus $n(h) = n(0)$ is the constant function $\mathbf{0}$ but h is not equal to 0 as $h(p) = 1$. This contradicts $n's$ being a ring isomorphism, so our claim holds.

Claim 2. $s : X_\delta \to Y$ is a homeomorphism.

Proof of the claim. We know that $a : Y \to X_\delta$ is continuous and since $as = 1$, the restriction $a|s[X_\delta]$ is a homeomorphism. Since by claim 1, a maps a dense subset of Y (since as is the identity on X_δ) onto its image, it follows from [Gillman & Jerison (1960), 6.11] that $s[X_\delta] = Y$ and hence s is a homeomorphism.

Claim 3. $A = C(X_\delta)$.

Proof of the claim. It suffices to show that the inclusion $j : A \to C(X_\delta)$ maps A onto $C(X_\delta)$. But since by claim 2, $s : X_\delta \to Y$ is a homeomorphism, the map $k \to ks$ is a ring isomorphism from $C(Y)$ onto $C(X_\delta)$. Hence by (2), n maps $C(Y)$ onto $C(X_\delta)$. But n maps $C(Y)$ onto A, so $A = C(X_\delta)$. □

By replacing A with $G(X)$ in theorem 4.1 we have:

Corollary 4.2. *If X is realcompact and if $G(X) \cong C(Y)$ for some space Y, then X is an RG-space.*

Remarks 4.3.

There is an easy algebraic proof of corollary 4.2 using properties of epimorphisms, but not one for theorem 4.1. It is possible for an algebra strictly between $C(X)$ and $C(X_\delta)$ to be regular, uniformly closed in the sense of [Henriksen, Johnson (1961)] , and have X be realcompact. One example is given by the algebra of Baire functions on the real line X discussed in [Henriksen, Johnson (1961), 5.1].

Theorem 4.4. *Let X and Y be Tychonoff spaces for which $G(X)$ is ring-isomorphic to $C(Y)$. Then $C(Y)$ is ring isomorphic to $C((\upsilon X)_\delta)$ and $\upsilon Y \sim (\upsilon X)_\delta$. In particular an RG-space is $\upsilon\delta$-commuting.*

Proof. We know that $G(X) \cong G(\upsilon X)$ (see [Raphael & Woods (2006), 4.1]) so $G(\upsilon X) \cong G(X) \cong C(Y) \cong C(\upsilon Y)$. Hence by 4.2 it follows that $G(\upsilon X) \cong C((\upsilon X)_\delta)$. Applying the above isomorphisms, it follows that $C(\upsilon Y) \cong C(Y) \cong C((\upsilon X)_\delta)$. As noted in Lemma 1.1 above, $(\upsilon X)_\delta$ is realcompact, as is υY, so it follows from [Gillman & Jerison (1960), 8.3] that $\upsilon Y \sim (\upsilon X)_\delta$. □

Corollary 4.5. *A perfectly normal scattered space of finite Cantor-Bendixon index is $\upsilon\delta$-commuting. A perfectly normal scattered space of finite Cantor-Bendixon index and non-measurable cardinality is realcompact.*

Proof. The first assertion follows from [Henriksen, Raphael, & Woods (2002)][2.12] where these spaces are shown to be RG. The second assertion follows by part (b) of corollary 1.5.

Remark 4.6.

Note that the space Ψ is not $\upsilon\delta$-commuting by corollary 1.5 because it is not realcompact. But it is of countable pseudocharacter, scattered, and of finite Cantor-Bendixon index. Thus the corollary holds for perfectly normal spaces, but not for general spaces of countable pseudocharacter. Note also that the space of [Levy & Rice (1981)][Example 2] given under Lusin's Hypothesis is perfectly normal, hence $\upsilon\delta$-commuting by the corollary. It is therefore realcompact.

There are many examples of $\upsilon\delta$-commuting spaces that are not RG. Compact spaces are always $\upsilon\delta$-commuting but rarely RG (see [Henriksen, Raphael, & Woods (2002), 3.4]). The scattered space of [Henriksen, Raphael, & Woods (2002), Example 2.10] is realcompact hence $\upsilon\delta$-commuting but not RG.

We can now generalize 4.2 using $\upsilon\delta$-commuting spaces. (The analogous generalization of 4.1 also holds).

Theorem 4.7. *If X is an $\upsilon\delta$-commuting space, and if $G(X)$ is ring-isomorphic to $C(Y)$ for some space Y, then $G(X) = C(X_\delta)$ (i.e. X is an RG-space).*

Proof. It suffices to show that $C(X_\delta) \subseteq G(X)$. As noted in the proof of Theorem 4.4, $G(\upsilon X) = C((\upsilon X)_\delta)$, so by hypothesis $G(\upsilon X) = C(\upsilon(X_\delta))$. Let $f \in C(X_\delta)$. Then its Hewitt extension f^υ belongs to $G(\upsilon X)$. But the map $g \to g|X$ is a ring isomorphism from $G(\upsilon X)$ onto $G(X)$ (see [Raphael & Woods (2006), 4.1]). Thus $f = f^\upsilon|X \in G(X)$. ☐

Remark 4.8.

Since RG-spaces are $\upsilon\delta$-commuting the work in [Hrusak, Raphael, & Woods (2005)] shows that there are $\upsilon\delta$-commuting spaces that are pseudocompact, non compact, almost compact, and almost-P.

Our central result is as follows.

Theorem 4.9. *Let X be a Tychnoff space, and suppose that there exists a space Y such that $G(X)$ is ring-isomorphic to $C(Y)$. The following are equivalent:*

(a) $(\upsilon X)_\delta = \upsilon(X_\delta)$

(b) X is an RG-space.

If these equivalent conditions hold, then $\upsilon Y \sim (\upsilon X)_\delta$.

Proof. $(a) \Rightarrow (b)$. This is Proposition 2.5 above.

(b) $\Rightarrow (a)$. As noted in the proof of 4.4, $G(\upsilon X) = C(\upsilon X)_\delta)$. Hence if $f \in C(X_\delta)$, then $f^\upsilon \in C((\upsilon X)_\delta)$, and thus the space X_δ is dense and C-embedded in the realcompact space $(\upsilon X)_\delta$ (see 1.1). It follows that $\upsilon(X_\delta) = (\upsilon X)_\delta$. ☐

The next result will provide us with a technique for generating examples of spaces X for which $G(X) \cong C(Y)$ but X is not an RG-space (ie $G(X) \neq C(X_\delta)$).

Theorem 4.10. *Let X be a non-realcompact space for which υX is a Lindelöf scattered space of finite CB-index and X_δ is realcompact. Then $G(X)$ is ring-isomorphic to a $C(Y)$, but X is not an RG-space.*

Proof. By [Henriksen, Raphael, & Woods (2002), 2.12] υX is an RG-space and so $C((\upsilon X)_\delta) = G(\upsilon X)$. But $G(X) \cong G(\upsilon X)$, so $(\upsilon X)_\delta$ is the desired Y. By part (a) of 1.5 X is not an $\upsilon\delta$-commuting space. □

Examples 4.11.

(a) As observed in 3.11 (a), the Dieudonné plank A is a non-realcompact space for which A_δ is realcompact, and υA is the product of a compact scattered space of finite CB-index and a Lindelöf scattered space of finite CB-index. Hence A satisfies the hypotheses of 4.10.

(b) There are "Ψ-spaces" (see 2.10) that are almost compact, ie for which $|\beta\Psi\backslash\Psi| = 1$; see [Mrowka (1977)] and [Teresawa (1980)]. For such a Ψ, $\beta\Psi$ is a compact scattered space of CB-index 3. Hence if K is any compact scattered space of finite CB-index, $(\beta\Psi)\times K$ is also such a space, and as $\Psi \times K$ is pseudocompact, it follows that $(\upsilon\Psi)\times K = \beta(\Psi\times K) = (\beta\Psi)\times K$.

We conclude this paper by considering the epimorphic hull $H(X)$ of $C(X)$. One can characterize $H(X)$ as the smallest regular ring lying between $C(X)$ and $Q(X)$, its complete ring of quotients (see [Raphael & Woods (2000)] for background).

Recall that a Tychonoff space X is *almost-P* if its zero-sets are regular closed; equivalently, if X has no proper dense cozero-sets. As noted in [Raphael & Woods (2000), paragraph before 5.1], if X is an almost-P space then $H(X) = G(X)$. In 4.10 we exhibited a non RG-space X for which $G(X)$ was nevertheless isomorphic to a ring of continuous functions. We now want to find an almost-P space T with the same property. If so, we will have an example of a space for which $H(T)$ is ring-isomorphic to a $C(Y)$ but not to $C(T_\delta)$. To do this, we will first (in 4.12 below) describe a technique for producing an almost-P space $AP(X)$ from any Tychonoff space X. We then (in 4.13) apply this construction to specific sorts of X to produce spaces T with the properties described above.

Construction 4.12.

The following construction is based on ideas that appeared in [Dashiell, Henriksen & Hager (1980)] and in an earlier version of [Barr, Kennison & Raphael (2005)]. Let X be a Tychonoff space. Let α be a regular cardinal larger than $|X|$, let D be a set of cardinality α, and let $L(\alpha) = D \cup \{p\}$ (where $p \notin D$) topologized as follows: points of D are isolated, and if $p \in A \subseteq L(\alpha)$ then A is open if and only if $|D\backslash A| < \alpha$.

Let $AP(X) = L(\alpha) \times X$, topologized as follows. Its topology τ has $B_1 \cup B_2 = B$ as an open base where $B_1 = \{\{(d,x)\} : d \in D, x \in X\}$ and $B_2 = \{A \times V, p \in A, |D\backslash A| < \alpha, V \text{ is open in } X\}$.

Clearly nonempty intersections of members of B are in B, and hence B is a base for a topology τ which is clearly Hausdorff. It is completely regular, for if $\{(d,x)\} \in D\times X$ then the characteristic function $\chi_{\{(d,x)\}}$ witnesses this, while if $(p,x) \in A\times V$ where V is open in X, let $f \in C(X)$ such that $f(p) = 0$ and $f[X\backslash A] = \{1\}$. Define $F : AP(X) \to \mathbf{R}$ by letting $F((d,y)) = f(y)$ for each $y \in X$. Clearly $F \in C(AP(X))$, $F((p,x)) = 0$, and $F[AP(X)\backslash(A\times V)] = \{1\}$.

Next, note that $AP(X)$ is an almost-P space; for let G be a non-empty G_δ-set of $AP(X)$. If $G\backslash(\{p\}\times X)\neq\phi$ clearly the $AP(X)$-interior of G is non-empty. If $(p,y)\in G$, there is a countable family $(A_n\times V_n)_{n\in\omega}$ of basic open sets of $AP(X)$ for which $(p,y)\in\bigcap_n\{A_n\times V_n\}\in G$. Choose $z\in(\bigcap_{n\in\omega}A_n)\backslash\{p\}$. This is possible as α, being regular, has uncountable cofinality. Clearly $(z,y)\in \mathrm{int}_{AP(X)}(G)$. Thus non-empty G_δ-sets of $AP(X)$ have non-empty interiors and $AP(X)$ is an almost-P space.

Next observe that $\{p\}\times X$ is C-embedded in $AP(X)$. To see this, suppose that $f\in C(\{p\}\times X)$ and extend f to $F:AP(X)\to\mathbf{R}$ by letting $F(L(\alpha)\times\{y\})=f(p,y)$ for each $y\in Y$. As the topology on $AP(X)$ is stronger than the topology on the product $L(\alpha)\times X$, and as F is continuous with respect to the product topology, clearly $F\in C(AP(X))$.

Construction 4.13.

We now apply the construction described above to produce an example of a space T with the properties described in the paragraph preceding 4.12. Let us suppose that X is a non-realcompact space for which $|\upsilon X\backslash X|=1$ and υX is a scattered Lindelöf space of finite CB-index. (Note that the Dieudonné plank, or an almost compact version of Ψ, satisfies the above hypotheses on the space X). Then $\mathrm{cl}_{\upsilon AP(X)}(\{p\}\times X)\backslash(\{p\}\times X)=\{q\}$ for some $q\in\upsilon AP(X)\backslash AP(X)$. Let $T=AP(X)\cup\{q\}$ and $E=(\{p\}\times X)\cup\{q\}$. We will show that E is C-embedded in T.

Let $g\in C(E)$. Define $G:AP(X)\to R$ by : $G(x,y)=g(p,y)$ for all $x\in L(\alpha)$ and $y\in X$. It is clear (as in the previous paragraph) that $G\in C(AP(X))$. Hence as $AP(X)$ is C-embedded in T (as $q\in\upsilon AP(X)$) there is a continuous extension G' of G to T. Now $G'|\{p\}\times X=g|\{p\}\times X$ and $\{p\}\times X$ is dense in E, so $G'|E=g$. Thus G' is a continuous extension of g to T, so E is C-embedded in T as claimed.

Thus T is the union of the discrete space $D\times X$ and the realcompact C-embedded (hence closed) subspace E. It follows (as in the proof of 2.20) that T is realcompact.

We will now show that T is an RG-space, and that $AP(X)_\delta$ is realcompact. We can then argue as in the proof of 4.10 that $H(AP(X))$ is ring-isomorphic to a $C(Y)$, but $AP(X)$ is not an RG-space.

To show that T is an RG-space, suppose $h\in C(T_\delta)$. Then $h|E\in C(E_\delta)$. Now E is homeomorphic to υX, a scattered Lindelöf space of finite CB-index. Thus E is an RG-space and so $h|E\in G(E)$. Thus there are $f_i,g_i\in C(E)$ such that $h=\Sigma(f_i)(g_i^*)$. As E is C-embedded in T there exist $F_i,G_i\in C(T)$ such that $f_i=F_i|E$ and $g_i=G_i|E$. By an argument similar to one used earlier there exists a subset M of D such that M is clopen in $L(\alpha)$ and h agrees with $\Sigma F_iG_i^*$ on $T\backslash((D\backslash M)\times X)$. Now $hm\in C(T)$ where m is the characteristic function of $(D\backslash M)\times X$, and clearly $h=hm+(1-m)\Sigma F_iG_i^*$ represents h as a member of $G(T)$. Thus $C(T_\delta)=G(T)$ and so T is an RG-space.

Finally, note that $(\{p\}\times X)_\delta\sim E_\delta$ and hence E_δ is realcompact as X_δ is hypothesized to be realcompact. It is easily seen that if $g\in C(E_\delta)$ then $G:AP(X)\to\mathbf{R}$ defined by $G((d,y))=g(p,y)$ is in $C(AP(X)_\delta)$, so $AP(X)_\delta$ is the union of a realcompact C-embedded subset $(\{p\}\times E)_\delta$ and a discrete (hence realcompact) space $D\times X$. Thus $AP(X)_\delta$ is realcompact. It follows that $H(AP(X))$ is ring-isomorphic to a $C(Y)$ but $AP(X)$ is not an RG-space.

5 Remarks and Questions 1. Let T denote the category of Tychonoff spaces and continuous functions. As the realcompact spaces define a reflective full subcategory of T

and the Tychonoff P-spaces define a coreflective full subcategory of T it is reasonable to ask whether the $\upsilon\delta$-commuting spaces define yet another reflective and/or coreflective full subcategory of T. Now reflective subcategories of T are closed-hereditary -see [Walker (1974), 10.21] so by 2.7 the $\upsilon\delta$-commuting spaces do not form a reflective subcategory. Similarly, the objects of coreflective subcategories of T are preserved by quotient maps (see [Walker (1974), 10C.5]). However the space ω_1 of countable ordinals can be written as the continuous image under an open (hence quotient) map of a realcompact space (see [Gillman & Jerison (1960), 8 I]). Thus the $\upsilon\delta$-commuting spaces do not define a coreflective subcategory of T either.

2. Is a regular closed set in a $\upsilon\delta$-commuting space $\upsilon\delta$-commuting? In particular, is the closure of a cozero-set of an $\upsilon\delta$-commuting space another $\upsilon\delta$-commuting space?.

3. (a) Is the free union of any collection of $\upsilon\delta$-commuting spaces $\upsilon\delta$-commuting? (i.e. is the property preserved by free unions of spaces indexed by measurable cardinals)? (See Remark 2.13).

(b) Are all paracompact spaces $\upsilon\delta$-commuting? (i.e. must paracompact spaces of measurable cardinality be $\upsilon\delta$-commuting ?)

4. Let $Y = X \cup K$, where X is a $\upsilon\delta$-commuting space and K is compact. Must Y be $\upsilon\delta$-commuting? The disjoint case, the case where X is z-embedded, and the case where X is a P-space appear above in section 2. The case where $X \cap K$ is a Baire set of X follows from the disjoint case and 2.1. The result is also clear if $X\backslash K$ is $\upsilon\delta$-commuting (again by the disjoint case).

5. Is there an example of a space Y that is not $\upsilon\delta$-commuting, but is of the form $X \cup K$, where X is $\upsilon\delta$-commuting and z-embedded in Y, and K is realcompact and C^*-embedded in Y? Is there such an example, where X and K are disjoint, and K is C^*-embedded? Note (see 4.8) that in the example in [Levy & Rice (1981)] the space D is C^*-embedded and the space is $\upsilon\delta$-commuting.

6. Is there an example of a perfect irreducible pre-image of an $\upsilon\delta$-commuting space that fails to be $\upsilon\delta$-commuting?

7. If "z-embedding" is dropped from the hypotheses of 2.18, does the resulting assertion remain true? What happens if one removes the demand for denseness?

8. The concepts of being a P-space, and being realcompact, have well-know generalizations to higher cardinals. Let α be an uncountable cardinal. A subset of a Tychonoff space X is a G_α-set if it can be written as the intersection of fewer than α open subsets of X. A Tychonoff space is a P_α-space if each G_α-set is open. The P_α-coreflection of X (denoted X_α) is obtained by declaring the G_α-sets of X to be an open base for a new topology. See [Comfort & Negrepontis (1974), chapter 2] for an extensive discussion of these ideas.

A z-ultrafilter A on X has the α-intersection property if the intersection of fewer than α members of A is non-empty. We recall that Herrlich [Herrlich, (1967)] has called a space X α-compact if each z-ultrafilter with the α-intersection property is fixed. The α-compactification $\upsilon_\alpha X$ of X is the subspace $X \cup \{A \in \beta X\backslash X\}$ where the A have the α-intersection property, and as in [Gillman & Jerison (1960)] we regard the points of βX as the z-ultrafilters on X). Note that if $\alpha = \aleph_1$, the above concepts reduce to the notions of P-space, X_δ, realcompactness, and υX.

Professor W. Comfort has raised the question of when $\upsilon_\alpha(X_\gamma) = (\upsilon_\alpha X)_\gamma$ for any pair (α, γ) of uncountable cardinals. This is a question of obvious interest that is beyond the scope of this paper. We hope to address it in a forthcoming paper.

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