EXPLICIT FORMULAS FOR THE REPRODUCING KERNELS OF THE SPACE OF HARMONIC POLYNOMIALS IN THE CASE OF CLASSICAL REAL RANK 1

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Abstract. We give the explicit formulas of the reproducing kernels of the space of harmonic polynomials of $\mathfrak{p} \subset \mathfrak{g}$ in the case of classical real rank 1, which are generalizations of the well-known reproducing formulas of classical harmonic polynomials on the unit sphere or any other $SO(p)$-orbits in $\mathbf{C}^p$. These formulas are expressed as integrals on a single orbit, simplifying our previous results that are expressed as double integrals on some family of nilpotent orbits.

Introduction.

For harmonic functions on $\mathbf{R}^p$ there are many studies. Especially, the following reproducing formula on the unit sphere S^{p-1} is well-known:

$$\delta_{n,m} f(s) = \dim H_{n,p} \int_{S^{p-1}} f(s_1) P_{n,p}(s \cdot s_1) ds_1 \qquad (s \in S^{p-1}, f \in H_{m,p}),$$

where $H_{n,p}$ is the space of spherical harmonics of degree n in dimension p, and $P_{n,p}(t) = \frac{(p-3)!n!}{(n+p-3)!} C_n^{\frac{p-2}{2}}(t)$ is the Legendre polynomial of degree n in dimension p and $C_n^\nu(t)$ is the Gegenbauer function (cf. [1], [7], [8], [11], etc). We denote by $H_n(\mathbf{C}^p)$ the space of polynomials f on $\mathbf{C}^p$ of degree n which satisfy $\sum_{j=1}^p \frac{\partial^2}{\partial {z_j}^2} f = 0$. Then homogeneous harmonic polynomials on $\mathbf{R}^p$ of degree n are uniquely extended to the element of $H_n(\mathbf{C}^p)$. The reproducing formulas of $H_n(\mathbf{C}^p)$ on any non-trivial $SO(p)$-orbit in $\mathbf{C}^p$ are also known in addition to the above case S^{p-1} (cf. [2], [9], [10], [17], [21]). For details on harmonic polynomials and harmonic functions on $\mathbf{R}^p$, see also [15], [16].

In this paper, we further generalize these formulas from the Lie algebraic standpoint in a unified way. According to the formulation of [5], harmonic polynomials on $\mathbf{R}^p$ can be canonically identified with harmonic polynomials on the vector space $\mathfrak{p}$, where $\mathfrak{p}$ is the complexification of $\mathfrak{p}_\mathbf{R}$ appeared in a Cartan decomposition of the Lie algebra $\mathfrak{g}_\mathbf{R} = \mathfrak{so}(p,1)$.

In this situation, any $SO(p)$-orbit in $\mathbf{C}^p$ corresponds to a $K_\mathbf{R}$-orbit in $\mathfrak{p}$, where $K_\mathbf{R}$ is a Lie subgroup of $GL(\mathfrak{p})$ generated by $\exp \operatorname{ad} X$ $(X \in \mathfrak{k}_\mathbf{R})$. Thus, the integral formulas of harmonic polynomials on $\mathbf{R}^p$ can be rewritten explicitly as integral representation formulas on each $K_\mathbf{R}$-orbits (cf. Appendix of [18]).

In [20] we generalize these formulas to the case where the Lie algebra $\mathfrak{g}_\mathbf{R}$ is real rank 1: i.e. $\mathfrak{g}_\mathbf{R} = \mathfrak{so}(p,1)$ $(p \geq 2)$, $\mathfrak{su}(p,1)$, $\mathfrak{sp}(p,1)$ $(p \geq 1)$ or $\mathfrak{f}_{4(-20)}$ by constructing the reproducing kernels for each case (cf. Theorems 1.2 and 1.3). We denote by $\mathfrak{g}_\mathbf{R} = \mathfrak{k}_\mathbf{R} + \mathfrak{p}_\mathbf{R}$ a Cartan decomposition of $\mathfrak{g}_\mathbf{R}$ and put $K_\mathbf{R} = \exp \operatorname{ad} \mathfrak{k}_\mathbf{R}$. In [20] we express these formulas as integrals of the reproducing kernels on a single $K_\mathbf{R}$-orbit in a unified way, simplifying the

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formulas previously obtained in [18], [19], where the integral formulas for two cases $\mathfrak{su}(p,1)$ and $\mathfrak{sp}(p,1)$ are expressed in the form of double integrals on some family of nilpotent $K_{\mathbf{R}}$-orbits. In particular the reproducing kernels are expressed in simple forms for nilpotent orbits . In this paper we give a complete proof of these results for the case $\mathfrak{sp}(p,1)$ which is omitted in [20], together with that of the case $\mathfrak{su}(p,1)$ for the sake of completeness.

Concerning reproducing formulas, the results of Nagel-Rudin [12] and Rudin [13] are also known. Their results correspond to our formula for the case $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$. Let $\widetilde{H}_n(\mathbf{C}^p)$ be the space of homogeneous polynomials f on $\mathbf{C}^p \cong \mathbf{R}^{2p}$ of degree n in the variables $z_1, z_2, \cdots, z_p, \overline{z}_1, \overline{z}_2, \cdots, \overline{z}_p$ which satisfy $\sum_{j=1}^{p} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} f = 0$. For nonnegative integers k and l we denote by $S^{k,l}$ the space of polynomials on $\mathbf{C}^p$ which have total degree k in the variables $z_1, z_2, \cdots, z_p$ and total degree l in the variables $\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_p$. Set $H^{k,l} = S^{k,l} \cap \widetilde{H}_{k+l}(\mathbf{C}^p)$. Then the Lie group $U(p)$ naturally acts on the space $\widetilde{H}_n(\mathbf{C}^p)$, and $H^{k,n-k}$ is a $U(p)$-invariant subspace of $\widetilde{H}_n(\mathbf{C}^p)$. The sum $\widetilde{H}_n(\mathbf{C}^p) = \bigoplus_{k=0}^{n} H^{k,n-k}$ gives the $U(p)$-irreducible decomposition (cf. [16]). And the reproducing formulas of $H^{k,l}$ on the unit sphere $\{z \in \mathbf{C}^p ; {}^t z \overline{z} = 1\}$ of $\mathbf{C}^p$ are explained in detail in [12], [13]. In this setting the element of $\widetilde{H}_n(\mathbf{C}^p)$ corresponds to a harmonic polynomial on $\mathfrak{p}$ and the unit sphere of $\mathbf{C}^p$ corresponds to one $K_{\mathbf{R}}$-orbit for the case $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$.

The plan of this paper is roughly stated as follows: In § 1 we recall the definitions and some fundamental properties of harmonic polynomials on $\mathfrak{p}$ which we use in this paper, mainly following the results stated in [20]. In § 2 we review the principal results for the case $\mathfrak{su}(p,1)$, which is previously stated in [20]. In § 3 – § 5 we consider the case $\mathfrak{sp}(p,1)$. In § 3 we review some known results on harmonic polynomials on $\mathfrak{p}$ when $\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p,1)$. In § 4 we give the $K_{\mathbf{R}}$-irreducible decomposition of harmonic polynomials on $\mathfrak{p}$, which is the principal part of this paper and state the main theorem (Theorem 4.5) by using the properties of $K_{\mathbf{R}}$-irreducible components. Finally in Appendix, we determine the dimension of the $K_{\mathbf{R}}$-irreducible component.

Thus, we obtain the reproducing kernels on each $K_{\mathbf{R}}$-orbit for all cases of classical real rank 1. To obtain integral formulas of harmonic polynomials in cases of classical real rank 2 is our next theme.

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1. Harmonic polynomials on $\mathfrak{p}$.

In this section we fix several notations which we use in this paper, and recall the definitions and the known results on harmonic polynomials.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_{\mathbf{R}}$ be a noncompact real form of $\mathfrak{g}$, $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ be a Cartan decomposition of $\mathfrak{g}_{\mathbf{R}}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its complexification. We put $G = \exp \operatorname{ad} \mathfrak{g}$ and $K_\theta = \{g \in G\,;\, \theta g = g\theta\}$, where the involution $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined by $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$. Let K be the identity component of K_θ. Then we have $K = \exp \operatorname{ad} \mathfrak{k}$.

Now we define harmonic polynomials on $\mathfrak{p}$ as follows. We denote by S and S_n the spaces of polynomials on $\mathfrak{p}$ and homogeneous polynomials on $\mathfrak{p}$ of degree n, respectively. For $f \in S$ and $g \in K_\theta$, we define $gf \in S$ by $(gf)(X) = f(g^{-1}X)$ $(X \in \mathfrak{p})$. We denote by J the ring of K-invariant polynomials on $\mathfrak{p}$ and put $J_+ = \{f \in J\,;\, f(0) = 0\}$. It is known that J is also K_θ-invariant. According to the definition in [5], a polynomial $f \in S$ is called harmonic if and only if $(\partial P)f = 0$ for any $P \in J_+$. We denote by $\mathcal{H}_n$ the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree n. In the following we put $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$. The following results are well known:

Theorem 1.1 (cf. [1], [5]). (i) *For any* $n \in \mathbf{Z}_+$ *we have*

$$S_n = (J_+S)_n \oplus \mathcal{H}_n,$$

where we put $(J_+S)_n = S_n \cap J_+S$.

(ii) *We put* $\mathcal{N} = \{X \in \mathfrak{p}\,;\, P(X) = 0 \text{ for any } P \in J_+\}$ *and let* $h(X,Y)$ *be a nondegenerate symmetric bilinear form on* $\mathfrak{p}$. *Then* $\mathcal{H}_n$ *is generated by* $\{h(\ \ ,Z)^n\,;\, Z \in \mathcal{N}\}$.

(iii) *Let* $\mathcal{O}$ *be a* K_θ*-orbit in* $\mathfrak{p}$ *of maximal dimension. Then the restriction mapping* $f \longrightarrow f|_{\mathcal{O}}$ *is a bijection from* $\mathcal{H}_n$ *onto* $\mathcal{H}_n|_{\mathcal{O}}$.

For further properties on harmonic polynomials on $\mathfrak{p}$, see [1], [5].

From now we consider the case where $\mathfrak{g}_\mathbf{R}$ is a classical simple Lie algebra with real rank 1, i.e. $\mathfrak{g}_\mathbf{R} = \mathfrak{so}(p,1)$ $(p \geq 2)$, $\mathfrak{su}(p,1)$ $(p \geq 1)$ or $\mathfrak{sp}(p,1)$ $(p \geq 1)$. Let $K_\mathbf{R}$ be the adjoint group of $\mathfrak{k}_\mathbf{R}$: $K_\mathbf{R} = \exp \operatorname{ad} \mathfrak{k}_\mathbf{R}$. Then it is known that $K_\mathbf{R}$ acts on the space $\mathcal{H}_n$, and we denote by

$$\mathcal{H}_n = \bigoplus_{k=0}^{N(n)} \mathcal{H}_{n,k}$$

the $K_\mathbf{R}$-irreducible decomposition of $\mathcal{H}_n$, where $N(n)+1$ is the number of $K_\mathbf{R}$-irreducible components. Now we assume that $\mathcal{H}_{n,k} \not\simeq \mathcal{H}_{m,l}$ if $(n,k) \neq (m,l)$. Then under this condition, the following results are proved in the previous paper [20].

Theorem 1.2 ([20] Theorem 1.3). *Up to a non-zero constant there exists a unique function* $\widetilde{H}_{n,k}(X,Y) \not\equiv 0$ $(0 \leq k \leq N(n))$ *defined on* $\mathfrak{p} \times \mathfrak{p}$ *such that*

$$\widetilde{H}_{n,k}(\ \ ,Y) \in \mathcal{H}_{n,k} \quad \text{for any } Y \in \mathfrak{p}, \tag{1.1}$$

$$\widetilde{H}_{n,k}(gX,gY) = \widetilde{H}_{n,k}(X,Y) \quad \text{for any } g \in K_\mathbf{R} \text{ and any } X,Y \in \mathfrak{p}, \tag{1.2}$$

$$\widetilde{H}_{n,k}(X,Y) = \overline{\widetilde{H}_{n,k}(Y,X)} \quad \text{for any } X,Y \in \mathfrak{p}. \tag{1.3}$$

Theorem 1.3 ([20] Theorem 1.3). *Let* $\widetilde{H}_{n,k}(X,Y) \not\equiv 0$ $(0 \leq k \leq N(n))$ *be a function which satisfies the conditions* (1.1)–(1.3). *Suppose* $X_0 \in \mathfrak{p}$ *and* $\widetilde{H}_{n,k}(X_0,X_0) \neq 0$. *Then for any* $f \in \mathcal{H}_{m,l}$ *and* $X \in \mathfrak{p}$ *the following reproducing formula of harmonic polynomials holds on each* $K_\mathbf{R}$*-orbit* $K_\mathbf{R}X_0$:

$$\delta_{n,m}\delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0,X_0)} \int_{K_\mathbf{R}} f(gX_0)\widetilde{H}_{n,k}(X,gX_0)dg. \tag{1.4}$$

Here dg *means the normalized Haar measure on* $K_\mathbf{R}$.

Remark 1.4. To prove Theorem 1.2 and Theorem 1.3 we need the assumption $\mathcal{H}_{n,k} \not\simeq \mathcal{H}_{m,l}$ $((n,k) \neq (m,l))$. In the case $\mathfrak{g}_\mathbf{R} = \mathfrak{su}(p,1)$ this fact is proved in Corollary of [16; p.241]. The proof for the case $\mathfrak{g}_\mathbf{R} = \mathfrak{sp}(p,1)$ will be given in Proposition 4.2 (ii) of this paper.

Remark 1.5. In the case $\mathfrak{g}_\mathbf{R} = \mathfrak{so}(p,1)$ the above equality (1.4) is already known as a formula of classical harmonic polynomials on $\mathbf{C}^p$ $(\simeq \mathfrak{p})$ and the above function $\widetilde{H}_{n,k}(X,Y)$ can be expressed explicitly in terms of the Legendre polynomial of degree n in dimension p (see, for example, [1], [2], [11], [17], [21]). When $\mathfrak{g}_\mathbf{R} = \mathfrak{su}(p,1)$, the equality (1.4) is known as a formula of polynomials on the space $H^{k,l}$ if $X_0 \in \mathfrak{p}_{\mathbf{R}}$ and $\mathrm{Tr}\,({X_0}^2) = 2$ (cf. [12], [13]). But for the remaining cases of $\mathfrak{su}(p,1)$, including $\mathfrak{sp}(p,1)$ and $\mathfrak{f}_{4(-20)}$, the function $\widetilde{H}_{n,k}(X,Y)$ defined in [20] is expressed as a double integral of some inexplicit functions and is not so clear. In this paper we express $\widetilde{H}_{n,k}(X,Y)$ as an integral of explicitly given polynomials on a single $K_{\mathbf{R}}$-orbit of $\mathfrak{p}$ for two cases $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$ and $\mathfrak{sp}(p,1)$.

2. Integral formulas of harmonic polynomials: The case of $\mathfrak{su}(p,1)$.

In this section we give the reproducing kernel of each irreducible subspace of $\mathcal{H}_n$ on $K_{\mathbf{R}}$-orbits in $\mathfrak{p}$ for the case $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$ $(p \geq 1)$ (Theorem 2.1). The principal results for this case are already stated in [20]. The reproducing kernel $\widetilde{H}_{n,k}(X,Y)$ takes a somewhat simpler form in case X or Y is contained in nilpotent orbits. Here we also give a proof of this fact.

In the case $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$, we have

$$\begin{aligned}
\mathfrak{k}_{\mathbf{R}} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} ;\, A \in \mathfrak{u}(p),\ \alpha \in \mathfrak{u}(1),\ \mathrm{Tr}\,A + \alpha = 0 \right\}, \\
\mathfrak{p}_{\mathbf{R}} &= \left\{ \begin{pmatrix} 0 & x \\ {}^t\overline{x} & 0 \end{pmatrix} ;\, x \in \mathbf{C}^p \right\}, \\
\mathfrak{k} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} ;\, A \in M(p,\mathbf{C}),\ \mathrm{Tr}\,A + \alpha = 0 \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & x \\ {}^ty & 0 \end{pmatrix} ;\, x, y \in \mathbf{C}^p \right\},
\end{aligned}$$

and $K_{\mathbf{R}} = \mathrm{Ad}\,S(U(p) \times U(1)) = \{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} ;\, A \in U(p)\}$. Let $B(\ ,\ )$ be the Killing form on $\mathfrak{p}$. For $X = \begin{pmatrix} 0 & x \\ {}^ty & 0 \end{pmatrix} \in \mathfrak{p}$, the polynomial

$$P(X) = (4p+4)^{-1}B(X,X) = \frac{1}{2}\mathrm{Tr}\,(X^2) = {}^tyx,$$

gives a generator of J. We put

$$\mathcal{N} = \{X \in \mathfrak{p}\,;\, P(X) = 0\},$$

$$\Sigma = \{X \in \mathfrak{p}\,;\, P(X) = 1\},$$

and

$$\Sigma_{\mathbf{R}} = \Sigma \cap \mathfrak{p}_{\mathbf{R}}.$$

We denote by $\mathcal{H}_n = \{f \in S_n\,;\, \sum_{j=1}^p \frac{\partial^2}{\partial x_j \partial y_j} f = 0\}$ the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree n. For $X = \begin{pmatrix} 0 & x \\ {}^ty & 0 \end{pmatrix} \in \mathfrak{p}$, we define the bijection $\Psi : \mathfrak{p} \longrightarrow \mathbf{C}^{2p}$ by $\Psi(X) = \frac{1}{2}\begin{pmatrix} x+y \\ i(y-x) \end{pmatrix}$, and let $H_n(\mathbf{C}^{2p})$ be the space of homogeneous polynomials on $\mathbf{C}^{2p}$ of degree n which satisfy $\sum_{j=1}^p \frac{\partial^2}{\partial {z_j}^2} f = 0$. Then $f \in \mathcal{H}_n$ if and only if $f \circ \Psi^{-1} \in H_n(\mathbf{C}^{2p})$, and we have

$$\dim\mathcal{H}_n = \dim H_n(\mathbf{C}^{2p}) = \frac{2(n+p-1)(n+2p-3)!}{n!(2p-2)!}.$$

Remark that the restriction mapping $\Psi : \Sigma_{\mathbf{R}} \longrightarrow S^{2p-1}$ is also bijective. This implies that $P_{n,2p}\left(\frac{\mathrm{Tr}\,{}^tX\overline{Y}}{2\sqrt{P(X)}}\right)(P(X))^{n/2}$ $(X \in \mathfrak{p}, Y \in \Sigma_{\mathbf{R}})$ is the reproducing kernel of $\mathcal{H}_n$ on $\Sigma_{\mathbf{R}}$, where $P_{n,q}(t)$ is the Legendre polynomial of degree n in dimension q (cf. [8], [11], etc). Note that the Legendre polynomial is related to the Gegenbuar function $C_n^{\nu}(t)$ by the equality $P_{n,q}(t) = \frac{(q-3)!n!}{(n+q-3)!}C_n^{\frac{q-2}{2}}(t)$.

In the rest of this section we assume $p \geq 2$. For the case $p = 1$, see Remark 2.3 at the end of this section. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$ and $g = \mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in K_{\mathbf{R}}$ $(A \in U(p))$ we have $gX = \begin{pmatrix} 0 & Ax \\ {}^t(\overline{A}y) & 0 \end{pmatrix}$. We put

$$\begin{aligned} E_1 &= \begin{pmatrix} 0 & e_1 \\ {}^t e_1 & 0 \end{pmatrix} \in \Sigma_{\mathbf{R}}, \\ \widetilde{E}_{r,q} &= \begin{pmatrix} 0 & re_1 \\ {}^t(\frac{1}{r}e_1 + qe_2) & 0 \end{pmatrix} \in \Sigma \quad (r > 0, q \geq 0), \\ \widetilde{E}_r &= \begin{pmatrix} 0 & re_1 \\ \sqrt{1-r^2}\,{}^t e_2 & 0 \end{pmatrix} \in \mathcal{N} \quad (0 \leq r \leq 1), \end{aligned}$$

where $e_1 = {}^t(1\,0\cdots 0)$, and $e_2 = {}^t(0\,1\cdots 0)$. Remark that

$$K_{\mathbf{R}}E_1 = \Sigma_{\mathbf{R}}, \quad \mathfrak{p} = \mathcal{N} \cup \bigcup_{\lambda \in \mathbf{C}\setminus\{0\}} \lambda\Sigma,$$

and the $K_{\mathbf{R}}$-orbit decompositions of Σ and $\mathcal{N}$ are given by

$$\Sigma = \bigcup_{q\geq 0,\, r>0} K_{\mathbf{R}}\widetilde{E}_{r,q} \quad \text{and} \quad \mathcal{N} = \bigcup_{\rho\geq 0,\, 0\leq r\leq 1} K_{\mathbf{R}}(\rho\widetilde{E}_r).$$

We put $\Lambda = \{(n,k)\,;\, n \in \mathbf{Z}_+,\, 0 \leq k \leq n\}$. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & a \\ {}^t b & 0 \end{pmatrix} \in \mathfrak{p}$ we put

$$\widetilde{K}_{n,k}(X,Y) = (x \cdot \overline{a})^k (y \cdot \overline{b})^{n-k} \quad ((n,k) \in \Lambda),$$

where $z \cdot w = {}^t zw$ for $z, w \in \mathbf{C}^p$. It is clear that

$$\begin{aligned} &\widetilde{K}_{n,k}(X,Y) = \overline{\widetilde{K}_{n,k}(Y,X)} && (X, Y \in \mathfrak{p}), \\ &\widetilde{K}_{n,k}(gX, gY) = \widetilde{K}_{n,k}(X,Y) && (g \in K_{\mathbf{R}}), \\ &\widetilde{K}_{n,k}(\ \ ,Y) \in \mathcal{H}_n && (Y \in \mathcal{N}). \end{aligned}$$

Let $\mathcal{H}_{n,k}$ be the subspace of $\mathcal{H}_n$ which is spanned by the elements $\widetilde{K}_{n,k}(\ \ ,Y)$ $(Y \in \mathcal{N})$. From Theorem 14.4 in [16] we can easily see that $\mathcal{H}_n = \oplus_{k=0}^n \mathcal{H}_{n,k}$ gives the $K_{\mathbf{R}}$-irreducible decomposition of $\mathcal{H}_n$ and

$$\dim \mathcal{H}_{n,k} = \frac{(p+n-1)(k+p-2)!(n-k+p-2)!}{(p-1)!(p-2)!k!(n-k)!}.$$

Now we put $E_0 = \begin{pmatrix} 0 & e_1 \\ {}^t e_2 & 0 \end{pmatrix}$, and by using $\widetilde{K}_{n,k}$, we define a function $\widetilde{H}_{n,k}(X,Z)$ $(X, Z \in \mathfrak{p})$ by

$$\widetilde{H}_{n,k}(X,Z) = \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(X, gE_0)\widetilde{K}_{n,k}(gE_0, Z)dg,$$

where dg is the normalized Haar measure on $K_{\mathbf{R}}$. For $f, h \in \mathcal{H}_n$, we define the $K_{\mathbf{R}}$-invariant inner product $(\ \ ,\ \ )$ by

$$(f,h) = \int_{K_{\mathbf{R}}} f(gE_0)\overline{h(gE_0)}dg.$$

Then we see that $\mathcal{H}_{n,k} \perp \mathcal{H}_{n,l}$ $(k \neq l)$. Therefore it is easy to show that $\widetilde{H}_{n,k} \in \mathcal{H}_{n,k}$. The following theorem asserts that the function $\widetilde{H}_{n,k}$ explicitly defined above gives the reproducing kernel of $\mathcal{H}_n$.

Theorem 2.1. *Let $X_0 \in \mathfrak{p}$ and assume $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ $({}^\forall(n,k) \in \Lambda)$. Let $f \in \mathcal{H}_{m,l}$ and $X \in \mathfrak{p}$. Then the following integral formula holds:*

$$\delta_{n,m}\delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0, X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{n,k}(X, gX_0)dg. \tag{2.1}$$

Especially if $X \in \mathcal{N}$ or $Y \in \mathcal{N}$, we have

$$\widetilde{H}_{n,k}(X, Y) = \widetilde{K}_{n,k}(X, Y). \tag{2.2}$$

And therefore the polynomial $\widetilde{K}_{n,k}(X, Y)$ itself gives a reproducing kernel on nilpotent orbits $K_{\mathbf{R}}X_0$ $(X_0 \in \mathcal{N}$):

$$\delta_{n,m}\delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\widetilde{K}_{n,k}(X_0, X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{K}_{n,k}(X, gX_0)dg. \tag{2.3}$$

Proof. We can easily show that the function $\widetilde{H}_{n,k}(X, Y)$ satisfies the conditions (1.1)–(1.3) in Theorem 1.2, and hence we obtain the formula (2.1).

Now we show (2.2) and (2.3). For $X, Y \in \mathfrak{p}$ we put

$$F_{n,k}(X, Y) = \dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} \widetilde{H}_{n,k}(X, gE_0)\widetilde{K}_{n,k}(gE_0, Y)dg.$$

Then the function $F_{n,k}(X, Y)$ also satisfies the conditions (1.1)–(1.3). Hence from Theorem 1.2 (i) there exists some $c_{n,k} \in \mathbf{C}$ such that

$$F_{n,k}(X, Y) = c_{n,k}\widetilde{H}_{n,k}(X, Y) \qquad (X, Y \in \mathfrak{p}). \tag{2.4}$$

On the other hand for $Y \in \mathcal{N}$ we have from (2.1)

$$F_{n,k}(X, Y) = \widetilde{H}_{n,k}(E_0, E_0)\widetilde{K}_{n,k}(X, Y) \tag{2.5}$$

because $\widetilde{K}_{n,k}(\ \ , Y)$ belongs to $\mathcal{H}_{n,k}$. The equalities (2.4) and (2.5) imply

$$c_{n,k}\widetilde{H}_{n,k}(X, Y) = \widetilde{H}_{n,k}(E_0, E_0)\widetilde{K}_{n,k}(X, Y). \tag{2.6}$$

Since

$$\widetilde{K}_{n,k}(E_0, E_0) = 1 \neq 0,$$

we have

$$\widetilde{H}_{n,k}(E_0, E_0) = \int_{K_{\mathbf{R}}} |\ \widetilde{K}_{n,k}(gE_0, E_0)\ |^2\ dg \neq 0.$$

Therefore from (2.6) we have $c_{n,k} = 1$ and hence by the property (1.3) the equality

$$\widetilde{H}_{n,k}(X, Y) = \widetilde{H}_{n,k}(E_0, E_0)\widetilde{K}_{n,k}(X, Y)$$

holds if $X \in \mathcal{N}$ or $Y \in \mathcal{N}$. From this and (2.1) we have easily (2.2) and (2.3). Q.E.D.

Remark 2.2. We have

$$\widetilde{H}_{n,k}(X_0, X_0) = C \int_{K_{\mathbf{R}}} |\ \widetilde{H}_{n,k}(gX_0, E_1)\ |^2\, dg \quad (X_0 \in \mathfrak{p}),$$

where $C = (\int_{K_{\mathbf{R}}} |\widetilde{K}_{n,k}(gX_0, E_1)|^2 dg)^{-1}$. Since $\widetilde{K}_{n,k}(\ \ , E_0) \not\equiv 0$ on $\Sigma_{\mathbf{R}}$, we have $C > 0$. Therefore the following two conditions (2.7) and (2.8) are equivalent.

$$\widetilde{H}_{n,k}(X_0, X_0) = 0, \tag{2.7}$$

$$\mathcal{H}_{n,k}|_{K_{\mathbf{R}}X_0} = \{0\}. \tag{2.8}$$

This implies that the assumption $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ in Theorem 2.1 holds for any $(n, k) \in \Lambda$ if and only if $X_0 \notin \lambda K_{\mathbf{R}}\widetilde{E}_1$ and $X_0 \notin \lambda K_{\mathbf{R}}\widetilde{E}_0$ for any $\lambda \in \mathbf{C}$.

Remark 2.3. We consider the case $p = 1$. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & x' \\ {}^t y' & 0 \end{pmatrix} \in \mathfrak{p}$ we put $\widetilde{H}_{n,1}(X, Y) = (x \cdot \overline{x'})^n$ and $\widetilde{H}_{n,2}(X, Y) = (y \cdot \overline{y'})^n$. We denote by $\mathcal{H}_{n,k}$ the subspace of $\mathcal{H}_n$ which is generated by $\{\widetilde{H}_{n,k}(\ \ , E_1)\}$ $(k = 1, 2)$. Then we have the $K_{\mathbf{R}}$-irreducible decomposition $\mathcal{H}_n = \mathcal{H}_{n,1} \oplus \mathcal{H}_{n,2}$. It is easy to show that $\widetilde{H}_{n,k}(X, Y)$ satisfies (1.1)–(1.3) in Theorem 1.2, and therefore Theorem 2.1 also holds in case $p = 1$.

3. Harmonic polynomials on $\mathfrak{p}$ in the case $\mathfrak{sp}(p, 1)$.

In the rest of this paper we consider the Lie algebra $\mathfrak{sp}(p, 1)$ and give the explicit formula of the reproducing kernel of harmonic polynomials on each $K_{\mathbf{R}}$-orbit (Theorem 4.5). In this case the expressions of matrices becomes much more complicated than the case of $\mathfrak{su}(p, 1)$, because the complexification $\mathfrak{sp}(p+1, \mathbf{C})$ of the real Lie algebra $\mathfrak{sp}(p, 1)$ can not be realized as a subalgebra of the quaternion general linear Lie algebra $\mathfrak{gl}(p + 1, \mathbf{H})$. (Note that in the case $\mathfrak{su}(p, 1)$, its complexification can be naturally identified with $\mathfrak{sl}(p + 1, \mathbf{C})$).

The construction of the reproducing kernel is also complicated for the case $\mathfrak{sp}(p, 1)$, and in this section we first settle the notations and state basic formulas on harmonic polynomials on $\mathfrak{p}$ for the Lie algebra $\mathfrak{sp}(p, 1)$. Since the Lie algebra $\mathfrak{sp}(1, 1)$ is isomorphic to $\mathfrak{so}(4, 1)$, we always assume $p \geq 2$ in the following argument. From now we put $\mathfrak{g} = \mathfrak{sp}(p + 1, \mathbf{C})$, $\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p, 1)$,

$$\mathfrak{k}_{\mathbf{R}} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & b \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{b} & 0 & \overline{a} \end{pmatrix} ; \begin{array}{l} A \in \mathfrak{u}(p),\ a \in \mathfrak{u}(1),\ b \in \mathbf{C} \\ B \ \text{ is } p \times p \ \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{p}_{\mathbf{R}} = \left\{ \begin{pmatrix} 0 & x & 0 & y \\ {}^t\overline{x} & 0 & {}^t y & 0 \\ 0 & \overline{y} & 0 & -\overline{x} \\ {}^t\overline{y} & 0 & -{}^t x & 0 \end{pmatrix} ; x, y \in \mathbf{C}^p \right\}.$$

Then we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ C & 0 & -{}^tA & 0 \\ 0 & \gamma & 0 & -\alpha \end{pmatrix} ; \begin{array}{l} A, B, C \in M(p, \mathbf{C}) \\ {}^tB = B, \ {}^tC = C \\ \alpha, \beta, \gamma \in \mathbf{C} \end{array} \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} ; x, y, z, w \in \mathbf{C}^p \right\},$$

and

$$K_{\mathbf{R}} = \left\{ \mathrm{Ad} \begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{\beta} & 0 & \overline{\alpha} \end{pmatrix} \in \mathrm{Ad}\, U(2p+2) ; \begin{array}{l} {}^tA\overline{A} + {}^t\overline{B}B = I_p, \\ {}^tA\overline{B} = {}^t\overline{B}A, \\ \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \end{array} \right\}.$$

For $X = \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} \in \mathfrak{p}$, the polynomial defined by

$$P(X) = \frac{1}{8(p+2)} B(X, X) = \frac{1}{4} \mathrm{Tr}\,(X^2) = {}^txy + {}^tzw$$

gives a generator of J and $\mathcal{H}_n$ is given by $\mathcal{H}_n = \{f \in S_n \,;\, \sum_{j=1}^p \left(\frac{\partial^2}{\partial x_j \partial y_j} + \frac{\partial^2}{\partial z_j \partial w_j} \right) f = 0\}$.

For $X \in \mathfrak{p}$ we define the bijective mapping $\Psi : \mathfrak{p} \longrightarrow \mathbf{C}^{4p}$ by $\Psi(X) = \frac{1}{2} \begin{pmatrix} x+y \\ z+w \\ i(y-x) \\ i(w-z) \end{pmatrix}$.
We can see that $f \in \mathcal{H}_n$ if and only if $f \circ \Psi^{-1} \in H_n(\mathbf{C}^{4p})$ and from this fact, we have

$$\dim \mathcal{H}_n = \dim H_n(\mathbf{C}^{4p}) = \frac{2(n+2p-1)(n+4p-3)!}{n!(4p-2)!}.$$

We put

$$\mathcal{N} = \{X \in \mathfrak{p} \,;\, P(X) = 0\},$$
$$\Sigma = \{X \in \mathfrak{p} \,;\, P(X) = 1\},$$

and

$$\Sigma_{\mathbf{R}} = \Sigma \cap \mathfrak{p}_{\mathbf{R}}.$$

Remark that $\Psi : \Sigma_{\mathbf{R}} \simeq S^{4p-1}$ and $\widetilde{H}_n(X, Y) = P_{n,4p} \left(\frac{\mathrm{Tr}\, {}^tX\overline{Y}}{4\sqrt{P(X)}} \right) (P(X))^{n/2}$ $(X \in \mathfrak{p}, Y \in \Sigma_{\mathbf{R}})$ gives the reproducing kernel on $\Sigma_{\mathbf{R}}$. Furthermore it is known that the restriction mapping $f \longmapsto f|_{\mathcal{N}}$ is also a bijection from $\mathcal{H}_n$ onto $\mathcal{H}_n|_{\mathcal{N}}$.

Let $g = \mathrm{Ad} \begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{\beta} & 0 & \overline{\alpha} \end{pmatrix} \in K_{\mathbf{R}}$ and $X = \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} \in \mathfrak{p}$. If we put $\Phi(X) = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbf{C}^{4p}$, we have

$$\Phi(gX) = \begin{pmatrix} A(\overline{\alpha}x + \overline{\beta}w) + B(\overline{\alpha}z - \overline{\beta}y) \\ \overline{B}(-\beta x + \alpha w) + \overline{A}(\alpha y + \beta z) \\ -\overline{B}(\overline{\alpha}x + \overline{\beta}w) + \overline{A}(\overline{\alpha}z - \overline{\beta}y) \\ A(-\beta x + \alpha w) - B(\alpha y + \beta z) \end{pmatrix}. \tag{3.1}$$

In the following, we often simply write $\alpha(g)=\alpha$ and $\beta(g)=\beta$, though α and β depend on $g\in K_{\mathbf{R}}$. We put

$$\widetilde{E}_r=\Phi^{-1}\begin{pmatrix} re_1\\ 0\\ 0\\ \sqrt{1-r^2}\,e_2\end{pmatrix}\in\mathcal{N}\quad(0\le r\le 1),$$

$$\widetilde{E}_{r,q}=\Phi^{-1}\begin{pmatrix} re_1\\ \frac{1}{r}\,e_1+qe_2\\ 0\\ 0\end{pmatrix}\in\Sigma\quad(r>0,\ q\ge 0).$$

In addition we put $E_1=\widetilde{E}_{1,0}$.

It is clear that $\mathfrak{p}=\mathcal{N}\cup\bigcup_{\lambda\in\mathbf{C}\setminus\{0\}}\lambda\Sigma$. Remark that

$$\mathcal{N}=\bigcup_{q\ge 0,\ \frac{1}{\sqrt{2}}\le r\le 1}K_{\mathbf{R}}(q\widetilde{E}_r),\qquad \Sigma=\bigcup_{q\ge 0,\ r>0}K_{\mathbf{R}}\widetilde{E}_{r,q}\tag{3.2}$$

give the $K_{\mathbf{R}}$-orbit decompositions of $\mathcal{N}$ and Σ, respectively. For $X=\Phi^{-1}\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix}$, $Y=\Phi^{-1}\begin{pmatrix} x'\\ y'\\ z'\\ w'\end{pmatrix}\in\mathfrak{p}$, we put $\langle X,Y\rangle=\frac{1}{2}\mathrm{Tr}\,({}^tX\overline{Y})=x\cdot\overline{x'}+y\cdot\overline{y'}+z\cdot\overline{z'}+w\cdot\overline{w'}$. Then we can easily see that $\langle\ ,\ \rangle$ is $K_{\mathbf{R}}$-invariant.

Next we put

$$H_1=\left\{\mathrm{Ad}\begin{pmatrix} A & 0 & B & 0\\ 0 & 1 & 0 & 0\\ -\overline{B} & 0 & \overline{A} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\in K_{\mathbf{R}}\right\}$$

and

$$H_2=\left\{\mathrm{Ad}\begin{pmatrix} I_p & 0 & 0 & 0\\ 0 & \alpha & 0 & \beta\\ 0 & 0 & I_p & 0\\ 0 & -\overline{\beta} & 0 & \overline{\alpha}\end{pmatrix}\in K_{\mathbf{R}}\right\}.$$

Then H_1 and H_2 are subgroups of $K_{\mathbf{R}}$, and for any $g\in K_{\mathbf{R}}$ there exist unique $h_j\in H_j$ $(j=1,2)$ such that $g=h_1h_2$. Furthermore, if $g_j\in H_j$ $(j=1,2)$, we have $g_1g_2=g_2g_1$. We denote by dh_j the normalized Haar measure on H_j and by $C(H_j)$ the space of continuous functions on H_j $(j=1,2)$. Remark that if we put $h_2=\mathrm{Ad}\begin{pmatrix} I_p & 0 & 0 & 0\\ 0 & \alpha & 0 & \beta\\ 0 & 0 & I_p & 0\\ 0 & -\overline{\beta} & 0 & \overline{\alpha}\end{pmatrix}$, $\alpha=\rho e^{i\theta}$, $\beta=\sqrt{1-\rho^2}\,e^{i\varphi}$, then for any $f\in C(H_2)$ we have

$$\int_{H_2}f(h_2)dh_2=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1\widetilde{f}(\rho,\theta,\varphi)\rho\,d\rho\,d\varphi\,d\theta,\tag{3.3}$$

where $\widetilde{f}(\rho,\theta,\varphi)=f(h_2)$.

For $h_1 = \mathrm{Ad}\begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_1$ we define the mapping $\phi:\ H_1\widetilde{E}_1 \longrightarrow S^{4p-1}$ by

$$\phi(h_1\widetilde{E}_1) = \begin{pmatrix} \mathrm{Re}\, a_1 \\ \mathrm{Im}\, a_1 \\ \mathrm{Re}\,(-\overline{b_1}) \\ \mathrm{Im}\,(-\overline{b_1}) \end{pmatrix},$$

where $h_1\widetilde{E}_1 = \Phi^{-1}\begin{pmatrix} a_1 \\ 0 \\ -\overline{b_1} \\ 0 \end{pmatrix}$ and $a_1 = Ae_1$, $b_1 = Be_1$. (If $h_1, h_1' \in H_1$ satisfy $h_1\widetilde{E}_1 = h_1'\widetilde{E}_1$, then we can easily prove $\phi(h_1\widetilde{E}_1) = \phi(h_1'\widetilde{E}_1)$. And this fact implies that the mapping ϕ is well defined.) From the definition of H_1 we see that ϕ is bijective and the equality

$$\int_{H_1} f(h_1\widetilde{E}_1)dh_1 = \int_{S^{4p-1}} f\circ\phi^{-1}(s)ds \tag{3.4}$$

holds for any $f \in C(H_1)$, where ds is the normalized $O(4p)$-invariant measure on S^{4p-1}. For $X = \Phi^{-1}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $Y = \Phi^{-1}\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} \in \mathfrak{p}$ we put

$$K_2(X,Y) = (x\cdot\overline{x'} + z\cdot\overline{z'})(y\cdot\overline{y'} + w\cdot\overline{w'}) + (x\cdot\overline{w'} - z\cdot\overline{y'})(y\cdot\overline{z'} - w\cdot\overline{x'}),$$
$$\widetilde{K}_m(X,Y) = \frac{(m+2p-1)!}{m!(2p-1)!}\int_{K_{\mathbf{R}}} \langle g\widetilde{E}_1, Y\rangle^m \langle X, g\widetilde{E}_1\rangle^m dg,$$
$$\widetilde{K}_{n,k}(X,Y) = \widetilde{K}_{n-2k}(X,Y)\{K_2(X,Y)\}^k$$

$(m, n \in \mathbf{Z}_+,\ k = 0, 1, \cdots, [n/2])$. These functions play an important role in constructing the function $\widetilde{H}_{n,k}(X,Y)$. Remark that the equalities

$$\widetilde{K}_{n,k}(X,Y) = \overline{\widetilde{K}_{n,k}(Y,X)}, \tag{3.5}$$

$$\widetilde{K}_{n,k}(X,Y) = \widetilde{K}_{n,k}(gX,gY) \tag{3.6}$$

hold for any $X, Y \in \mathfrak{p}$, $g \in K_{\mathbf{R}}$.

4. Decomposition of the space $\mathcal{H}_n$ and the integral formula for the case $\mathfrak{sp}(p,1)$.

In this section we first show that $\widetilde{K}_{n,k}(\ \ ,Y) \in \mathcal{H}_n$ if $Y \in \mathcal{N}$, and next by using this property, we define $K_{\mathbf{R}}$-irreducible subspaces $\mathcal{H}_{n,k}$ of $\mathcal{H}_n$ $(k = 0, 1, \cdots, [n/2])$. And finally we state our main theorem for the case $\mathfrak{sp}(p,1)$ (Theorem 4.5). As before we always assume $p \geq 2$.

First, for $k = 0, 1, \cdots, [n/2]$, we introduce the polynomial $K_{n,k}$ to simplify the following calculations:

$$K_{n,k}(X,Y) = \frac{1}{n-2k+1}\langle X,Y\rangle^{n-2k}\{K_2(X,Y)\}^k \qquad (X, Y \in \mathfrak{p}).$$

We can see that $K_{n,k}(\ \ ,Y) \in \mathcal{H}_n$ if $Y \in \mathcal{N}$.

Now we prove that $\widetilde{K}_{n,k}(\ \ ,Y) \in \mathcal{H}_n$ $(Y \in \mathcal{N})$. For this purpose we need the following

Proposition 4.1. (i) *For* $X = \Phi^{-1}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $Y = \Phi^{-1}\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} \in \mathfrak{p}$ *the following formula holds*:

$$\widetilde{K}_m(X,Y) = \frac{1}{m+1} \sum_{m_1+m_2+2m_3=m} \frac{(m_1+m_3)!(m_2+m_3)!}{m_1!m_2!(m_3!)^2}(x\cdot\overline{x'}+z\cdot\overline{z'})^{m_1} \times (y\cdot\overline{y'}+w\cdot\overline{w'})^{m_2}(z\cdot\overline{y'}-x\cdot\overline{w'})^{m_3}(y\cdot\overline{z'}-w\cdot\overline{x'})^{m_3}. \tag{4.1}$$

(ii) *There exist* $a_{m,q} \in \mathbf{R}$ $(q = 1, 2, \cdots, [m/2])$ *such that*

$$\langle X, Y\rangle^m = (m+1)\widetilde{K}_m(X,Y) + \sum_{q=1}^{[m/2]} a_{m,q}K_{m,q}(X,Y) \quad (X, Y \in \mathfrak{p}). \tag{4.2}$$

(iii) *There exist* $b_{m,q} \in \mathbf{R}$ $(q = 1, 2, \cdots, [m/2])$ *such that*

$$\langle X, Y\rangle^m = (m+1)\widetilde{K}_m(X,Y) + \sum_{q=1}^{[m/2]} b_{m,q}\widetilde{K}_{m,q}(X,Y) \quad (X, Y \in \mathfrak{p}). \tag{4.3}$$

Proof. (i) Assume $a, b \in \mathbf{C}^{4p}$ and $a\cdot a = b\cdot b = 0$. Then the following equality holds (see [11]):

$$\int_{S^{4p-1}} (s\cdot a)^m(s\cdot\overline{b})^m ds = \frac{m!(2p-1)!}{2^m(m+2p-1)!}\,(a\cdot\overline{b})^m.$$

From this formula and from (3.1), (3.4) we have

$$\begin{aligned}
\widetilde{K}_m(X,Y) &= \frac{(m+2p-1)!}{m!(2p-1)!}\int_{K_{\mathbf{R}}} \langle g\widetilde{E}_1, Y\rangle^m\langle X, g\widetilde{E}_1\rangle^m dg \\
&= \frac{(m+2p-1)!}{m!(2p-1)!}\int_{H_2}\Bigl(\int_{H_1}\{a_1\cdot(\overline{\alpha}\overline{x'}-\beta\overline{w'})-\overline{b_1}\cdot(\overline{\alpha}\overline{z'}+\beta\overline{y'})\}^m \\
&\qquad \times \{\overline{a_1}\cdot(\alpha x-\overline{\beta}w)-b_1\cdot(\alpha z+\overline{\beta}y)\}^m dh_1\Bigr) dh_2
\end{aligned} \tag{4.4}$$

$$= \frac{(m+2p-1)!}{m!(2p-1)!}\int_{H_2}\left(\int_{S^{4p-1}}\left\{s\cdot\begin{pmatrix}\overline{\alpha}\overline{x'}-\beta\overline{w'} \\ i(\overline{\alpha}\overline{x'}-\beta\overline{w'}) \\ \beta\overline{y'}+\overline{\alpha}\overline{z'} \\ i(\beta\overline{y'}+\overline{\alpha}\overline{z'})\end{pmatrix}\right\}^m\left\{s\cdot\begin{pmatrix}\alpha x-\overline{\beta}w \\ -i(\alpha x-\overline{\beta}w) \\ \overline{\beta}y+\alpha z \\ -i(\overline{\beta}y+\alpha z)\end{pmatrix}\right\}^m ds\right) dh_2$$

$$= \int_{H_2}\{(\overline{\alpha}\overline{x'}-\beta\overline{w'})\cdot(\alpha x-\overline{\beta}w)+(\overline{\alpha}\overline{z'}+\beta\overline{y'})\cdot(\alpha z+\overline{\beta}y)\}^m dh_2.$$

The last expression of (4.4) equals

$$\begin{aligned}
&\int_{H_2}\{|\alpha|^2(x\cdot\overline{x'}+z\cdot\overline{z'})+|\beta|^2(w\cdot\overline{w'}+y\cdot\overline{y'}) \\
&\qquad +\alpha\beta(z\cdot\overline{y'}-x\cdot\overline{w'})+\overline{\alpha}\overline{\beta}(y\cdot\overline{z'}-w\cdot\overline{x'})\}^m dh_2 \\
&= \sum_{m_1+m_2+m_3+m_4=m}\frac{m!}{m_1!m_2!m_3!m_4!}\left(\int_{H_2}|\alpha|^{2m_1}|\beta|^{2m_2}(\alpha\beta)^{m_3}(\overline{\alpha}\overline{\beta})^{m_4}dh_2\right) \\
&\qquad \times (x\cdot\overline{x'}+z\cdot\overline{z'})^{m_1}(y\cdot\overline{y'}+w\cdot\overline{w'})^{m_2}(z\cdot\overline{y'}-x\cdot\overline{w'})^{m_3}(y\cdot\overline{z'}-w\cdot\overline{x'})^{m_4}.
\end{aligned}$$

Putting $\alpha = te^{i\theta}$ and $\beta = \sqrt{1-t^2}e^{i\varphi}$, we have from (3.3)

$$\begin{aligned}&\int_{H_2} |\alpha|^{2m_1}|\beta|^{2m_2}(\alpha\beta)^{m_3}(\overline{\alpha}\overline{\beta})^{m_4}dh_2\\&= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 t^{2m_1+m_3+m_4}(1-t^2)^{(2m_2+m_3+m_4)/2}(e^{i\theta}e^{i\varphi})^{m_3-m_4}t\,dt\,d\theta\,d\varphi\\&= \delta_{m_3,m_4}\frac{(m_1+m_3)!(m_2+m_3)!}{(m+1)!}.\end{aligned}$$

Therefore we obtain (4.1).

(ii) We prove the following formulas by induction on n:

$$\left\{\begin{array}{l}\langle X,Y\rangle^{2n-1} = (2n)\widetilde{K}_{2n-1}(X,Y) + \sum_{q=1}^{n-1} a_{2n-1,q}K_{2n-1,q}(X,Y),\\ \langle X,Y\rangle^{2n} = (2n+1)\widetilde{K}_{2n}(X,Y) + \sum_{q=1}^{n} a_{2n,q}K_{2n,q}(X,Y),\end{array}\right. \tag{4.5}$$
$$(a_{2n-1,q}, a_{2n,q} \in \mathbf{R},\ n = 1,2,\cdots).$$

When $n = 1$, we have (4.5) because (4.1) gives

$$\begin{aligned}2\widetilde{K}_1(X,Y) &= \langle X,Y\rangle,\\ 3\widetilde{K}_2(X,Y) &= \langle X,Y\rangle^2 - K_2(X,Y).\end{aligned}$$

Assume that (4.5) is valid for $n = 1,2,\cdots,k$. By this assumption and by (4.1), we obtain the following equality after some calculations:

$$\begin{aligned}\langle X,Y\rangle^{2k+1} &= \{(2k+1)\widetilde{K}_{2k}(X,Y) + \sum_{q=1}^{k} a_{2k,q}K_{2k,q}(X,Y)\}\langle X,Y\rangle\\ &= \sum_{q=1}^{k} a'_{2k,q}K_{2k+1,q}(X,Y) + \{\sum_{m_1+m_2+2m_3=2k} \frac{(m_1+m_3)!(m_2+m_3)!}{m_1!m_2!(m_3!)^2}\\ &\quad\times(x\cdot\overline{x'}+z\cdot\overline{z'})^{m_1}(y\cdot\overline{y'}+w\cdot\overline{w'})^{m_2}(z\cdot\overline{y'}-x\cdot\overline{w'})^{m_3}(y\cdot\overline{z'}-w\cdot\overline{x'})^{m_3}\}\langle X,Y\rangle\\ &= \sum_{q=1}^{k} a'_{2k,q}K_{2k+1,q}(X,Y) + (2k+2)\widetilde{K}_{2k+1}(X,Y) + 2k\widetilde{K}_{2k-1}(X,Y)K_2(X,Y),\end{aligned}$$

where $a'_{2k,q} = a_{2k,q}(2k-2q+2)(2k-2q+1)^{-1}$. By the assumption of induction we have

$$\begin{aligned}2k\widetilde{K}_{2k-1}(X,Y)K_2(X,Y) &= K_2(X,Y)\{\langle X,Y\rangle^{2k-1} - \sum_{q=1}^{k-1} a_{2k-1,q}K_{2k-1,q}(X,Y)\}\\ &= 2kK_{2k+1,1}(X,Y) - \sum_{q=1}^{k-1} a_{2k-1,q}K_{2k+1,q+1}(X,Y).\end{aligned}$$

Hence there exist some $a_{2k+1,q} \in \mathbf{R}$ $(q = 1,2,\cdots,k)$ such that

$$\langle X,Y\rangle^{2k+1} = (2k+2)\widetilde{K}_{2k+1}(X,Y) + \sum_{q=1}^{k} a_{2k+1,q}K_{2k+1,q}(X,Y).$$

In the same way we can show the second equality of (4.5) for $n = k+1$.

(iii) Using (4.2), we can prove (4.3) easily. Q.E.D.

From (4.2) there exist $a_{n-2k,q} \in \mathbf{R}$ $(q = 1, 2, \cdots, [n/2] - k)$ such that

$$(n-2k+1)\widetilde{K}_{n-2k}(X,Y) = \langle X, Y\rangle^{n-2k} - \sum_{q=1}^{[n/2]-k} a_{n-2k,q} K_{n-2k,q}(X,Y) \quad (X, Y \in \mathfrak{p}).$$

From the definitions of $\widetilde{K}_{n,k}(X,Y)$ and $K_{n,k}(X,Y)$ and from this formula, there exist $c_{n,q} \in \mathbf{R}$ $(q = k, k+1, \cdots, [n/2])$ such that

$$\widetilde{K}_{n,k}(X,Y) = \sum_{q=k}^{[n/2]} c_{n,q} K_{n,q}(X,Y).$$

Hence we see that $\widetilde{K}_{n,k}(\ \ ,Y) \in \mathcal{H}_n$ because $K_{n,k}(\ \ ,Y) \in \mathcal{H}_n$ $(Y \in \mathcal{N})$.

We denote by $\mathcal{H}_{n,k}$ the subspace of $\mathcal{H}_n$ which is generated by $\{\widetilde{K}_{n,k}(\ \ ,Z)\,;\, Z \in \mathcal{N}\}$. Then from (3.6) it is clear that the space $\mathcal{H}_{n,k}$ is $K_{\mathbf{R}}$-invariant. From now we put $E_0 = \Phi^{-1}\begin{pmatrix} e_1 \\ 0 \\ 0 \\ e_2 \end{pmatrix} \in \mathcal{N}$. To show our main theorem, we must prepare the following proposition.

Proposition 4.2. (i) *For any* $X, Y \in \mathfrak{p}$ *we have*

$$\int_{K_{\mathbf{R}}} \widetilde{K}_{n,l}(gE_0, Y)\widetilde{K}_{n,k}(X, gE_0)dg = 0 \quad (l \neq k). \tag{4.6}$$

(ii) $\mathcal{H}_n = \oplus_{k=0}^{[n/2]} \mathcal{H}_{n,k}$ *gives the* $K_{\mathbf{R}}$*-irreducible decomposition of* $\mathcal{H}_n$*. Furthermore,* $\mathcal{H}_{n,k}$ *and* $\mathcal{H}_{m,l}$ *are not equivalent as* $K_{\mathbf{R}}$*-representation spaces if* $(n,k) \neq (m,l)$.

To prove this proposition, we need the following

Lemma 4.3. (i) *For any* $h_2 \in H_2$ *and* $X, Y \in \mathfrak{p}$ *it is valid that* $K_2(h_2X, Y) = K_2(X, Y)$.
(ii) *If* $n, m \in \mathbf{Z}_+$ *and* $n > m$*, we have for any* $X, Y \in \mathfrak{p}$

$$\int_{H_2} \langle h_2X, E_0\rangle^m \langle Y, h_2\widetilde{E}_1\rangle^n dh_2 = 0, \tag{4.7}$$

and

$$\int_{H_2} \widetilde{K}_n(h_2E_0, X)\widetilde{K}_m(Y, h_2E_0)dh_2 = 0. \tag{4.8}$$

Proof. If we put $\Phi(X) = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbf{C}^{4p}$ and $h_2 = \mathrm{Ad}\begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & I_p & 0 \\ 0 & -\overline{\beta} & 0 & \overline{\alpha} \end{pmatrix}$ $(\alpha, \beta \in \mathbf{C},\ \alpha\overline{\alpha} + \beta\overline{\beta} = 1)$, from (3.1) we have

$$\Phi(h_2X) = \begin{pmatrix} \overline{\alpha}x + \overline{\beta}w \\ \alpha y + \beta z \\ \overline{\alpha}z - \overline{\beta}y \\ -\beta x + \alpha w \end{pmatrix}. \tag{4.9}$$

By using (4.9), it is easy to show (i). We will prove (ii). From (4.9) there exist some $t, s, r, q, \mu, \nu \in \mathbf{C}$ such that

$$\langle h_2 X, E_0\rangle^m = (\alpha r + \beta q + \overline{\alpha}\mu + \overline{\beta}\nu)^m,$$

and

$$\langle Y, h_2 \widetilde{E}_1\rangle^n = (\alpha t + \overline{\beta} s)^n.$$

These formulas give that

$$\begin{aligned}&\int_{H_2} \langle h_2 X, E_0\rangle^m \langle Y, h_2 \widetilde{E}_1\rangle^n dh_2 \\ &= \sum_{k=0}^{n} \sum_{m_1+m_2+m_3+m_4=m} C_{m_1,m_2,m_3,m_4,n,k}(t,s,r,q,\mu,\nu) \int_{H_2} \alpha^{m_1+k}\beta^{m_2}\,\overline{\alpha}^{m_3}\,\overline{\beta}^{n-k+m_4} dh_2,\end{aligned}$$

where $C_{m_1,m_2,m_3,m_4,n,k}(t,s,r,q,\mu,\nu)$ is a polynomial of t,s,r,q,μ,ν. Putting $\alpha = \rho e^{i\theta}$ and $\beta = \sqrt{1-\rho^2}\, e^{i\varphi}$, we have

$$\begin{aligned}(4.10)\qquad &\int_{H_2} \alpha^{m_1+k}\beta^{m_2}\,\overline{\alpha}^{m_3}\,\overline{\beta}^{n-k+m_4} dh_2 \\ &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 \{\rho^{m_1+k+m_3}(1-\rho^2)^{(m_2+m_4+n-k)/2}(e^{i\theta})^{m_1+k-m_3} \\ &\qquad\qquad \times (e^{i\varphi})^{m_2-m_4-n+k}\,\rho\}\, d\rho\, d\theta\, d\varphi.\end{aligned}$$

If $n > m$, we have $m_1+k-m_3 \neq 0$ or $m_2-m_4-n+k \neq 0$ because

$$(m_1+k-m_3)-(m_2-m_4-n+k) = n+m_1-m_3+m_4-m_2 \geq n-m > 0.$$

Therefore we obtain (4.7) from (4.10). From the definition of $\widetilde{K}_n(\ ,\ )$ we have for some $C_{n,m} \in \mathbf{R}$

$$\begin{aligned}(4.11)\qquad &\int_{H_2} \widetilde{K}_n(h_2E_0, X)\widetilde{K}_m(Y, h_2E_0)dh_2 \\ &= C_{n,m}\int_{H_2}\int_{K_{\mathbf{R}}}\int_{K_{\mathbf{R}}} \langle g\widetilde{E}_1, X\rangle^n \langle h_2E_0, g\widetilde{E}_1\rangle^n \langle g_0\widetilde{E}_1, h_2E_0\rangle^m \langle Y, g_0\widetilde{E}_1\rangle^m dg\, dg_0\, dh_2.\end{aligned}$$

We put $g = g_2 g_1$ ($g_i \in H_i, i = 1,2$). By changing variables and by using the property $k_1k_2 = k_2k_1 (k_i \in H_i, i = 1,2)$ we have from (4.7)

$$\begin{aligned}&\int_{H_2} \langle h_2E_0, g\widetilde{E}_1\rangle^n \langle g_0\widetilde{E}_1, h_2E_0\rangle^m dh_2 \\ &= \int_{H_2} \langle E_0, h_2^{-1}g_2g_1\widetilde{E}_1\rangle^n \langle h_2^{-1}g_0\widetilde{E}_1, E_0\rangle^m dh_2 \\ &= \int_{H_2} \langle g_1^{-1}E_0, h_2\widetilde{E}_1\rangle^n \langle h_2g_2^{-1}g_0\widetilde{E}_1, E_0\rangle^m dh_2 = 0 \ \ (n > m).\end{aligned}$$

Hence (4.11) implies (4.8). Q.E.D.

Proof of Proposition 4.2. (i) From (3.6) we have

$$\begin{aligned}\int_{K_{\mathbf{R}}} \widetilde{K}_{n,l}(gE_0,Y)\widetilde{K}_{n,k}(X,gE_0)dg &= \int_{H_1}\int_{H_2} \widetilde{K}_{n,l}(h_1h_2E_0,Y)\widetilde{K}_{n,k}(X,h_1h_2E_0)dh_2\, dh_1 \\ &= \int_{H_1}\int_{H_2} \widetilde{K}_{n,l}(h_2E_0,h_1^{-1}Y)\widetilde{K}_{n,k}(h_1^{-1}X,h_2E_0)dh_2\, dh_1.\end{aligned} \tag{4.12}$$

Assume $k > l$. Then from (4.8) it is valid that for any $X_1, Y_1 \in \mathfrak{p}$

$$\begin{aligned}\int_{H_2} \widetilde{K}_{n,l}(h_2E_0,Y_1)\widetilde{K}_{n,k}(X_1,h_2E_0)dh_2 &= \{\widetilde{K}_2(E_0,Y_1)\}^l\{\widetilde{K}_2(X_1,E_0)\}^k \\ &\times \int_{H_2} \widetilde{K}_{n-2l}(h_2E_0,Y_1)\widetilde{K}_{n-2k}(X_1,h_2E_0)dh_2 = 0.\end{aligned} \tag{4.13}$$

Therefore, by (4.12) and (4.13) we have (4.6). When $k < l$, we obtain (4.6) because

$$\int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(X,gE_0)\widetilde{K}_{n,l}(gE_0,Y)dg = \overline{\int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(gE_0,X)\widetilde{K}_{n,l}(Y,gE_0)dg} = 0.$$

(ii) We define the inner product of $L^2(K_{\mathbf{R}}E_0)$ by

$$(f,h) = \int_{K_{\mathbf{R}}} f(gE_0)\overline{h(gE_0)}dg$$

for $f, h \in L^2(K_{\mathbf{R}}E_0)$. Then from (4.6) we have $\mathcal{H}_{n,k} \perp \mathcal{H}_{n,l}$ for $k \neq l$ with respect to the inner product $(\ ,\ )$. To prove $\mathcal{H}_n = \bigoplus_{k=0}^{[n/2]} \mathcal{H}_{n,k}$, we have only to show that the number of $K_{\mathbf{R}}$-irreducible components of $\mathcal{H}_n$ is $[n/2]+1$ because $\mathcal{H}_{n,k} \neq \{0\}$ and $\mathcal{H}_{n,k} \perp \mathcal{H}_{n,l}$ for $k \neq l$. We denote by $S^n(\mathbf{C}^{2p} \otimes \mathbf{C}^2)$ the n-th symmetric tensor product space of $\mathbf{C}^{2p} \otimes \mathbf{C}^2$. Then the sum

$$S^n(\mathbf{C}^{2p} \otimes \mathbf{C}^2) = \sum_{\lambda} S_\lambda(\mathbf{C}^{2p}) \otimes S_\lambda(\mathbf{C}^2) \tag{4.14}$$

gives the irreducible decomposition of $S^n(\mathbf{C}^{2p} \otimes \mathbf{C}^2)$ with respect to the natural action of $GL(2p,\mathbf{C}) \times GL(2,\mathbf{C})$, where $S_\lambda(\mathbf{C}^{2p})$ and $S_\lambda(\mathbf{C}^2)$ denote the GL-irreducible representation space corresponding to the partition $\lambda = (\lambda_1, \lambda_2)$ ($\lambda_1 \geq \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = n$). Then using the branching rule from $GL(2p,\mathbf{C})$ to $Sp(p)$ stated in [4; p.507], we can see that $S_\lambda(\mathbf{C}^2)$ is always irreducible as an $Sp(1)$-module and $S_\lambda(\mathbf{C}^{2p})$ splits into $\lambda_2 + 1$ $Sp(p)$-irreducible components with highest weight $(\lambda_1 - \lambda_2 + k)\varepsilon_1 + k\varepsilon_2 = (\lambda_1 - \lambda_2)\Lambda_1 + k\Lambda_2$ ($k = 0 \sim \lambda_2$), where we use the usual numbering. Since λ_2 moves from 0 to $[n/2]$, it follows that the number of $K_{\mathbf{R}}$-irreducible components of (4.14) is $1 + 2 + \cdots + ([n/2]+1)$, which is equal to the number of $K_{\mathbf{R}}$-irreducible subspaces of S_n. Let J_m be the space of $K_{\mathbf{R}}$-invariant homogeneous polynomials of degree m. In this case we have $J_{2m-1} = \{0\}$ and $\dim J_{2m} = 1$ ($m \in \mathbf{Z}_+$). Then, from the formula $S_n = \oplus_{k=0}^n \mathcal{H}_k J_{n-k}$ (cf. Theorem1.1 (i)) we can easily show that the number of $K_{\mathbf{R}}$-irreducible subspaces of $\mathcal{H}_n$ is $[n/2]+1$ and this shows $\mathcal{H}_n = \bigoplus_{k=0}^{[n/2]} \mathcal{H}_{n,k}$.

Next we show that $\mathcal{H}_{n,k}$ and $\mathcal{H}_{m,l}$ are not $K_{\mathbf{R}}$-equivalent if $(n,k) \neq (m,l)$. By using the results $S_n = \oplus_{k=0}^{n} \mathcal{H}_k J_{n-k}$ and $S_\lambda(\mathbf{C}^{2p})$ is a sum of $Sp(p)$-irreducible components with highest weight $(\lambda_1 - \lambda_2)\Lambda_1 + k\Lambda_2$, we can show that $\mathcal{H}_n$ is a sum of $Sp(p) \times Sp(1)$-irreducible components with highest weight $\{(n-2k)\Lambda_1 + k\Lambda_2\} \otimes (n-2k)\Lambda_1$ $(k = 0 \sim [n/2])$. From this fact we can easily see that $\mathcal{H}_{n,k} \simeq \mathcal{H}_{m,l}$ if and only if $n = m$ and $k = l$, because two irreducible representations are equivalent if and only if their highest weights coincide.
Q.E.D.

Remark 4.4. The irreducible decomposition of S_n and the generators of irreducible components of this representation are also stated in [3], though the number of irreducible components in [3] was misprinted. However the generators given in [3] are not fitted to our purpose, and we give here a new proof for the sake of completeness.

Now we put $\Lambda = \{(n,k)\,;\, n \in \mathbf{Z}_+,\, 0 \leq k \leq [n/2]\}$. Under these preliminaries we define the function $\widetilde{H}_{n,k}(X,Y)$ on $\mathfrak{p} \times \mathfrak{p}$ as follows:

$$\widetilde{H}_{n,k}(X,Y) = \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(X, gE_0)\widetilde{K}_{n,k}(gE_0, Y)dg. \tag{4.15}$$

From the definition it is clear that $\widetilde{H}_{n,k}(\ \ , Y) \in \mathcal{H}_{n,k}$ for any $Y \in \mathfrak{p}$. Therefore we can easily show that $\widetilde{H}_{n,k}(\ \ , Y)$ satisfies the conditions (1.1)–(1.3) in Theorem 1.2. Then we can show the following theorem completely in the same way as in the case of $\mathfrak{su}(p,1)$ (Theorem 2.1).

Theorem 4.5. *Let $X_0 \in \mathfrak{p}$ and assume that $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ $({}^\forall(n,k) \in \Lambda)$. Then for any $f \in \mathcal{H}_{m,l}$ and $X \in \mathfrak{p}$* we have

$$\delta_{n,m}\delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0, X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{n,k}(X, gX_0)dg. \tag{4.16}$$

Especially for any $X_0 \in \mathcal{N}$ and $f \in \mathcal{H}_{n,k}$ we have

$$\widetilde{H}_{n,k}(X, X_0) = \widetilde{K}_{n,k}(X, X_0) \tag{4.17}$$

and

$$\delta_{n,m}\delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\widetilde{K}_{n,k}(X_0, X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{K}_{n,k}(X, gX_0)dg. \tag{4.18}$$

Remark 4.6. For any $Z_0 \in \Sigma_{\mathbf{R}}$ we have

$$\widetilde{H}_{n,k}(Z_0, Z_0) = \int_{K_{\mathbf{R}}} |\widetilde{K}_{n,k}(Z_0, gE_0)|^2 dg = \int_{K_{\mathbf{R}}} |\widetilde{K}_{n,k}(gZ_0, E_0)|^2 dg.$$

Since $\widetilde{K}_{n,k}(E_0, E_0) = 1$, we have $\widetilde{K}_{n,k}(X, E_0) \not\equiv 0$ on $\mathfrak{p}$. From this we see $\widetilde{K}_{n,k}(X, E_0)|_{\Sigma_{\mathbf{R}}} \not\equiv 0$ and $\int_{K_{\mathbf{R}}} |\widetilde{K}_{n,k}(gZ_0, E_0)|^2 dg \neq 0$ because $\widetilde{K}_{n,k}(\ \ , E_0) \in \mathcal{H}_n$. Therefore we can see that $\widetilde{H}_{n,k}(Z_0, Z_0) \neq 0$ and $\frac{\widetilde{H}_{n,k}(X,Y)}{\widetilde{H}_{n,k}(Z_0,Z_0)}$ satisfies (1.7) in [20].

Remark 4.7. We have $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ for any $(n, k) \in \Lambda$ if and only if $X_0 \notin \lambda K_{\mathbf{R}} \widetilde{E}_1$ and $X_0 \notin \lambda K_{\mathbf{R}} \widetilde{E}_0$ for any $\lambda \in \mathbf{C}$.

Remark 4.8. To write down $\widetilde{H}_{n,k}(X, Y)$ for the cases $\mathfrak{su}(p, 1)$ and $\mathfrak{sp}(p, 1)$ in a simple form by using some special functions is our subject.

Appendix.

In this section we will get the dimension of $\mathcal{H}_{n,k}$ for the case $\mathfrak{sp}(p, 1)$.

Proposition A.1. *When* $\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p, 1)$ $(p \geq 2)$, *we have*

$$\dim \mathcal{H}_{n,k} = \frac{(n-2k+1)^2\,(2p+n-1)\,(2p+n-k-2)!\,(2p+k-3)!}{(n-k+1)!\,k!(2p-3)!\,(2p-1)!}. \tag{A.1}$$

Furthermore the highest weight of $\mathcal{H}_{n,k}$ *is* $\{(n-2k)\Lambda_1 + k\Lambda_2\} \otimes (n-2k)\Lambda_1$ $(k = 0 \sim [n/2])$.

To prove this proposition we use the following lemma.

Lemma A.2 (cf. [19] Theorem 2.2). *Assume* $p \geq 2$. *For any* $f \in \mathcal{H}_n$ *and any* $X \in \mathfrak{p}$ *we have*

$$f(X) = \dim \mathcal{H}_n \int_0^1 \rho(t) \left(\int_{K_{\mathbf{R}}} f(g\widetilde{E}_t) \langle X, g\widetilde{E}_t \rangle^n dg \right) dt, \tag{A.2}$$

where we put

$$\rho(t) = 2^{4p-3} \frac{\Gamma(2p - \frac{1}{2})}{\sqrt{\pi}(2p-3)!} t^{4p-5} (1-t^2)^{2p-3} (2t^2-1)^2 \quad (0 \leq t \leq 1).$$

For the proof of this lemma see [19].

Proof of Proposition A.1. We can see that there exist $a_{n,q} \in \mathbf{R}$ $(q = 1, 2, \cdots, [n/2]-k)$ such that

$$K_{n,k}(X, Y) = \widetilde{K}_{n,k}(X, Y) + \sum_{q=1}^{[n/2]-k} a_{n,q} \widetilde{K}_{n,q+k}(X, Y) \quad (X, Y \in \mathfrak{p}) \tag{A.3}$$

by (4.3). (4.18) and (A.3) give that

$$\begin{aligned}(\dim \mathcal{H}_{n,k})^{-1} f(X) &= \int_{K_{\mathbf{R}}} f(gE_0) \widetilde{K}_{n,k}(X, gE_0) dg \\ &= \int_{K_{\mathbf{R}}} f(gE_0) K_{n,k}(X, gE_0) dg,\end{aligned} \tag{A.4}$$

because $\widetilde{K}_{n,k}(E_0, E_0) = 1$. From (A.2) and (4.3) we have for any $X \in \mathfrak{p}$ and $Y \in \mathcal{N}$

$$\text{(A.5)} \qquad (\dim \mathcal{H}_n)^{-1}\widetilde{K}_{n,k}(X,Y) = \int_0^1 \rho(t) \left(\int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(g\widetilde{E}_t, Y)\langle X, g\widetilde{E}_t\rangle^n dg \right) dt$$
$$= B_{n,k} \int_0^1 \rho(t) \left(\int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(g\widetilde{E}_t, Y)\widetilde{K}_{n,k}(X, g\widetilde{E}_t) dg \right) dt$$
$$= A_{n,k}B_{n,k}(\dim \mathcal{H}_{n,k})^{-1}\widetilde{K}_{n,k}(X,Y)$$
$$= A_{n,k}B_{n,k} \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(gE_0, Y)\widetilde{K}_{n,k}(X, gE_0) dg$$
$$= A_{n,k} \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(gE_0, Y)\langle X, gE_0\rangle^n dg,$$

where

$$A_{n,k} = \int_0^1 \widetilde{K}_{n,k}(\widetilde{E}_t, \widetilde{E}_t)\rho(t) dt$$

and

$$\langle X, Y\rangle^n = \sum_{q=0}^{[n/2]} B_{n,q}\widetilde{K}_{n,q}(X,Y) \quad (X, Y \in \mathfrak{p}).$$

Since

$$\widetilde{K}_{n,k}(\widetilde{E}_t, \widetilde{E}_t) = \begin{cases} \dfrac{t^{2k}(1-t^2)^k\{(1-t^2)^{n-2k+1} - t^{2(n-2k+1)}\}}{(n-2k+1)(1-2t^2)} & (t \neq \frac{1}{\sqrt{2}}), \\ 2^{-n} & (t = \frac{1}{\sqrt{2}}), \end{cases}$$

we get

$$A_{n,k} = 2^{4p-3}\frac{\Gamma(2p-\frac{1}{2})(2p+n-k-2)!\,(2p+k-3)!}{\sqrt{\pi}(2p-3)!\,(4p+n-3)!}.$$

By (A.5) we get for any $f \in \mathcal{H}_{n,k}$

$$\text{(A.6)} \qquad (\dim \mathcal{H}_n)^{-1} f(X) = A_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\langle X, gE_0\rangle^n dg.$$

Now we introduce the following polynomial to simplify the calculations:

$$h_{n,k}(X) = \langle X, \widetilde{E}_1\rangle^{n-2k}\{K_2(X, E_0)\}^k \quad (X \in \mathfrak{p}).$$

Then we have $h_{n,k} \in \mathcal{H}_n$. By using (4.6) we can see that

$$\int_{K_{\mathbf{R}}} h_{n,k}(gE_0)\widetilde{K}_{n,l}(X, gE_0)dg = 0 \quad (k \neq l, X \in \mathfrak{p})$$

and this and (4.18) show that $h_{n,k}$ belongs to $\mathcal{H}_{n,k}$. Hence (A.4) and (A.6) imply

$$(\dim \mathcal{H}_{n,k})^{-1} = \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)K_{n,k}(E_0, gE_0)dg$$

and

$$(\dim \mathcal{H}_n)^{-1} = A_{n,k} \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)\langle E_0, gE_0\rangle^n dg$$

because $h_{n,k}(E_0) = 1$. In order to compute $\dim \mathcal{H}_{n,k}$, we compare the values of the right hand sides of these two formulas. By some calculations we obtain

$$\begin{aligned}
&\int_{K_{\mathbf{R}}} h_{n,k}(gE_0)K_{n,k}(E_0, gE_0)dg \\
&= \frac{1}{n-2k+1} \int_{H_1} |K_2(X, E_0)|^{2k} \int_{H_2} \langle h_2X, \widetilde{E}_1\rangle^{n-2k}\langle E_0, h_2X\rangle^{n-2k}\, dh_2\, dh_1 \\
&= \frac{1}{2(n-2k+1)^2} \int_{H_1} (|x_1|^2 + |w_1|^2)^{n-2k}|K_2(X, E_0)|^{2k} dh_1
\end{aligned}$$

and

$$\begin{aligned}
&\int_{K_{\mathbf{R}}} h_{n,k}(gE_0)\langle E_0, gE_0\rangle^n dg \\
&= \int_{H_1} K_2(X, E_0)^k \int_{H_2} \langle h_2X, \widetilde{E}_1\rangle^{n-2k}\langle E_0, h_2X\rangle^{n}\, dh_2\, dh_1 \\
&= \frac{n!}{2k!\,(n-k+1)!} \int_{H_1} (|x_1|^2 + |w_1|^2)^{n-2k}|K_2(X, E_0)|^{2k} dh_1,
\end{aligned}$$

where we put $X = \Phi^{-1}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = h_1E_0$ and $x_i = x \cdot e_i, w_i = w \cdot e_i$ $(i = 1, 2)$. Hence we obtain

$$\dim \mathcal{H}_{n,k} = A_{n,k} \frac{n!\,(n-2k+1)^2}{k!\,(n-k+1)!} \dim \mathcal{H}_n.$$

From this we get (A.1).

In the proof of Proposition 4.2 (ii) we showed that $\mathcal{H}_n$ is a direct sum of $K_{\mathbf{R}}$-irreducible components with highest weight $\{(n-2k)\Lambda_1 + k\Lambda_2\} \otimes (n-2k)\Lambda_1$ $(k = 0 \sim [n/2])$. By using Weyl's dimension formula, we know that the dimension of the irreducible component corresponding to $\{(n-2k)\Lambda_1 + k\Lambda_2\} \otimes (n-2k)\Lambda_1$ just coincides with (A.1). Hence the highest weight of $\mathcal{H}_{n,k}$ is given by $\{(n-2k)\Lambda_1 + k\Lambda_2\} \otimes (n-2k)\Lambda_1$. Q.E.D.

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