ON INTUITIONISTIC FUZZY SEMIPRIME IDEALS IN SEMIGROUPS

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy semiprimality in a semigroup, which is an extension of fuzzy semiprimality and investigate some properties of intuitionistic fuzzification of the concept of several ideals.

1 Introduction After the introduction of fuzzy sets by L. A. Zadeh [8], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by K. T. Atanassov [1], as a generalization of the notion of fuzzy set. In [4], N. Kuroki gave some properties of fuzzy ideals and fuzzy semiprime ideals in semigroups. In this paper, we introduce the notion of intuitionistic fuzzy semiprimality in a semigroup, which is an extension of fuzzy semiprimality and investigate some properties of intuitionistic fuzzification of the concept of several ideals.

2 Preliminaries Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that $A^2 \subseteq A$, and by a *left* (*right*) *ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$. A semigroup S is said to be *regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$. A semigroup S is said to be *completely regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$ and $xy = yx$.

By a *fuzzy set* μ in a non-emptyset S we mean a function $\mu : S \to [0, 1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in S given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$.

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

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3 Intuitionistic Fuzzy ideals In what follows, let S denote a semigroup unless otherwise specified.

Definition 3.1. An IFS $A = (\mu_A, \gamma_A)$ in S is called an *intuitionistic fuzzy subsemigroup* of S if
(IF1) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
(IF2) $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,
for all $x, y \in S$.

Proposition 3.2. If IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subsemigroup of S, then the set

$$S_A := \{x \in S \mid \mu_A(x) = \mu_A(0), \gamma_A(x) = \gamma_A(0)\}$$

is a subsemigroup of S.

Proof. Let $x, y \in S_A$. Then $\mu_A(x) = \mu_A(y) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(0)$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subsemigroup of S, it follows that

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} = \mu_A(0),$$

and

$$\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\} = \gamma_A(0).$$

So, we have $\mu_A(xy) = \mu_A(0)$ and $\gamma_A(xy) = \gamma_A(0)$. Thus $xy \in S_A$. This proves the theorem. □

Definition 3.3. [2] An IFS $A = (\mu_A, \gamma_A)$ in S is called an *intuitionistic fuzzy left ideal* of S if
(IF3) $\mu_A(xy) \geq \mu_A(y)$,
(IF4) $\gamma_A(xy) \leq \gamma_A(y)$, for all $x, y \in S$. An intuitionistic fuzzy right ideal of S is defined in an analogous way. An IFS $A = (\mu_A, \gamma_A)$ in S is called an *intuitionistic fuzzy ideal* of S if it is both an intuitionistic fuzzy right and an intuitionistic fuzzy left ideal of S.

It is clear that any intuitionistic fuzzy left (right) ideal of S is an intuitionistic fuzzy subsemigroup of S.

Lemma 3.4. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subsemigroup of S such that $\mu_A(x) \geq \mu_A(y)$ (resp. $\mu_A(y) \geq \mu_A(x)$) and $\gamma_A(x) \leq \gamma_A(y)$ (resp. $\gamma_A(y) \leq \gamma_A(x)$) for all $x, y \in S$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left (resp. *right*) ideal of S.

Proof. Let $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(x) \geq \mu_A(y)$. For $x, y \in S$, we have

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} = \mu_A(y),$$

and

$$\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\} = \gamma_A(y).$$

Therefore, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S. In a similar way, it is easy to prove that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S. □

Definition 3.5. [2] An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S is called an *intuitionistic fuzzy bi-ideal* of S if
(IF5) $\mu_A(xwy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
(IF6) $\gamma_A(xwy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$
for all $w, x, y \in S$.

Example 3.6. [2] Let $S := \{a, b, c, d, e\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Define an IFS $A = (\mu_A, \gamma_A)$ in S by $\mu_A(a) = 0.6, \mu_A(b) = 0.5, \mu_A(c) = 0.4, \mu_A(d) = \mu_A(e) = 0.3$, $\gamma_A(a) = \gamma_A(b) = 0.3, \gamma_A(c) = 0.4$ and $\gamma_A(d) = 0.5, \gamma_A(e) = 0.6$. By routine calculation, we can check that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S.

A subset A of a semigroup S is called *semiprime* if $a^2 \in A$ imply $a \in A$ for all $a \in S$.

Definition 3.7. An IFS $A = (\mu_A, \gamma_A)$ is called *intuitionistic fuzzy semiprime* if
(IF7) $\mu_A(x) \geq \mu_A(x^2)$,
(IF8) $\gamma_A(x) \leq \gamma_A(x^2)$,
for all $x \in S$.

Example 3.8. Let $S = \{0, e, f, a, b\}$ be a set with the following Cayley table:

$\cdot$	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	a	0
f	0	0	f	0	b
a	0	a	0	0	e
b	0	0	b	f	0

Then S is a semigroup. Define an IFS $A = (\mu_A, \gamma_A)$ in S by $\mu_A(e) = \mu_A(f) = 1$, $\mu_A(a) = \mu_A(b) = \mu_A(0) = 0$, $\gamma_A(e) = \gamma_A(f) = 0$, and $\gamma_A(a) = \gamma_A(b) = \gamma_A(0) = 1$. By routine calculations we know that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy semiprime of S.

Theorem 3.9. If X is semiprime, then, an IFS $\tilde{A} = (\chi_X, \bar{\chi}_X)$ is intuitionistic fuzzy semiprime.

Proof. Let a be any element of S. If $a^2 \in X$, then since X is semiprime, we have $a \in X$. Thus

$$\chi_X(a) = 1 \geq \chi_X(a^2)$$

and

$$\bar{\chi}_X(a) = 1 - \chi_X(a) \leq 1 - \chi_X(a^2) = \bar{\chi}_X(a^2).$$

If $a^2 \notin X$, then we have $\chi_X(a^2) = 0$. Therefore,

$$\chi_X(a) \geq 0 = \chi_X(a^2)$$

and

$$\bar{\chi}_X(a^2) = 1 - \chi_X(a^2) \geq 1 - \chi_X(a) = \bar{\chi}_X(a).$$

This proves the theorem. □

Theorem 3.10. Let X be a nonempty subset of S. If an IFS $\tilde{A} = (\chi_X, \bar{\chi}_X)$ satisfy (IF7) or (IF8), then X is semiprime.

Proof. Suppose that $\tilde{A} = (\chi_X, \bar{\chi}_X)$ satisfy (IF7). Let $a^2 \in X$. Then, $\chi_X(a) \geq \chi_X(a^2) = 1$. So, $a \in X$. Hence X is semiprime. Now suppose that $\tilde{A} = (\chi_X, \bar{\chi}_X)$ satisfy (IF8). Let $a^2 \in X$. Then $\bar{\chi}_X(a) \leq \bar{\chi}_X(a^2) = 1 - \chi_X(a^2) = 1 - 1 = 0$, i.e, $\chi_X(a) = 1$. This show that $a \in X$. This proves the theorem. □

Theorem 3.11. For any intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S, if $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime, $A(a) = A(a^2)$ holds.

Proof. Let a be an element of S. Then, since μ_A is a fuzzy subsemigroup of S, we have

$$\mu_A(a) \geq \mu_A(a^2) = \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a),$$

and so we have $\mu_A(a) = \mu_A(a^2)$. Also, we have

$$\gamma_A(a) \leq \gamma_A(a^2) = \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a).$$

Thus $\gamma_A(a) = \gamma_A(a^2)$. This proves the theorem. □

A semigroup S is called *left* (resp. *right*) regular if, for each element a of S, there exists an element x in S such that $a = xa^2$ (resp. $a = a^2x$).

Theorem 3.12. Let S be left regular. Then, for every intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ of S, $A(a) = A(a^2)$ holds for all $a \in S$.

Proof. Let a be any element of S. Then since S is left regular, there exists an element x in S such that $a = xa^2$. Then we have

$$\mu_A(a) = \mu_A(xa^2) \geq \mu_A(a^2) \geq \mu_A(a),$$

and so we have $\mu_A(a) = \mu_A(a^2)$. Also, we have

$$\gamma_A(a) = \gamma_A(xa^2) \leq \gamma_A(a^2) \leq \gamma_A(a).$$

Thus $\gamma_A(a) = \gamma_A(a^2)$. So, $A(a) = A(a^2)$. This proves the theorem. □

Theorem 3.13. Let S be left regular. Then, every intuitionistic fuzzy left ideal of S is intuitionistic fuzzy semiprime.

Proof. Let IFS $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S and let $a \in S$. Then, there exists an element x in S such that $a = xa^2$ since S is left regular. So, we have $\mu_A(a) = \mu_A(xa^2) \geq \mu_A(a^2)$, and $\gamma_A(a) = \gamma_A(xa^2) \leq \gamma_A(a^2)$. This proves the theorem. □

A semigroup S is called *intra-regular* if, for each element a of S, there exist elements x and y in S such that $a = xa^2y$.

Definition 3.14. [3] An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S is called an *intuitionistic fuzzy interior ideal* of S if
(IF9) $\mu_A(xay) \geq \mu_A(a)$,
(IF10) $\gamma_A(xay) \leq \gamma_A(a)$,
for all $x, y, a \in S$.

Theorem 3.15. Let $A = (\mu_A, \gamma_A)$ be an IFS in an intra-regular semigroup S. Then, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of S.

Proof. Let a, b be any elements of S, and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of S. Then, since S is intra-regular, there exist elements x, y, u and v in S such that $a = xa^2y$ and $b = ub^2y$. Then, since μ_A is a fuzzy interior ideal of S, we have

$$\mu_A(ab) = \mu_A((xa^2y)b) = \mu_A((xa)a(yb)) \geq \mu_A(a)$$

and

$$\mu_A(ab) = \mu_A(a(ub^2v)) = \mu_A((au)b(bv)) \geq \mu_A(b).$$

Also, we have

$$\gamma_A(ab) = \gamma_A((xa^2y)b) = \gamma_A((xa)a(yb)) \leq \gamma_A(a)$$

and

$$\gamma_A(ab) = \gamma_A(a(ub^2v)) = \gamma_A((au)b(bv)) \leq \gamma_A(b).$$

On the other hand, let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of S. Then, since μ_A is a fuzzy ideal of S, we have

$$\mu_A(xay) = \mu_A(x(ay)) \geq \mu_A(ay) \geq \mu_A(a),$$

and

$$\gamma_A(xay) = \gamma_A(x(ay)) \leq \gamma_A(ay) \leq \gamma_A(a)$$

for all x, a and $y \in S$. This proves the theorem. □

Theorem 3.16. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of S. If S is intra-regular, then $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime.

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. So, we have

$$\mu_A(a) = \mu_A(xa^2y) \geq \mu_A(a^2y) \geq \mu_A(a^2),$$

and

$$\gamma_A(a) = \gamma_A(xa^2y) \leq \gamma_A(a^2y) \leq \gamma_A(a^2).$$

This proves the theorem. □

Theorem 3.17. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of S. If S is an intra-regular, then $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime.

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. So, we have

$$\mu_A(a) = \mu_A(xa^2y) \geq \mu_A(a^2),$$

and

$$\gamma_A(a) = \gamma_A(xa^2y) \leq \gamma_A(a^2).$$

This proves the theorem. □

Theorem 3.18. Let S be intra-regular. Then, for all intuitionistic fuzzy interior ideal $A = (\mu_A, \gamma_A)$ and for all $a \in S$, $A(a) = A(a^2)$ holds

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. So, we have

$$\mu_A(a) = \mu_A(xa^2y) \geq \mu_A(a^2) = \mu_A((xa^2y)(xa^2y)) = \mu_A((xa)a(yxa^2y)) \geq \mu_A(a),$$

and

$$\gamma_A(a) = \gamma_A(xa^2y) \leq \gamma_A(a^2) = \gamma_A((xa^2y)(xa^2y)) = \gamma_A((xa)a(yxa^2y)) \leq \gamma_A(a).$$

So, we have $A(a) = A(a^2)$. This proves the theorem. □

Theorem 3.19. Let S be intra-regular. Then, for all intuitionistic fuzzy interior ideal $A = (\mu_A, \gamma_A)$ and for all $a, b \in S$, $A(ab) = A(ba)$ holds

Proof. Let a be any element of S. Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. So, we have

$$\mu_A(ab) = \mu_A((ab)^2 = \mu_A(a(ba)b) \geq \mu_A(ba) = \mu_A((ba)^2)\mu_A(b(ab)a) \geq \mu_A(ab),$$

and

$$\gamma_A(ab) = \gamma_A((ab)^2 = \gamma_A(a(ba)b) \leq \gamma_A(ba) = \gamma_A((ba)^2)\gamma_A(b(ab)a) \leq \gamma_A(ab).$$

So, we have $A(ab) = A(ba)$. This proves the theorem. □

A semigroup S is called *archimedean* if, for any elements a, b, there exists a positive integer n such that $a^n \in SbS$.

Theorem 3.20. Let S be an archimedean semigroup. Then, every intuitionistic fuzzy semiprime fuzzy ideal of S is a constant function.

Proof. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy semiprime fuzzy ideal of S and $a, b \in S$. Then since S is archimedean, there exist x and y in S such that $a^n = xby$ for some integer n. Then, we have

$$\mu_A(a) = \mu_A(a^n) = \mu_A(xby) \geq \mu_A(b),$$

and

$$\mu_A(b) = \mu_A((b^n) = \mu_A(xay) \geq \mu_A(a).$$

Thus, we have $\mu_A(a) = \mu_A(b)$. Also, we have

$$\gamma_A(a) = \gamma_A(a^n) = \gamma_A(xby) \leq \gamma_A(b),$$

and

$$\gamma_A(b) = \gamma_A(b^n) = \gamma_A(xay) \leq \gamma_A(a).$$

Therefore, we have $A(a) = A(b)$ for all $a, b \in S$. This proves the theorem. □

References

[1] K. T. Atanassov, "Intuitionistic fuzzy sets" *Fuzzy sets and Systems* **20** (1986), 87-96.

[2] K. H. Kim and Y. B. Jun, "Intuitionistic fuzzy ideals of semigroups," *Indian J. Pure Appl. Math.* **33(4)** (2002), 443-449.

[3] K. H. Kim and Y. B. Jun, "Intuitionistic fuzzy interior ideals of semigroups," *Int. J. Math. Math. Sci.* **33(4)** (2002), 443-449.

[4] N. Kuroki, "On fuzzy ideals and fuzzy bi-ideals in semigroups," *Fuzzy Sets and Systems* **5** (1981), 203-215.

[5] N. Kuroki, "Fuzzy semiprime ideals in semigroups," *Fuzzy Sets and Systems* **8** (1982), 71-79.

[6] A. Rosenfeld, "Fuzzy groups," *J. Math. Anal. Appl.* **35** (1971), 512-517.

[7] Y. H. Yon and K. H. Kim, *On intuitionistic fuzzy filters and ideals of lattices,* Far East J Math. Sci.(FJMS) **1 (3)**, (1999), 429-442.

[8] L. A. Zadeh, "Fuzzy sets," *Information and Control.* **8** (1965), 338-353.

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