RIEMANN ZETA FUNCTION, BERNOULLI POLYNOMIALS AND THE BEST CONSTANT OF SOBOLEV INEQUALITY

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Abstract. Green function for periodic boundary value problem of $2M$-th order ordinary differential equation is found by symmetric orthogonalization method under a suitable solvability condition. As an application, the best constants and the best functions of the Sobolev inequalities in a certain series of Hilbert spaces are found and expressed by means of the well-known Bernoulli polynomials. This result has clarified the variational meaning of the special values $\zeta(2M)$ $(M = 1, 2, 3, \cdots)$ of Riemann zeta function $\zeta(z)$.

1 Conclusion In this paper we clarified the variational meaning of the special values $\zeta(2M)$ $(M = 1, 2, 3, \cdots)$ of Riemann zeta function $\zeta(z)$. A constant multiple of $\zeta(2M)$ is a supremum of M-th Sobolev functional $S_M(u)$ in a suitable function space H_M.

As a preparation, we explain briefly about Riemann zeta function, Bernoulli polynomial and Bernoulli number. Riemann zeta function is a meromorphic function defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad (\mathrm{Re}\ z > 1) \tag{1.1}$$

It has only one simple pole at $z = 1$. All its nontrivial zeros lie on a straight line $\mathrm{Re}\, z = 1/2$, which is a famous Riemann hypothesis. Bernoulli polynomial $b_n(x)$ is defined by the following recurrence relation.

$$b_0(x) = 1 \tag{1.2}$$

$$b_n'(x) = b_{n-1}(x), \quad \int_0^1 b_n(x)\, dx = 0 \qquad (n = 1, 2, 3, \cdots) \tag{1.3}$$

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That is to say $b_n(x)$ is a primitive of $b_{n-1}(x)$ having mean value 0 on an interval $0 < x < 1$.

$$
\begin{aligned}
&b_0(x) = 1, \qquad b_1(x) = x - \frac{1}{2}, \qquad b_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}, \\
&b_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x, \qquad b_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720} \\
&b_5(x) = \frac{1}{120}x^5 - \frac{1}{48}x^4 + \frac{1}{72}x^3 - \frac{1}{720}x \\
&b_6(x) = \frac{1}{720}x^6 - \frac{1}{240}x^5 + \frac{1}{288}x^4 - \frac{1}{1440}x^2 + \frac{1}{30240} \\
&b_7(x) = \frac{1}{5040}x^7 - \frac{1}{1440}x^6 + \frac{1}{1440}x^5 - \frac{1}{4320}x^3 + \frac{1}{30240}x \\
&b_8(x) = \frac{1}{40320}x^8 - \frac{1}{10080}x^7 + \frac{1}{8640}x^6 - \frac{1}{17280}x^4 + \frac{1}{60480}x^2 - \frac{1}{1209600} \\
&\cdots
\end{aligned}
$$

Bernoulli number is defined by

$$B_M = (2M)!\,(-1)^{M-1}\,b_{2M}(0) \qquad (M = 1, 2, 3, \cdots) \tag{1.4}$$

It can be obtained by the following recurrence relation

$$\begin{cases} \displaystyle\sum_{j=0}^{n-1} (-1)^j \binom{2n}{2j} B_j = -n \qquad (n = 1, 2, 3, \cdots) \\ B_0 = -1 \end{cases} \tag{1.5}$$

Bernoulli numbers are positive rational numbers. We know that

$$
\begin{aligned}
&B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \\
&B_5 = \frac{5}{66}, \quad B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}, \quad \cdots
\end{aligned}
$$

In order to present our main theorems, we prepare a sequence of function spaces

$$
\begin{aligned}
H_M = \Big\{ u(x) \,\Big|\, & u^{(M)}(x) = (d/dx)^M u(x) \in L^2(0,1), \\
& u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \quad \int_0^1 u(x)dx = 0 \Big\}
\end{aligned} \tag{1.6}
$$

for $M = 1, 2, 3, \cdots$ and Sobolev functionals

$$S_M(u) = \left(\sup_{0 \le y \le 1} |u(y)| \right)^2 \Big/ \int_0^1 \left| u^{(M)}(x) \right|^2 dx \tag{1.7}$$

The main theorems we have obtained in this paper are as follows.

Theorem 1.1 *For $M = 1, 2, 3, \cdots$ we have the following conclusions.*

(1) $$\sup_{\substack{u \in H_M \\ u \not\equiv 0}} S_M(u) = C_M = \frac{2\,\zeta(2M)}{(2\pi)^{2M}} = \frac{B_M}{(2M)!} = \int_0^1 |\, b_M(x)\,|^2 dx \tag{1.8}$$

(2) *For any fixed* y *satisfying* $0 \le y \le 1$ *we have*

$$S_M\,(\,b_{2M}(\,|x-y|\,)\,) \;=\; C_M \tag{1.9}$$

(3)
$$\inf_{\substack{u\in H_M\\ u\not\equiv 0}} S_M(u) \;=\; 0 \tag{1.10}$$

The above theorem is rewritten equivalently in the following manner.

Theorem 1.2 *For each fixed* $M = 1, 2, 3, \cdots$ *and for every function* $u(x) \in H_M$*, we have a suitable positive constant* C *which is independent of* $u(x)$ *such that the following Sobolev inequality holds.*

$$\left(\sup_{0\le y\le 1}|u(y)|\right)^2 \;\le\; C\int_0^1 \left|u^{(M)}(x)\right|^2 dx \tag{1.11}$$

Among such C *the best constant* C_M *is given in the previous theorem.*

In the above inequality if we replace C *by* C_M*, the equality holds for* $u(x) = const.\, b_{2M}(\,|x-y|\,)$ *, where* y *is an arbitrarily fixed number satisfying* $0 \le y \le 1$.

These main Theorems are proved in the later sections but the proof of (3) of Theorem 1.1 is very simple. In fact for $n = 1, 2, 3, \cdots$ we have $\cos(2\pi nx) \in H_M$ and

$$S_M(\,\cos(2\pi nx)\,) \;=\; \frac{2}{(2\pi)^{2M}}\,\frac{1}{n^{2M}} \quad \xrightarrow[n\to\infty]{} \quad 0$$

This shows (3) of Theorem 1.1. Positive definiteness of Sobolev energy $\displaystyle\int_0^1 \left|u^{(M)}(x)\right|^2 dx$ is shown later.

For the sake of comparison we present the well-known theorem concerning Wirtinger inequality.

Theorem 1.3 (Wirtinger) *For each fixed* $M = 1, 2, 3, \cdots$ *and for every function* $u(x) \in H_M$*, we have a suitable positive constant* C *which is independent of* $u(x)$ *such that the following Wirtinger inequality holds.*

$$\int_0^1 |\,u(x)\,|^2\,dx \;\le\; C\int_0^1 \left|u^{(M)}(x)\right|^2 dx \tag{1.12}$$

Among such C *the best constant* $\widetilde{C}_M$ *is given by*

$$\widetilde{C}_M \;=\; 1\,/\,(2\pi)^{2M} \tag{1.13}$$

In the above inequality if we replace C *by* $\widetilde{C}_M$*, the equality holds for a special function*

$$u(x) \;=\; const.\,\cos(2\pi x) \,+\, const.\,\sin(2\pi x) \qquad (0 < x < 1) \tag{1.14}$$

2 Bernoulli polynomials In this section, we explain important aspects of Bernoulli polynomials which are used frequently in this paper. We omit their proofs, some of which are given in appendix.

We start with definitions of $(n+1)\times(n+1)$ nilpotent matrix

$$\boldsymbol{N} = \left(\;\delta_{i,j-1}\;\right) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

and its exponential function

$$\boldsymbol{E}(x) = \exp(x\boldsymbol{N}) = \Big(\; E_{j-i} \; \Big)(x) = \begin{pmatrix} E_0 & \cdots & E_n \\ & \ddots & \vdots \\ & & E_0 \end{pmatrix}(x) \tag{2.1}$$

where $\delta_{i,j}$ is a Kronecker delta symbol defined by

$$\delta_{i,j} = 1 \quad (i = j), \qquad 0 \quad (i \neq 0)$$

and $E_i(x)$ $(i = 0, \pm 1, \pm 2, \cdots)$ are monomials defined by

$$E_i(x) = x^i/i! \quad (i = 0, 1, 2, \cdots), \qquad 0 \quad (i = -1, -2, \cdots) \tag{2.2}$$

We also use the following abbreviation.

$$E_i = E_i(1) \qquad (i = 0, 1, 2, \cdots)$$

$\boldsymbol{E}(x)$ satisfies an initial value problem

$$(d/dx)\, \boldsymbol{E}(x) = \boldsymbol{N}\, \boldsymbol{E}(x), \quad \boldsymbol{E}(0) = \boldsymbol{I}$$

and an addition rule

$$\boldsymbol{E}(x + y) = \boldsymbol{E}(x)\, \boldsymbol{E}(y) \qquad (x, y \in \boldsymbol{C})$$

Lemma 2.1

(1) $$\frac{e^{xt}}{t^{-1}(e^t - 1)} = \sum_{i=0}^{\infty} b_i(x)\, t^i \qquad (\,|\,t\,| < 2\pi\,) \tag{2.3}$$

(2) $$\frac{\cos(\,(1 - 2x)t\,)}{t^{-1}\sin(t)} = \sum_{j=0}^{\infty} (-1)^j\, b_{2j}(x)\, (2t)^{2j} \qquad (\,|\,t\,| < \pi\,) \tag{2.4}$$

Lemma 2.2

$$b_i(1 - x) = (-1)^i\, b_i(x) \qquad (i = 0, 1, 2, \cdots) \tag{2.5}$$

Lemma 2.3

$$b_{i+1}(x + 1) - b_{i+1}(x) = E_i(x) \qquad (i = 0, 1, 2, \cdots) \tag{2.6}$$

Lemma 2.4

$$b_i(1) - b_i(0) = \delta_{i,1} \qquad (i = 0, 1, 2, \cdots) \tag{2.7}$$

Lemma 2.5

$$b_n(x) = \sum_{j=0}^{n} b_{n-j}(0)\, E_j(x) \qquad (n = 0, 1, 2, \cdots) \tag{2.8}$$

$$b_{2n}(x) = \sum_{j=0}^{n} b_{2(n-j)}(0)\, E_{2j}(x) - \frac{1}{2}\, E_{2n-1}(x) \qquad (n = 1, 2, 3, \cdots) \tag{2.9}$$

$$b_{2n+1}(x) = \sum_{j=0}^{n} b_{2(n-j)}(0)\, E_{2j+1}(x) - \frac{1}{2}\, E_{2n}(x) \qquad (n = 0, 1, 2, \cdots) \tag{2.10}$$

Lemma 2.6

$$b_{2i+1}(0) \;=\; -\frac{1}{2}\delta_{i,0} \qquad (i=0,1,2,\cdots) \tag{2.11}$$

$$b_{2i+1}\,(1/2) \;=\; 0 \qquad (i=0,1,2,\cdots) \tag{2.12}$$

From lemma 2.5 we have

Lemma 2.7

$$\boldsymbol{E}(x) \;=\; \begin{pmatrix} E_0 & \cdots & E_n \\ & \ddots & \vdots \\ & & E_0 \end{pmatrix}(x) \;=\; \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}(x) \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}^{-1}(0) \tag{2.13}$$

We introduce a matrix $\boldsymbol{E}_1(x)$ defined by

$$\boldsymbol{E}_1(x) \;=\begin{pmatrix} E_1 & \cdots & E_{n+1} \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix}(x) \tag{2.14}$$

and its inverse

$$\breve{\boldsymbol{E}}_1(x) \;=\; \boldsymbol{E}_1(x)^{-1} \;=\begin{pmatrix} \breve{E}_1 & \cdots & \breve{E}_{n+1} \\ & \ddots & \vdots \\ & & \breve{E}_1 \end{pmatrix}(x) \tag{2.15}$$

The following lemma holds.

Lemma 2.8 *The inverse of the matrix $\boldsymbol{E}_1(1)$ is given by*

$$\breve{\boldsymbol{E}}_1(1) \;=\; \boldsymbol{E}_1(1)^{-1} \;=\; \begin{pmatrix} \breve{E}_1 & \cdots & \breve{E}_{n+1} \\ & \ddots & \vdots \\ & & \breve{E}_1 \end{pmatrix}(1) \;=\; \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}(0) \tag{2.16}$$

that is

$$b_i(0) \;=\; \breve{E}_{i+1} \qquad (i=0,1,2,\cdots) \tag{2.17}$$

Lemma 2.9

$$b_n(x) \;=\; \left|\begin{array}{ccc|c} E_1 & & & E_0(x) \\ \vdots & \ddots & & \vdots \\ E_n & \cdots & E_1 & E_{n-1}(x) \\ E_{n+1} & \cdots & E_2 & E_n(x) \end{array}\right| \qquad (n=1,2,3,\cdots) \tag{2.18}$$

Lemma 2.10

$$\begin{pmatrix} b_{n+1}(x)-b_{n+1}(0) \\ \vdots \\ b_1(x)-b_1(0) \end{pmatrix} = \breve{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{n+1} \\ \vdots \\ E_1 \end{pmatrix}(x) \qquad (n=0,1,2,\cdots) \tag{2.19}$$

From the relation $\boldsymbol{E}(x+y) = \boldsymbol{E}(x)\,\boldsymbol{E}(y)$ we have the following lemma.

Lemma 2.11

$$\begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}(x+y) = \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}(x)\,\breve{\boldsymbol{E}}_1(1) \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}(y)$$

$$(n=1,2,3,\cdots) \tag{2.20}$$

Lemma 2.12

$$\left(\, b_1(x)-b_1(0),\ \cdots\ ,b_n(x)-b_n(0)\,\right) \boldsymbol{E}_1(1) \begin{pmatrix} b_n(y)-b_n(0) \\ \vdots \\ b_1(y)-b_1(0) \end{pmatrix} =$$

$$b_{n+1}(x+y) - b_{n+1}(x) - b_{n+1}(y) + b_{n+1}(0) \qquad (n=1,2,3,\cdots) \tag{2.21}$$

Next we derive the Fourier expansion formula of $b_i(\{x\})$, where $\{x\} = x - [x]$ denotes a decimal part of a real number x.

Lemma 2.13 *If we expand $b_i(\{x\})$ in Fourier series as*

$$b_i(\{x\}) = \sum_{n=-\infty}^{\infty} \widehat{b}_i(n)\exp(\sqrt{-1}\,2\pi n x) \tag{2.22}$$

its Fourier coeficients

$$\widehat{b}_i(n) = \int_0^1 b_i(\{x\})\,\exp\left(-\sqrt{-1}\,2\pi n x\right)\,dx \tag{2.23}$$

are given as follows.

$$\widehat{b}_0(n) = \delta_{n,0} \qquad (n=0,1,2,\cdots) \tag{2.24}$$

For $i=1,2,3,\cdots$

$$\widehat{b}_i(n) = \begin{cases} 0 & (n=0) \\ -\left(\sqrt{-1}\,2\pi n\right)^{-i} & (n=\pm1,\pm2,\cdots) \end{cases} \tag{2.25}$$

Lemma 2.14 *For $i=1,2,3,\cdots,$ Fourier series*

$$b_i(\{x\}) = -\sum_{n\neq 0} \left(\sqrt{-1}\,2\pi n\right)^{-i} \exp\left(\sqrt{-1}\,2\pi n x\right) \tag{2.26}$$

can be differentiated with respect to x termwise in the sence of distribution as

$$\left(\frac{d}{dx}\right)^j b_i(\{x\}) \;=\; b_{i-j}(\{x\}) \;=\; -\sum_{n\neq 0} \left(\sqrt{-1}\,2\pi n\right)^{j-i} \exp\left(\sqrt{-1}\,2\pi n x\right) \tag{2.27}$$

$$(0 \le j \le i-1)$$

The right hand side converges in $L^2(0,1)$.

In the real form we have

$$b_{2i}(\{x\}) \;=\; (-1)^{i-1}\, 2 \sum_{n=1}^{\infty} (2\pi n)^{-2i} \cos(2\pi n x) \tag{2.28}$$

$$b_{2i+1}(\{x\}) \;=\; (-1)^{i-1}\, 2 \sum_{n=1}^{\infty} (2\pi n)^{-(2i+1)} \sin(2\pi n x) \tag{2.29}$$

Lemma 2.15

$$(-1)^{i-1}\, b_{2i}(0) \;=\; \frac{2}{(2\pi)^{2i}}\, \zeta(2i) \qquad (i=1,2,3,\cdots) \tag{2.30}$$

$$b_{2i}(1/2) \;=\; -\left(1-2^{-(2i-1)}\right) b_{2i}(0) \qquad (i=1,2,3,\cdots) \tag{2.31}$$

Especially we have

$$|\, b_{2i}(0)\,| \;>\; |\, b_{2i}(1/2)\,| \;>\; 0 \tag{2.32}$$

where $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} n^{-s}$ $(\mathrm{Re}\ s > 1)$ *is Riemann zeta function.*

From the Parseval identity, we have the following lemma.

Lemma 2.16

$$\int_0^1 |\, b_i(x)\,|^2\, dx \;=\; \sum_{n\neq 0} (2\pi n)^{-2i} \;=\; 2\sum_{n=1}^{\infty} (2\pi n)^{-2i} \;=\; \frac{2}{(2\pi)^{2i}}\, \zeta(2i) \tag{2.33}$$

$$(i=1,2,3,\cdots)$$

Lemma 2.17

$$(-1)^{n+1}\, b_{2n+1}(x) \;=\; \int_0^{\frac{1}{2}} \left(x\wedge y \,-\, 2xy\right) (-1)^n\, b_{2n-1}(y)\, dy \;>\; 0 \tag{2.34}$$

$$(0<x<1/2, \quad n=1,2,3,\cdots)$$

Lemma 2.18 *For $n=1,2,3,\cdots,$ We have*

$$\max_{0\le x\le 1} |\, b_{2n}(x)\,| \;=\; (-1)^{n-1}\, b_{2n}(0) \tag{2.35}$$

3 Boundary value problem In this section, we treat the following boundary value problem.

BVP

$$\begin{cases} (-1)^M u^{(2M)} = f(x) & (0<x<1) \quad (3.1) \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le 2M-1) \quad (3.2) \\ \int_0^1 u(x)\,dx = 0 & \quad (3.3) \end{cases}$$

At first, we remark that the corresponding eigenvalue problem

EVP

$$\begin{cases} (-1)^M u^{(2M)} = \lambda\, u & (0<x<1) \quad (3.4) \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le 2M-1) \quad (3.5) \end{cases}$$

has an eigenvalue $\lambda = 0$. Corresponding normalized eigen function is given by

$$\varphi(x) = 1 \qquad (0<x<1) \tag{3.6}$$

The aim of this section is to prove the following theorem concerning the solvability of this BVP.

Theorem 3.1 *For any bounded continuous function $f(x)$ on an interval $0<x<1$, if there exists a classical solution $u(x)$ to* BVP *then we have*

$$\int_0^1 f(y)\,dy = \int_0^1 f(y)\,\varphi(y)\,dy = 0 \tag{3.7}$$

and

$$u(x) = \int_0^1 g(x,y)\,f(y)\,dy + const. \qquad (0<x<1) \tag{3.8}$$

where const. is a suitable constant and

$$g(x,y) = (-1)^M \frac{1}{2}\Bigg[E_{2M-1}(|x-y|) - (E_1,\cdots,E_{2M-1})(x)\, \breve{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(1-y) - (E_1,\cdots,E_{2M-1})(y)\, \breve{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(1-x) \Bigg] \tag{3.9}$$

We call the above function $g(x,y)$ the proto Green function.

Proof of Therome 3.1 If we introduce $2M$ new functions by

$$u_i(x) \,=\, u^{(i)}(x) \qquad (0 \le i \le 2M-1,\ 0 < x < 1) \tag{3.10}$$

then we have

$$\begin{cases} u_i' \,=\, u_{i+1} \qquad (0 \le i \le 2M-2) & (3.11) \\ u_{2M-1}' \,=\, (-1)^M f(x) & (3.12) \end{cases}$$

Introducing notations

$$\boldsymbol{u} = {}^t(u_0, \cdots, u_{2M-1}), \qquad \boldsymbol{e} = {}^t(0, \cdots, 0, 1), \quad \boldsymbol{N} = \Big(\, \delta_{i,j-1} \,\Big) \tag{3.13}$$

we have

$$\boldsymbol{u}' \,=\, \boldsymbol{N}\,\boldsymbol{u} \,+\, (-1)^M \boldsymbol{e}\, f(x) \qquad (0 < x < 1) \tag{3.14}$$

$\boldsymbol{N}$ is a nilpotent matrix. Solving (3.14), we have

$$\boldsymbol{u}(x) \,=\, \boldsymbol{E}(x)\,\boldsymbol{u}(0) \,+\, \int_0^x (-1)^M\,\boldsymbol{E}(x-y)\,\boldsymbol{e}\,f(y)\,dy \tag{3.15}$$

$$\boldsymbol{u}(x) \,=\, \boldsymbol{E}(x-1)\,\boldsymbol{u}(1) \,-\, \int_x^1 (-1)^M\,\boldsymbol{E}(x-y)\,\boldsymbol{e}\,f(y)\,dy \tag{3.16}$$

where $\boldsymbol{E}(x) \,=\, \exp(x\,\boldsymbol{N})$. Comparing the 0-th component of the above relations, we have

$$u_0(x) \,=\, u_0(0) \,+\, (E_1, \cdots, E_{2M-1})(x) \begin{pmatrix} u_1 \\ \vdots \\ u_{2M-1} \end{pmatrix}(0) + \\ \int_0^x (-1)^M\,E_{2M-1}(|x-y|)\,f(y)\,dy \tag{3.17}$$

$$u_0(x) \,=\, u_0(1) \,+\, (E_1, \cdots, E_{2M-1})(x-1) \begin{pmatrix} u_1 \\ \vdots \\ u_{2M-1} \end{pmatrix}(1) + \\ \int_x^1 (-1)^M\,E_{2M-1}(|x-y|)\,f(y)\,dy \tag{3.18}$$

From the boundary condition we have

$$\boldsymbol{u}(0) \,=\, \boldsymbol{u}(1) \,=\, \boldsymbol{E}(1)\,\boldsymbol{u}(0) \,+\, \int_0^1 (-1)^M\,\boldsymbol{E}(1-y)\,\boldsymbol{e}\,f(y)\,dy$$

$$\boldsymbol{u}(1) \,=\, \boldsymbol{u}(0) \,=\, \boldsymbol{E}(-1)\,\boldsymbol{u}(1) \,-\, \int_0^1 (-1)^M\,\boldsymbol{E}(-y)\,\boldsymbol{e}\,f(y)\,dy$$

These are rewritten equivalently in the following matrix forms.

$$\begin{pmatrix} 0 & E_1 & \cdots & E_{2M-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & E_1 \\ & & & 0 \end{pmatrix}(1) \begin{pmatrix} u_0 \\ \vdots \\ u_{2M-1} \end{pmatrix}(0) = -\int_0^1 (-1)^M \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_0 \end{pmatrix}(1-y)\, f(y)\, dy$$

$$\begin{pmatrix} 0 & E_1 & \cdots & E_{2M-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & E_1 \\ & & & 0 \end{pmatrix}(-1) \begin{pmatrix} u_0 \\ \vdots \\ u_{2M-1} \end{pmatrix}(1) = \int_0^1 (-1)^M \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_0 \end{pmatrix}(-y)\, f(y)\, dy$$

Noticing $E_0(x) = 1$, we have

$$\int_0^1 f(y)\, dy = 0 \tag{3.19}$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_{2M-1} \end{pmatrix}(0) = -\int_0^1 (-1)^M \breve{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(1-y)\, f(y)\, dy \tag{3.20}$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_{2M-1} \end{pmatrix}(1) = \int_0^1 (-1)^M \breve{\boldsymbol{E}}_1(-1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(-y)\, f(y)\, dy \tag{3.21}$$

Now we proved that the solvability condition (3.7) is a necessary condition for the existence of the classical solution to BVP.

From (3.17), (3.18), (3.20), (3.21) we have

$$u_0(x) = u_0(0) + \int_0^x (-1)^M E_{2M-1}(|x-y|)\, f(y)\, dy - \int_0^1 (-1)^M (E_1, \cdots, E_{2M-1})(x) \breve{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(1-y)\, f(y)\, dy \tag{3.22}$$

$$u_0(x) = u_0(1) + \int_x^1 (-1)^M E_{2M-1}(|x-y|)\, f(y)\, dy + \int_0^1 (-1)^M (E_1, \cdots, E_{2M-1})(x-1) \breve{\boldsymbol{E}}_1(-1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(-y)\, f(y)\, dy \tag{3.23}$$

Taking the average of (3.22) and (3.23), we have

$$u_0(x) \;=\; \frac{1}{2}\Big(u_0(0)+u_0(1)\Big) \;+\; \int_0^1 g(x,y)\,f(y)\,dy \tag{3.24}$$

where $g(x,y)$ is given by

$$\begin{aligned} g(x,y) \;=\; (-1)^M\,\frac{1}{2}\Bigg[\, & E_{2M-1}(|x-y|) \;- \\ & (E_1,\cdots,E_{2M-1})(x)\breve{\boldsymbol{E}}_1(1)\begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(1-y) + \\ & (E_1,\cdots,E_{2M-1})(x-1)\breve{\boldsymbol{E}}_1(-1)\begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(-y)\Bigg] \end{aligned} \tag{3.25}$$

Since

$$E_i(-x) \;=\; (-1)^i\,E_i(x) \qquad (i=0,1,2,\cdots)$$

we have

$$(E_1,\cdots,E_{2M-1})(x-1) \;=\; -\,(E_1,\cdots,E_{2M-1})(1-x)\begin{pmatrix} (-1)^i\delta_{i,j} \end{pmatrix} \tag{3.26}$$

$$\boldsymbol{E}_1(-1) \;=\; -\begin{pmatrix} (-1)^i\delta_{i,j} \end{pmatrix}\boldsymbol{E}_1(1)\begin{pmatrix} (-1)^i\delta_{i,j} \end{pmatrix} \tag{3.27}$$

$$\begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(-y) \;=\; -\begin{pmatrix} (-1)^i\delta_{i,j} \end{pmatrix}\begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix}(y) \tag{3.28}$$

Thus we have

$$(E_1,\cdots,E_{2M-1})(x-1)\,\breve{\boldsymbol{E}}_1(-1)\begin{pmatrix}E_{2M-1}\\ \vdots\\ E_1\end{pmatrix}(-y)\;=$$

$$-\,(E_1,\cdots,E_{2M-1})(1-x)\,\breve{\boldsymbol{E}}_1(1)\begin{pmatrix}E_{2M-1}\\ \vdots\\ E_1\end{pmatrix}(y)\;=$$

$$-\,(E_{2M-1},\cdots,E_1)(y)\,{}^t\breve{\boldsymbol{E}}_1(1)\begin{pmatrix}E_1\\ \vdots\\ E_{2M-1}\end{pmatrix}(1-x)\;=$$

$$-\,(E_1,\cdots,E_{2M-1})(y)\begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}{}^t\breve{\boldsymbol{E}}_1(1)\begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}\begin{pmatrix}E_{2M-1}\\ \vdots\\ E_1\end{pmatrix}(1-x)\;=$$

$$-\,(E_1,\cdots,E_{2M-1})(y)\,\breve{\boldsymbol{E}}_1(1)\begin{pmatrix}E_{2M-1}\\ \vdots\\ E_1\end{pmatrix}(1-x)$$

We note that $\breve{E}_1(1)\,=\,E_1(1)\,=\,1$. From Lemma 2.8, we have

$$\breve{E}_{i+1}(1)\,=\,b_i(0)\qquad(i=0,1,2,\cdots)\tag{3.29}$$

Finally we have obtained the expression

$$g(x,y)\,=\,(-1)^M\,\frac{1}{2}\Bigg[\,E_{2M-1}(|x-y|)\,-$$
$$(E_1,\cdots,E_{2M-1})(x)\,\breve{\boldsymbol{E}}_1(1)\begin{pmatrix}E_{2M-1}\\ \vdots\\ E_1\end{pmatrix}(1-y)\,-$$
$$(E_1,\cdots,E_{2M-1})(y)\,\breve{\boldsymbol{E}}_1(1)\begin{pmatrix}E_{2M-1}\\ \vdots\\ E_1\end{pmatrix}(1-x)\Bigg]\tag{3.30}$$

which completes the proof. ■

4 Proto Green function In this section, we show the following theorem concerning the proto Green function $g(x,y)$ introduced in the previous section.

Theorem 4.1 *If $f(x)$ is a bounded continuous function on an interval $0<x<1$, then*

$$u(x) = \int_0^1 g(x,y)\, f(y)\, dy \qquad (0<x<1) \tag{4.1}$$

satisfies

$$\begin{cases} (-1)^M u^{(2M)} = f(x) & (0<x<1) \quad (4.2) \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le 2M-2) \quad (4.3) \\ u^{(2M-1)}(1) - u^{(2M-1)}(0) = (-1)^M \int_0^1 f(y)\, dy & \quad (4.4) \end{cases}$$

Before proof, we present several expressions of $g(x,y)$.

Lemma 4.1 *Proto Green function $g(x,y)$ is expressed in the following 4 ways.*

(1) $$g(x,y) = (-1)^M \frac{1}{2} \Bigg[E_{2M-1}(|x-y|) - (E_1, \cdots, E_{2M-1})(x)\, \check{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix} (1-y) - (E_1, \cdots, E_{2M-1})(y)\, \check{\boldsymbol{E}}_1(1) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix} (1-x) \Bigg] = g(y,x) \qquad (0<x,y<1) \tag{4.5}$$

(2) $$g(x,y) = (-1)^M \frac{1}{2} \Bigg[E_{2M-1}(|x-y|) - (\, b_1(x)-b_1(0), \cdots, b_{2M-1}(x)-b_{2M-1}(0)\,) \begin{pmatrix} E_{2M-1} \\ \vdots \\ E_1 \end{pmatrix} (1-y) - (\, E_1, \cdots, E_{2M-1}\,)(y) \begin{pmatrix} b_{2M-1}(1-x)-b_{2M-1}(0) \\ \vdots \\ b_1(1-x)-b_1(0) \end{pmatrix} \Bigg]$$
$$(0<x,y<1) \tag{4.6}$$

(3) $$g(x,y) = (-1)^M \frac{1}{2}\Bigg[E_{2M-1}(|x-y|) - (b_1(x)-b_1(0),\cdots,b_{2M-1}(x)-b_{2M-1}(0))\, \boldsymbol{E}_1(1) \begin{pmatrix} b_{2M-1}(1-y)-b_{2M-1}(0) \\ \vdots \\ b_1(1-y)-b_1(0) \end{pmatrix} - (b_1(y)-b_1(0),\cdots,b_{2M-1}(y)-b_{2M-1}(0))\, \boldsymbol{E}_1(1) \begin{pmatrix} b_{2M-1}(1-x)-b_{2M-1}(0) \\ \vdots \\ b_1(1-x)-b_1(0) \end{pmatrix} \Bigg]$$
$$(0<x,y<1) \tag{4.7}$$

(4) $$g(x,y) = (-1)^{M-1}\Big[b_{2M}(|x-y|) - b_{2M}(x) - b_{2M}(y) + b_{2M}(0) \Big] \qquad (0<x,y<1) \tag{4.8}$$

Proof of Lemma 4.1 (1) was obtained in the previous section. (2) and (3) follow from Lemma 2.10. Applying Lemma 2.12 to (3), we have

$$\begin{aligned} g(x,y) &= (-1)^M \frac{1}{2}\Big[E_{2M-1}(|x-y|) - \\ &\quad b_{2M}(x+1-y) + b_{2M}(x) + b_{2M}(1-y) - b_{2M}(0) - \\ &\quad b_{2M}(y+1-x) + b_{2M}(y) + b_{2M}(1-x) - b_{2M}(0) \Big] = \\ &(-1)^M \frac{1}{2}\Big[E_{2M-1}(|x-y|) - b_{2M}(1+|x-y|) - b_{2M}(1-|x-y|) + \\ &\quad b_{2M}(x) + b_{2M}(1-x) + b_{2M}(y) + b_{2M}(1-y) - 2\,b_{2M}(0) \Big] \end{aligned}$$

Since we have

$$b_{2M}(1+|x-y|) = b_{2M}(|x-y|) + E_{2M-1}(|x-y|)$$

from Lemma 2.3 and

$$\begin{aligned} &b_{2M}(1-|x-y|) = b_{2M}(|x-y|), \\ &b_{2M}(1-x) = b_{2M}(x), \quad b_{2M}(1-y) = b_{2M}(y) \end{aligned}$$

from Lemma 2.2, we finally obtain

$$g(x,y) = (-1)^{M-1}\Big[b_{2M}(|x-y|) - b_{2M}(x) - b_{2M}(y) + b_{2M}(0) \Big]$$

(4) is proved. ■

Lemma 4.2

(1) $$g(x,y) = g(y,x) \qquad (0<x,y<1) \tag{4.9}$$

(2) $$\partial_x^i g(x,y)\Big|_{y=x-0} - \partial_x^i g(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases}$$

$$(0 < x < 1) \tag{4.10}$$

(3) $$\partial_x^i g(x,y)\Big|_{x=y+0} - \partial_x^i g(x,y)\Big|_{x=y-0} = \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases}$$

$$(0 < y < 1) \tag{4.11}$$

(4) $$g(1,y) = g(0,y) = 0 \qquad (0 < y < 1) \tag{4.12}$$

(5) $$\partial_x^{i+1} g(x,y)\Big|_{x=1} = \partial_x^{i+1} g(x,y)\Big|_{x=0} = (-1)^M \left[(-1)^i \, b_{2M-1-i}(y) + b_{2M-1-i}(0) \right]$$
$$(0 \le i \le 2M-3, \quad 0 < y < 1) \tag{4.13}$$

(6) $$\partial_x^{2M-1} g(x,y)\Big|_{x=1} = (-1)^M \left(b_1(y) + \frac{1}{2} \right) \tag{4.14}$$

$$\partial_x^{2M-1} g(x,y)\Big|_{x=0} = (-1)^M \left(b_1(y) - \frac{1}{2} \right) \tag{4.15}$$

$$\partial_x^{2M-1} g(x,y)\Big|_{x=1} - \partial_x^{2M-1} g(x,y)\Big|_{x=0} = (-1)^M \qquad (0 < y < 1) \tag{4.16}$$

(7) $$\partial_x^{2M} g(x,y) = 0 \qquad (0 < x, y < 1, \quad x \ne y) \tag{4.17}$$

Proof of Lemma 4.2 (1) is obvious.

From Lemma 4.1 (1), we have

$$\partial_x^i g(x,y)\Big|_{y=x-0} - \partial_x^i g(x,y)\Big|_{y=x+0} =$$
$$(-1)^M \frac{1}{2} \left[\partial_x^i E_{2M-1}(|x-y|)\Big|_{y=x-0} - \partial_x^i E_{2M-1}(|x-y|)\Big|_{y=x+0} \right] =$$
$$\begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases}$$

which shows (2). (3) follows from (2).

From Lemma 4.1 (4), we have

$$g(1,y) = (-1)^{M-1} \left[b_{2M}(1-y) - b_{2M}(1) - b_{2M}(y) + b_{2M}(0) \right] = 0$$

$$g(0,y) = (-1)^{M-1} \left[b_{2M}(y) - b_{2M}(0) - b_{2M}(y) + b_{2M}(0) \right] = 0$$

which proves (4).

Taking x-derivatives on both sides of Lemma 4.1 (4), we have

$$\partial_x^{i+1} g(x,y) \;=\; (-1)^{M-1} \left[(\,\mathrm{sgn}(x-y)\,)^{i+1}\, b_{2M-1-i}(|x-y|) - \, b_{2M-1-i}(x) \right]$$

$$(0 \le i \le 2M-2)$$

Especially if $i = 2M-2$,

$$\partial_x^{2M-1} g(x,y) \;=\; (-1)^{M-1} \left[\mathrm{sgn}(x-y)\, b_1(|x-y|) - \, b_1(x) \right]$$

Putting $x = 1$ and $x = 0$, we have (5) and (6).

Differentiating the above equality with respect to x, we obtain (7) and this completes the proof of Lemma 4.2. ■

Theorem 4.1 follows from Lemma 4.2.

5 Symmetric orthogonalization In this section, we construct Green function $G(x,y)$ of BVP under the following condition.

$$\int_0^1 f(x)dx \;=\; 0 \tag{5.1}$$

Finally we show that

$$u(x) \;=\; \int_0^1 G(x,y)\, f(y)\, dy \qquad (0 < x < 1) \tag{5.2}$$

is a true solution to BVP. Starting from a proto Green function $g(x,y)$, we can construct Green function $G(x,y)$ by the following formula.

$$\begin{aligned} G(x,y) \;=& g(x,y) \;-\; \varphi(x) \int_0^1 \varphi(x')\, g(x',y)\, dx' \;-\; \int_0^1 g(x,y')\, \varphi(y')\, dy'\, \varphi(y) \;+ \\ & \varphi(x) \int_0^1 \int_0^1 \varphi(x') g(x',y') \varphi(y')\, dy'\, dx'\, \varphi(y) \qquad (0 < x, y < 1) \end{aligned} \tag{5.3}$$

$\varphi(x)$ is the normalized eigenfunction of EVP in section 3 corresponding to the eigen value $\lambda = 0$. In this case we have $\varphi(x) = 1$ and $G(x,y)$ is expressed as follows.

$$\begin{aligned} G(x,y) \;=\; g(x,y) \;-\; \int_0^1 g(x',y)\, dx' \;-\; \int_0^1 g(x,y')\, dy' \;+\; \int_0^1 \int_0^1 g(x',y')\, dy'\, dx' \\ (0 < x, y < 1) \end{aligned} \tag{5.4}$$

Since $G(x,y)$ thus obtained has both symmetric and orthogonal properties, as is shown later in Theorem 5.1, we call this procedure gerenating $G(x,y)$ from $g(x,y)$ the symmetric orthogonalization method.

At first we show the following Lemma.

Lemma 5.1 *The function*

$$\psi(x) \;=\; \int_0^1 g(x,y)\, \varphi(y)\, dy \;=\; \int_0^1 g(x,y)\, dy \qquad (0 < x < 1) \tag{5.5}$$

is expressed as

$$\psi(x) \,=\, (-1)^M \Big[\, b_{2M}(x) -\, b_{2M}(0) \,\Big] \qquad (0<x<1) \tag{5.6}$$

and satisfies

$$\left\{ \begin{array}{l} (-1)^M \psi^{(2M)}(x) \,=\, \varphi(x) \,=\, 1 \qquad (0<x<1) \\ \psi^{(i)}(1) -\, \psi^{(i)}(0) \,=\, \left\{ \begin{array}{ll} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{array} \right. \end{array} \right. \tag{5.7}$$

Proof of Lemma 5.1 From Lemma 4.1 (4) we have

$$\begin{aligned} &\psi(x) \,=\, \int_0^1 g(x,y)\,dy \,= \\ &(-1)^{M-1} \int_0^1 \Big[\, b_{2M}(|x-y|) -\, b_{2M}(x) -\, b_{2M}(y) + b_{2M}(0) \,\Big]\,dy \,= \\ &(-1)^{M-1} \Big[\int_0^1 b_{2M}(|x-y|)\,dy -\, b_{2M}(x) + b_{2M}(0) - \int_0^1 b_{2M}(y)\,dy \Big] \end{aligned}$$

Noticing that

$$\begin{aligned} &\int_0^1 b_{2M}(|x-y|)\,dy \,=\, \int_0^x b_{2M}(x-y)\,dy + \int_x^1 b_{2M}(y-x)\,dy \,= \\ &-\, b_{2M+1}(x-y)\Big|_{y=0}^{y=x} + b_{2M+1}(y-x)\Big|_{y=x}^{y=1} \,= \\ &-\, b_{2M+1}(0) + b_{2M+1}(x) + b_{2M+1}(1-x) -\, b_{2M+1}(0) \,=\, 0 \\ &\int_0^1 b_{2M}(y)\,dy \,=\, 0 \end{aligned}$$

we have

$$\psi(x) \,=\, (-1)^M \Big[\, b_{2M}(x) -\, b_{2M}(0) \,\Big]$$

The latter half follows from Theorem 4.1. ■

Lemma 5.2

$$g_0 \,=\, \int_0^1 \int_0^1 g(x,y)\,dy\,dx \tag{5.8}$$

is expressed as

$$g_0 \,=\, \int_0^1 \psi(x)\,dx \,=\, (-1)^{M-1}\, b_{2M}(0) \tag{5.9}$$

Since the above lemma is shown through direct calculations, we omit its proof.

From Lemma 4.1(4), 5.1, 5.2 we have

$$G(x,y) \,=\, g(x,y) - \psi(x) - \psi(y) + g_0 \,=$$
$$(-1)^{M-1}\Big[\, b_{2M}(|x-y|) - b_{2M}(x) - b_{2M}(y) + b_{2M}(0) \,\Big] \,-$$
$$(-1)^{M}(\, b_{2M}(x) - b_{2M}(0)\,) - (-1)^{M}(\, b_{2M}(y) - b_{2M}(0)\,) + (-1)^{M-1} b_{2M}(0) \,=$$
$$(-1)^{M-1}\, b_{2M}(|x-y|)$$

The next Theorem shows that $G(x,y)$ is Green function of BVP under solvability condition (5.1).

Theorem 5.1

$$G(x,y) \,=\, (-1)^{M-1}\, b_{2M}(|x-y|) \qquad (0<x,y<1) \tag{5.10}$$

has the following properties.

(1) $$G(x,y) \,=\, G(y,x) \qquad (0<x,y<1) \tag{5.11}$$

(2) $$\partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} \,=\, \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases}$$
$$(0<x<1) \tag{5.12}$$

(3) $$\partial_x^i G(x,y)\Big|_{x=y+0} - \partial_x^i G(x,y)\Big|_{x=y-0} \,=\, \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases}$$
$$(0<y<1) \tag{5.13}$$

(4) $$\partial_x^i G(x,y)\Big|_{x=1} \,=\, \partial_x^i G(x,y)\Big|_{x=0} \,=\, (-1)^{M-1+i}\, b_{2M-i}(y) \tag{5.14}$$
$$(0<y<1, \quad 0 \le i \le 2M-1)$$

(5) $$\partial_x^{2M} G(x,y) \,=\, (-1)^{M-1} \qquad (0<x,y<1, \quad x \ne y) \tag{5.15}$$

(6) $$\int_0^1 G(x,y)\, dx \,=\, 0 \tag{5.16}$$

Proof of Theorem 5.1 (1) is obvious.
Since

$$\partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} \,=\, \partial_x^i g(x,y)\Big|_{y=x-0} - \partial_x^i g(x,y)\Big|_{y=x+0}$$

then (2) follows from Lemma 4.2 (2). (3) follows from (2).

Differentiating (5.10) i times with respect to x we have

$$\partial_x^i G(x,y) \;=\; (-1)^{M-1}(\,\mathrm{sgn}(x-y)\,)^i b_{2M-i}(|x-y|) \tag{5.17}$$

(4) follows from the following facts.

$$\partial_x^i G(x,y)\Big|_{x=1} \;=\; (-1)^{M-1} b_{2M-i}(1-y) \;=\; (-1)^{M-1-i} b_{2M-i}(y)$$

$$\partial_x^i G(x,y)\Big|_{x=0} \;=\; (-1)^{M-1+i} b_{2M-i}(y)$$

(5) and (6) are obvious. ■

From Theorem 5.1, we have the following existence theorem of solution to BVP.

Theorem 5.2 *For any bounded continuous function $f(x)$ on an interval $0<x<1$ which satisfies the solvability condition* (3.7)

$$u(x) \;=\; \int_0^1 G(x,y)\,f(y)\,dy \qquad (0<x<1) \tag{5.18}$$

is the solution to BVP.

From Lemma 2.14, we have the following conclusion.

Theorem 5.3

$$G(x,y) \;=\; (-1)^{M-1}\, b_{2M}(|x-y|) \;=\; 2\sum_{n=1}^{\infty}(2\pi n)^{-2M}\cos(\,2\pi n(x-y)\,) \tag{5.19}$$

$$(0<x,y<1)$$

Especially its diagonal part is given by

$$G(y,y) \;=\; (-1)^{M-1}\, b_{2M}(0) \;=\; 2\sum_{n=1}^{\infty}(2\pi n)^{-2M} \;=\; \frac{2}{(2\pi)^{2M}}\,\zeta(2M) \tag{5.20}$$

$$(0<y<1)$$

6 Reproducing kernel Green function $G(x,y)$ is at the same time a reproducing kernel for suitable set of a Hilbert space H_M and its inner product $(\cdot,\cdot)_M$.

Theorem 6.1 *For function space*

$$H_M \;=\; \Big\{ u(x) \,\Big|\, u^{(M)}(x)\in L^2(0,1), \quad u^{(i)}(1)-u^{(i)}(0)=0 \quad (0\le i\le M-1), \quad \int_0^1 u(x)dx = 0 \Big\} \tag{6.1}$$

a sesquilinear form

$$(u,v)_M \;=\; \int_0^1 u^{(M)}(x)\,\overline{v}^{(M)}(x)\,dx \tag{6.2}$$

is an inner product.

Proof of Theorem 6.1 By Fourier series expansion, we have

$$u(x) = \sum_{j\in\mathbf{Z}} \widehat{u}(j)\, \exp\bigl(\sqrt{-1}\,2\pi jx\bigr) \tag{6.3}$$

where

$$\widehat{u}(j) = \int_0^1 u(y)\, \exp\bigl(-\sqrt{-1}\,2\pi jy\bigr)\, dy \qquad (\, j\in\mathbf{Z}\,) \tag{6.4}$$

Especially

$$\widehat{u}(0) = \int_0^1 u(y)\, dy = 0 \tag{6.5}$$

The right hand side of the above expansion converges in $L^2(0,1)$. Differentiating (6.3) M times termwise with respect to x, we have

$$u^{(M)}(x) = \sum_{|j|\geq 1} \bigl(\sqrt{-1}\,2\pi j\,\bigr)^M\, \widehat{u}(j)\, \exp\bigl(\sqrt{-1}\,2\pi jx\bigr) \tag{6.6}$$

From Parseval equality we have

$$(u,u)_M = \int_0^1 \Bigl|\, u^{(M)}(x) \Bigr|^2 dx = \sum_{|j|\geq 1} (\, 2\pi|j|\,)^{2M}\, |\, \widehat{u}(j)\,|^2 \tag{6.7}$$

The above equality implies that $(u,u)_M = 0$ holds if and only if $\widehat{u}(j) = 0$ $(\, j\in\mathbf{Z}\,)$, that is to say $u(x)\equiv 0$. This shows that $(\cdot,\cdot)_M$ is an inner product in H_M. ■

From Theorem 5.1 and Theorem 6.1, we can show the following fact.

Theorem 6.2 (1) *If* $u^{(M)}(x) \in L^2(0,1)$ *then we have*

$$\int_0^1 u^{(M)}(x)\, \partial_x^M G(x,y)\, dx = u(y) - \int_0^1 u(x)\, dx + \sum_{j=0}^{M-1} (-1)^{M-1-j} \Bigl(u^{(j)}(1) - u^{(j)}(0) \Bigr)\, \partial_x^{2M-1-j} G(x,y) \Big|_{x=0} \qquad (0\leq y\leq 1) \tag{6.8}$$

Or equivalently

$$-\int_0^y u^{(M)}(x)\, b_M(y-x)\, dx - (-1)^M \int_y^1 u^{(M)}(x)\, b_M(x-y)\, dx = u(y) - \int_0^1 u(x)\, dx - \sum_{j=0}^{M-1} \Bigl(u^{(j)}(1) - u^{(j)}(0) \Bigr)\, b_{j+1}(y) \qquad (0\leq y\leq 1) \tag{6.9}$$

(2) *If* $u(x)\in H_M$ *then we have the following reproducing relation.*

$$\int_0^1 u^{(M)}(x)\, \partial_x^M G(x,y)\, dx = u(y) \qquad (0\leq y\leq 1) \tag{6.10}$$

Theorem 6.2 (2) shows that the Green function $G(x, y)$ is a reproducing kernel in a Hilbert space H_M with an inner product $(\cdot, \cdot)_M$.

Proof of Theorem 6.2 For functions $u = u(x)$ and $v = v(x) = G(x, y)$ with y arbitrarily fixed in $0 < y < 1$, we have

$$u^{(M)}\,\overline{v}^{(M)} \,-\, (-1)^M\, u\,\overline{v}^{(2M)} \,=\, \left(\sum_{j=0}^{M-1} (-1)^{M-1-j}\, u^{(j)}\, \overline{v}^{(2M-1-j)} \right)'$$

Integrating this with respect to x on intervals $0 < x < y$ and $y < x < 1$, we have

$$\begin{aligned}
&\int_0^1 u^{(M)}(x)\,\overline{v}^{(M)}(x)\,dx \,-\, \int_0^1 (-1)^M\, u(x)\,\overline{v}^{(2M)}(x)\,dx \,= \\
&\left[\sum_{j=0}^{M-1} (-1)^{M-1-j}\, u^{(j)}(x)\,\overline{v}^{(2M-1-j)}(x) \right] \left\{ \Big|_{x=0}^{x=y-0} + \Big|_{x=y+0}^{x=1} \right\} = \\
&\sum_{j=0}^{M-1} (-1)^{M-1-j} \Big(u^{(j)}(1) \,-\, u^{(j)}(0) \Big)\, \overline{v}^{(2M-1-j)}(0) \,+ \\
&\sum_{j=0}^{M-1} (-1)^{M-1-j}\, u^{(j)}(y) \Big(\overline{v}^{(2M-1-j)}(y-0) \,-\, \overline{v}^{(2M-1-j)}(y+0) \Big) \,=\, u(y)
\end{aligned}$$

Using Theorem 5.1 we have (6.8). (6.10) follows at once from (6.8). Theorem 6.2 is proved. ■

Another proof of Theorem 6.1 Applying Schwarz inequality to (6.10), we have

$$|\,u(y)\,|^2 \,\le\, \int_0^1 \left| u^{(M)}(x) \right|^2 dx \int_0^1 \left|\, \partial_x^M\, G(x,y)\,\right|^2 dx$$

Putting $u(x) = G(x, y)$ in (6.10), we have

$$\int_0^1 \left|\, \partial_x^M\, G(x,y)\,\right|^2 dx \,=\, G(y,y) \,=\, (-1)^{M-1}\, b_{2M}(0)$$

and therefore

$$|\,u(y)\,|^2 \,\le\, (-1)^{M-1}\, b_{2M}(0) \int_0^1 \left| u^{(M)}(x) \right|^2 dx$$

This inequality implies that $(u, u)_M \,=\, \int_0^1 \left| u^{(M)}(x) \right|^2 dx \,=\, 0$ holds if and only if $u(x) \equiv 0$, which shows that the form $(\cdot, \cdot)_M$ is an inner product in H_M. ■

7 The best constant of Sobolev inequality In this section, we prove the main Theorem 1.2 in this paper.

Proof of Theorem 1.2 As shown at the final part in the previous section, the inequality

$$\left(\sup_{0 \le y \le 1} |\,u(y)\,| \right)^2 \,\le\, \max_{0 \le y \le 1} G(y,y) \int_0^1 \left| u^{(M)}(x) \right|^2 dx \tag{7.1}$$

holds for any function $u(x) \in H_M$. Since $G(y,y) = (-1)^{M-1} b_{2M}(0)$ does not depend on y, for any y_0 satisfying $0 \le y_0 \le 1$ we have

$$\max_{0\le y\le 1} G(y,y) \;=\; G(y_0,y_0) \;=\; (-1)^{M-1} b_{2M}(0) \tag{7.2}$$

Now we showed that for any function $u(x) \in H_M$ there exists a positive constant C which is not depend on $u(x)$ such that the following Sobolev inequality holds.

$$\left(\sup_{0\le y\le 1} |\, u(y) \,| \right)^2 \;\le\; C \int_0^1 \left| u^{(M)}(x) \right|^2 dx \tag{7.3}$$

We denote by C_M the best constant among such C. The above inequality (7.1) shows that

$$C_M \;\le\; G(y_0,y_0) \tag{7.4}$$

Applying the Sobolev inequality (7.3) with $C = C_M$ to a special function $u(x) = G(x,y_0)$, we have

$$\left(\sup_{0\le y\le 1} |\, G(y,y_0) \,| \right)^2 \;\le\; C_M G(y_0,y_0)$$

Now we have the trivial relation

$$G(y_0,y_0)^2 \;\le\; \left(\sup_{0\le y\le 1} |\, G(y,y_0) \,| \right)^2$$

Combining these two inequalities we have

$$G(y_0,y_0)^2 \;\le\; \left(\sup_{0\le y\le 1} |\, G(y,y_0) \,| \right)^2 \;\le\; C_M\, G(y_0,y_0)$$

Combining this with (7.4), we finally have

$$C_M = G(y_0,y_0) \tag{7.5}$$

and

$$\left(\sup_{0\le y\le 1} |\, G(y,y_0) \,| \right)^2 \;=\; C_M\, G(y_0,y_0) \;=\; C_M \int_0^1 \left|\, \partial_x^M G(x,y_0) \,\right|^2 dx \tag{7.6}$$

If we replace C by C_M in the Sobolev inequality (7.3), the equality holds for $u(x) = G(x,y_0)$. This completes the proof of Theorem 1.2. ■

We give some interesting properties of the best constant C_i.

Lemma 7.1 *The best constant*

$$C_i \;=\; (-1)^{i-1}\, b_{2i}(0) \;=\; \frac{B_i}{(2i)!} \qquad (i = 0,1,2,\cdots) \tag{7.7}$$

which appeared in Theorem 1.1 satisfies the following recurrence relations.

$$\begin{cases} C_0 + \displaystyle\sum_{j=1}^{n-1} (-1)^j\, 2n(2n-1)\cdots(2n-2j+1)\, C_j \;=\; -\,n \qquad (n = 2,3,\cdots) \\ C_0 \;=\; -1 \end{cases} \tag{7.8}$$

The above recurrence relation is a direct consequence of (1.5). From this relation, we can easily calculate exact values of the best constants C_i.

$$
\begin{aligned}
C_1 &= \frac{1}{12}, \quad C_2 = \frac{1}{720}, \quad C_3 = \frac{1}{30240}, \quad C_4 = \frac{1}{1209600}, \\
C_5 &= \frac{1}{47900160}, \quad C_6 = \frac{691}{1307674368000}, \quad C_7 = \frac{1}{74724249600}, \\
C_8 &= \frac{3617}{10670622842880000}, \quad \cdots
\end{aligned}
$$

Lemma 7.2 *A generating function of C_i is given by*

$$\frac{-1}{t^{-1}\tan(t)} = \sum_{j=0}^{\infty} C_j\,(2t)^{2j} \qquad (\,|\,t\,| < \pi\,) \tag{7.9}$$

8 Application of Theorem 1.2 In this section, we consider an application of Theorem 1.2. We start with the following theorem, which is a special case $M = 1$ of Theorem 1.2.

Theorem 8.1 *For every function $u(x) \in H_1$, we have a suitable positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$\left(\sup_{0 \le y \le 1} |u(y)| \right)^2 \le C \int_0^1 |u'(x)|^2\,dx \tag{8.1}$$

Among such C the best constant is $C_1 = 1/12$. In the above inequality if we replace C by C_1, the equality holds for a special function $u(x) = const.\, b_2(\,|x-y|\,)$, where y is an arbitrary fixed number satisfying $0 \le y \le 1$.

For $a_i \ge 0$ $(0 \le i \le n-1,\ n = 1, 2, 3, \cdots)$, we assume that

$$u(x) = \sum_{i=0}^{n-1} \frac{i+2}{i+1}\, a_i \left[|2x-1|^{i+1} - \frac{1}{i+2} \right] \qquad (0 < x < 1) \tag{8.2}$$

Suppose $a_i = 0$ $(i \ne 1)$, then we have $u(x) = 12 a_1 b_2(x)$. $u(x)$ satisfies $u(1) - u(0) = 0$ and $\int_0^1 u(x)dx = 0$. That is to say $u(x) \in H_1$.

Since

$$u'(x) = 2 \sum_{i=0}^{n-1} (i+2)\, a_i\, (2x-1)^i \ge 0 \qquad (1/2 \le x \le 1)$$

and

$$\left| u(1/2) \right| = -u(1/2) = \sum_{i=0}^{n-1} \frac{1}{i+1}\, a_i \le \sum_{i=0}^{n-1} a_i = u(1)$$

we have

$$\sup_{0 \le y \le 1} |\,u(y)\,| = u(1)$$

On the other hand, we have

$$\int_0^1 |\,u'(x)\,|^2\,dx \;=\; 8\int_{1/2}^1 \left(\sum_{i=0}^{n-1}(i+2)\,a_i\,(2x-1)^i\right)^2 dx \;=$$
$$4\int_0^1 \left(\sum_{i=0}^{n-1}(i+2)\,a_i\,y^i\right)^2 dy \;=\; 4\int_0^1 \sum_{i,j=0}^{n-1}(i+2)\,(j+2)\,a_i\,a_j\,y^{i+j}dy \;=$$
$$4\sum_{i,j=0}^{n-1}\frac{(i+2)\,(j+2)}{i+j+1}\,a_i\,a_j$$

Applying (8.1) to $u(x)$ (8.2), we have a nontrivial inequality.

Theorem 8.2 *For $a_i \geq 0$ $(0 \leq i \leq n-1, n=1,2,\cdots)$, the following inequality holds.*

$$\left(\sum_{i=0}^{n-1} a_i\right)^2 \;\leq\; \frac{1}{3}\sum_{i,j=0}^{n-1}\frac{(i+2)\,(j+2)}{i+j+1}\,a_i\,a_j \tag{8.3}$$

Proof of Theorem 8.2 Although the above theorem is a direct consequence of Sobolev inequality, we prove the theorem directly. It is enough to prove the following inequality.

$$\sum_{i,j=0}^{n-1}\left(\frac{(i+2)\,(j+2)}{i+j+1} \;-\; 3\right)a_i\,a_j \;\geq\; 0 \tag{8.4}$$

Left hand side of the above inequality is calculated as follows.

$$\sum_{i,j=0}^{n-1}\frac{(i-1)\,(j-1)}{i+j+1}\,a_i\,a_j \;=$$
$$a_0^2 \;-\; 2a_0\sum_{i=2}^{n-1}\frac{i-1}{i+1}\,a_i \;+\; \sum_{i,j=2}^{n-1}\frac{(i-1)\,(j-1)}{i+j+1}\,a_i\,a_j \;=$$
$$\left(a_0 \;-\; \sum_{i=2}^{n-1}\frac{i-1}{i+1}\,a_i\right)^2 \;+\; \sum_{i,j=2}^{n-1}(i-1)\,(j-1)\left(\frac{1}{i+j+1} \;-\; \frac{1}{(i+1)(j+1)}\right)a_i\,a_j$$
$$\left(a_0 \;-\; \sum_{i=2}^{n-1}\frac{i-1}{i+1}\,a_i\right)^2 \;+\; \sum_{i,j=2}^{n-1}\frac{ij(i-1)\,(j-1)}{(i+j+1)(i+1)(j+1)}\,a_i\,a_j \;\geq\; 0$$

The equality in (8.3) holds if and only if $a_i = 0$ $(i \neq 1)$. ■

9 Appendix : Proofs of Lemmas in section 2 In this appendix, we give proofs of some key lemmas concerning Bernoulli polynomials in section 2.

Proof of Lemma 2.8 Replacing x by ξ in (2.13) of Lemma 2.7 and integrating it with respect to ξ on an interval $0<\xi<x$, we have

$$\begin{pmatrix} E_1 & \cdots & E_{n+1} \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix}(x) =$$

$$\begin{pmatrix} b_1(x)-b_1(0) & \cdots & b_{n+1}(x)-b_{n+1}(0) \\ & \ddots & \vdots \\ & & b_1(x)-b_1(0) \end{pmatrix} \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}^{-1}(0)$$

Putting $x=1$ and applying Lemma 2.4, we have

$$\begin{pmatrix} E_1 & \cdots & E_{n+1} \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix} = \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}^{-1}(0)$$

This proves Lemma 2.8. ■

Proof of Lemma 2.9 From Lemma 2.7, 2.8 we have

$$\boldsymbol{E}(x) = \begin{pmatrix} b_0 & \cdots & b_n \\ & \ddots & \vdots \\ & & b_0 \end{pmatrix}(x) \begin{pmatrix} E_1 & \cdots & E_{n+1} \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix}$$

Taking 0-th row we have

$$\begin{pmatrix} E_0, & \cdots, & E_n \end{pmatrix}(x) = \begin{pmatrix} b_0, & \cdots, & b_n \end{pmatrix}(x) \begin{pmatrix} E_1 & \cdots & E_{n+1} \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix} \tag{9.1}$$

or equivalently

$$\begin{pmatrix} E_1 & & \\ \vdots & \ddots & \\ E_{n+1} & \cdots & E_1 \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix}(x) = \begin{pmatrix} E_0 \\ \vdots \\ E_n \end{pmatrix}(x)$$

Solving the above equation with respect to $b_n(x)$, we have the expression in Lemma 2.9 from Cramer formula. ■

Proof of Lemma 2.10 Replacing x by ξ in (9.1) and integrating it with respect to ξ on an interval $0<\xi<x$, we have (2.19). ■

Proof of Lemma 2.12 From Lemma 2.11 we have

$$b_n(x+y) = \begin{pmatrix} b_0, & \cdots, & b_n \end{pmatrix}(x) \begin{pmatrix} E_1 & \cdots & E_{n+1} \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix} \begin{pmatrix} b_n \\ \vdots \\ b_0 \end{pmatrix}(y)$$

Replacing (n,x,y) by $(n-1,\xi,\eta)$ we have

$$b_{n-1}(\xi+\eta) = (\, b_0,\ \cdots\ , b_{n-1}\,)(\xi) \begin{pmatrix} E_1 & \cdots & E_n \\ & \ddots & \vdots \\ & & E_1 \end{pmatrix} \begin{pmatrix} b_{n-1} \\ \vdots \\ b_0 \end{pmatrix}(\eta)$$

If we integrate this with respect to ξ on an interval $0<\xi<x$, with respect to η on an interval $0<\eta<y$ then we have

$$\int_0^x \int_0^y b_{n-1}(\xi+\eta)\, d\eta\, d\xi = \int_0^x \Big(b_n(\xi+y) - b_n(\xi) \Big) d\xi =$$
$$b_{n+1}(x+y) - b_{n+1}(x) - b_{n+1}(y) + b_{n+1}(0)$$

■

Proof of Lemma 2.17 For

$$f(x) = -\,b_{2n-1}(x), \qquad u(x) = b_{2n+1}(x)$$

we have

$$\begin{cases} -\,u'' = f(x) & (0<x<1/2) \\ u(0) = u(1/2) = 0 \end{cases}$$

Using Green function

$$x \wedge y - 2xy = \min\{\,x,y\,\} - 2xy > 0 \qquad (0<x,y<1/2)$$

we have the formula (2.34). Starting with

$$-\,b_1(x) > 0 \qquad (0<x<1/2)$$

we can show

$$(-1)^{n+1}\, b_{2n+1}(x) > 0 \qquad (0<x<1/2, \quad n=1,2,3,\cdots)$$

by induction. ■

Proof of Lemma 2.18 Owing to Lemma 2.17, we have

$$\frac{d}{dx}\,(-1)^{n-1}\, b_{2n}(x) = (-1)^{n-1}\, b_{2n-1}(x) < 0 \qquad (0<x<1/2)$$

Considering that $b_{2n}(x)=b_{2n}(1-x)$, the function $(-1)^{n-1}\, b_{2n}(x)$ behaves as follows.

$$\begin{cases} (-1)^{n-1}\, b_{2n}(0) > 0 & (x=0) \\ \textit{monotone decreasing} & (0<x<1/2) \\ (-1)^{n-1}\, b_{2n}(1/2) < 0 & (x=1/2) \\ \textit{monotone increasing} & (1/2<x<1) \\ (-1)^{n-1}\, b_{2n}(1) = (-1)^{n-1}\, b_{2n}(0) > 0 & (x=1) \end{cases}$$

from which we have Lemma 2.18. ■

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