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REAL K-COHOMOLOGY OF COMPLEX PROJECTIVE SPACES

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Abstract. In this paper, we determine the structure of KO-cohomology of complex projective space $\boldsymbol{C}P^l$ and its product space $\boldsymbol{C}P^l\times\boldsymbol{C}P^m$ as algebras over the coefficient ring KO^*. We also give a description of the map $KO^*(\boldsymbol{C}P^{l+m}) \to KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ induced by the map that classifies the tensor product of the canonical line bundles and show that its image is not contained in the image of the cross product $KO^*(\boldsymbol{C}P^l)\otimes_{KO^*} KO^*(\boldsymbol{C}P^m) \to KO^*(\boldsymbol{C}P^l\times\boldsymbol{C}P^m)$ to see that non-existence of the formal group structure on $KO^*(\boldsymbol{C}P^\infty)$.

Introduction A commutative ring spectrum E is said to be complex oriented if an element x of the reduced E-cohomology of the infinite dimensional complex projective space $\boldsymbol{C}P^\infty$ is given such that x maps to a generator of the reduced E-cohomology of 1-dimensional complex projective space $\boldsymbol{C}P^1$ ([2]). We call such an element x a complex orientation of E. On the other hand, if E-homology E_*E of E is a flat over the coefficient ring E_*, E_*E has a structure of a Hopf algebroid and E-homology theory takes values in the category of E_*E-comodule, in other words, the category of representations of the groupoid represented by the affine groupoid scheme represented by E_*E ([1]).

If E is a complex oriented ring spectrum, the E-cohomology of the complex projective space is just a truncated polynomial algebra over E_* and it is shown that E-homology E_*E of E is a flat over E_*. Moreover the product structure of $\boldsymbol{C}P^\infty$ gives a one dimensional formal group law over E^* ([5]) which closely relates with the structure of the Hopf algebroid ([2]). The complex K-theory is one of the most basic examples of complex oriented cohomology theories. However, KO-spectrum representing the real K-theory is one of a few well-known examples of spectra E without any complex orientation such that E_*E is flat over E_* ([2], [7]). In fact, we see that KO-spectrum does not have any complex orientation by showing that the Atiyah-Hirzebruch spectral sequence converging to $KO^*(\boldsymbol{C}P^l)$ has a non-trivial differential (2.2).

The purpose of this paper is to determine the structure of KO-cohomology of complex projective space $\boldsymbol{C}P^l$ and its product space $\boldsymbol{C}P^l\times\boldsymbol{C}P^m$ as algebras over the coefficient ring KO^* in order to understand the behavior of the following map γ^*. Let us denote by $\gamma : \boldsymbol{C}P^l\times\boldsymbol{C}P^m \to \boldsymbol{C}P^{l+m}$ the map induced by the classifying map $\boldsymbol{C}P^\infty\times\boldsymbol{C}P^\infty\to\boldsymbol{C}P^\infty$ of the tensor product of the canonical line bundles. We give an explicit description of the map $\gamma^* : KO^*(\boldsymbol{C}P^{l+m}) \to KO^*(\boldsymbol{C}P^l\times\boldsymbol{C}P^m)$ and show that image of γ^* is not contained in the image of the cross product $KO^*(\boldsymbol{C}P^l)\otimes_{KO^*} KO^*(\boldsymbol{C}P^m) \to KO^*(\boldsymbol{C}P^l\times\boldsymbol{C}P^m)$ (3.13). This implies a negative result that the classifying map $\boldsymbol{C}P^\infty\times\boldsymbol{C}P^\infty\to\boldsymbol{C}P^\infty$ does not give a formal group structure on $KO^*(\boldsymbol{C}P^\infty)$. In [3], M. Fujii has described the structure of $KO^*(\boldsymbol{C}P^l)$ as a graded abelian group and the ring structure of the subring of $KO^*(\boldsymbol{C}P^l)$ consisting of even dimensional elements and our result on $KO^*(\boldsymbol{C}P^l)$ is slightly sharper than his result in the point that we give a complete description of $KO^*(\boldsymbol{C}P^l)$ as an algebra over KO^*.

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1 Preliminaries

We first recall the Bott periodicity

$$O \simeq \Omega(\boldsymbol{Z} \times BO), \quad O/U \simeq \Omega O, \quad U/Sp \simeq \Omega(O/U), \quad \boldsymbol{Z} \times BSp \simeq \Omega(U/Sp)$$

$$Sp \simeq \Omega(\boldsymbol{Z} \times BSp), \quad Sp/U \simeq \Omega Sp, \quad U/O \simeq \Omega(Sp/U), \quad \boldsymbol{Z} \times BO \simeq \Omega(U/O).$$

Thus the KO-spectrum $KO = (\varepsilon_n : SKO_n \to KO_{n+1})_{n \in \boldsymbol{Z}}$ is given as follows.

$$KO_{8n} = \boldsymbol{Z} \times BO, \quad KO_{8n+1} = U/O, \quad KO_{8n+2} = Sp/U, \quad KO_{8n+3} = Sp,$$

$$KO_{8n+4} = \boldsymbol{Z} \times BSp, \quad KO_{8n+5} = U/Sp, \quad KO_{8n+6} = O/U, \quad KO_{8n+7} = O.$$

We also recall that $K^* = \boldsymbol{Z}[t, t^{-1}]$, $KO^* = \boldsymbol{Z}[\alpha, x, y, y^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2 - 4y)$, where t, α, x and y are generators of $K^{-2} = \pi_2(K) \cong \boldsymbol{Z}$, $KO^{-1} = \pi_1(KO) \cong \boldsymbol{Z}/2\boldsymbol{Z}$, $KO^{-4} = \pi_4(KO) \cong \boldsymbol{Z}$, $KO^{-8} = \pi_8(KO) \cong \boldsymbol{Z}$, respectively. Note that t, α are the homotopy classes of the inclusion maps $S^2 = \boldsymbol{C}P^1 \to BU = K_0$, $S^1 = \boldsymbol{R}P^1 \to BO = KO_0$ to the bottom cells.

Let us denote by $h_2 : S^3 \to S^2$ the Hopf map, by $j : S^3 = Sp(1) \to Sp$, $i : S^2 = Sp(1)/U(1) \to Sp/U$ the inclusion maps of the bottom cells, and by $p : Sp \to Sp/U$ the quotient map. Then

$$\begin{array}{ccc} S^3 & \xrightarrow{h_2} & S^2 \\ \downarrow j & & \downarrow i \\ Sp & \xrightarrow{p} & Sp/U \end{array}$$

commutes.

Lemma 1.1 *The homotopy class of $ih_2 = pj$ generates $\pi_3(Sp/U) \cong \boldsymbol{Z}/2\boldsymbol{Z}$. Hence ih_2 represents $\alpha \in \pi_1(KO) \cong \pi_3(KO_2)$.*

Proof. By the commutativity of the above diagram, we have the following commutative diagram.

$$\begin{array}{ccc} \pi_3(S^3) & \xrightarrow[\cong]{h_{2*}} & \pi_3(S^2) \\ \cong \downarrow j_* & & \downarrow i_* \\ \pi_3(Sp) & \xrightarrow{p_*} & \pi_3(Sp/U) \end{array}$$

Since $p_* : \pi_3(Sp) \to \pi_3(Sp/U)$ is surjective, the assertion follows. *Q.E.D.*

Lemma 1.2 *Let n and m be integers such that $n \geq 2$. Then, the composition of*

$$(S^{n-2}h_2)^* : \widetilde{KO}^m(S^n) \to \widetilde{KO}^m(S^{n+1})$$

and the inverse of the suspension

$$\sigma^{-1} : \widetilde{KO}^m(S^{n+1}) \to \widetilde{KO}^{m-1}(S^n)$$

coincides with the multiplication map by α.

Proof. Let $f : S^n \to KO_m$ be a map which represents an element ξ of $\widetilde{KO}^m(S^n)$. Then, $\sigma^2(S^{n-2}h_2)^*(\xi)$ is represented by $S^{n+3} \xrightarrow{S^n h_2} S^{n+2} \xrightarrow{S^2 f} S^2 KO_m \xrightarrow{\varepsilon_{m+1} S\varepsilon_m} KO_{m+2}$. Since the diagram

$$\begin{array}{ccccccc} S^3 \wedge S^n & \xrightarrow{h_2 \wedge 1_{S^n}} & S^2 \wedge S^n & \xrightarrow{S^2 f} & S^2 \wedge KO_m & \xrightarrow{i \wedge 1_{KO_m}} & (Sp/U) \wedge KO_m \\ & & & & \downarrow{\scriptstyle S\varepsilon_m} & & \downarrow{\scriptstyle \mu_{2,m}} \\ & & & & SKO_{m+1} & \xrightarrow{\varepsilon_{m+1}} & KO_{m+2} \end{array}$$

commutes, $\sigma^3(\alpha\xi)$ is represented by $S^3 \wedge S^n \xrightarrow{h_2 \wedge 1_{S^n}} S^2 \wedge S^n \xrightarrow{S^2 f} S^2 \wedge KO_m \xrightarrow{\varepsilon_{m+1} S\varepsilon_m} KO_{m+2}$. We have seen that $S^n h_2$ is homotopic to $h_2 \wedge 1_{S^n}$. It follows that $\sigma^2(S^{n-2}h_2)^*(\xi) = \sigma^3(\alpha\xi)$ *Q.E.D.*

Lemma 1.3 *Let $\eta_s : S^{2s-1} \to S^{2s-2} = \boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2}$ be the attaching map of the $2s$-cell of $\boldsymbol{C}P^s/\boldsymbol{C}P^{s-2}$ $(s \geq 2)$. Then, η_s is null homotopic if s is odd and it is homotopic to $S^{2s-4}h_2$ if s is even.*

Proof. Let g_j $(j = 2s-2, 2s)$ be the generators of $H^j(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-2}; \boldsymbol{F}_2)$. Since

$$Sq^2 g_{2s-2} = \begin{cases} g_{2s} & s \text{ is even} \\ 0 & s \text{ is odd} \end{cases},$$

the assertion follows. *Q.E.D.*

Let us denote by $v_i \in \widetilde{KO}^i(S^i)$ $(i \geq 0)$ the canonical generators, that is, v_i's are given by $v_0 = 1$, $\sigma(v_i) = v_{i+1}$. For $s \geq 2$, consider the cofiber sequence

$$\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2} \xrightarrow{\iota} \boldsymbol{C}P^s/\boldsymbol{C}P^{s-2} \xrightarrow{\kappa} \boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}.$$

We have the long exact sequences associated with this cofiber sequence.

$$\cdots \to \widetilde{KO}^n(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) \xrightarrow{\kappa^*} \widetilde{KO}^n(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-2}) \xrightarrow{\iota^*} \widetilde{KO}^n(\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2}) \xrightarrow{\delta}$$
$$\widetilde{KO}^{n+1}(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) \to \cdots$$

Lemma 1.4 *The connecting homomorphism*

$$\delta : \widetilde{KO}^n(\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2}) \to \widetilde{KO}^{n+1}(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1})$$

is given by

$$\delta(v_{2s-2}) = \begin{cases} \alpha v_{2s} & s \text{ is even} \\ 0 & s \text{ is odd} \end{cases}.$$

Proof. Since the composition

$$\widetilde{KO}^n(\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2}) \xrightarrow{\delta} \widetilde{KO}^{n+1}(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) = \widetilde{KO}^{n+1}(S^{2s}) \xrightarrow{\sigma^{-1}} \widetilde{KO}^n(S^{2s-1})$$

coincides with the map induced by the attaching map η_s, the second formula follows from (1.3) and (1.2). *Q.E.D.*

The following result is known.

Proposition 1.5 *The complexification map* $\boldsymbol{c} : KO^*(X) \to K^*(X)$, *the realization map* $\boldsymbol{r} : K^*(X) \to KO^*(X)$ *and the conjugation map* $\Psi^{-1} : K^*(X) \to K^*(X)$ *are natural transformation of cohomology theories having the following properties.*

1) $\boldsymbol{c}$ *is a homomorphism of graded rings which maps* $\alpha \in KO^{-1}$ *to* 0, $x \in KO^{-4}$ *to* $2t^2$ *and* $y \in KO^{-8}$ *to* t^4.

2) $\boldsymbol{r}$ *is a homomorphism of graded abelian groups which maps* $t^{4i} \in K^{-8i}$ *to* $2y^i$, $t^{4i+1} \in K^{-8i-2}$ *to* $\alpha^2 y^i$ *and* $t^{4i+2} \in K^{-8i-4}$ *to* xy^i *for* $i \in \boldsymbol{Z}$.

3) Ψ^{-1} *is a ring homomorphism.*

4) $\boldsymbol{rc} = 2id_{KO^*(X)}$, $\boldsymbol{cr} = id_{K^*(X)} + \Psi^{-1}$ *and* $\Psi^{-1}\Psi^{-1} = id_{K^*(X)}$ *hold.*

By the above result, $\boldsymbol{cr}$ maps $t \in K^{-2}$ to $\boldsymbol{c}(\alpha)^2 = 0$. Thus we have $t + \Psi^{-1}(t) = \boldsymbol{cr}(t) = 0$, namely,

Corollary 1.6 $\Psi^{-1}(t) = -t$.

We denote by $B : \widetilde{K}^n(X) \to \widetilde{K}^{n-2}(X)$ the Bott periodicity map $B(a) = ta$ and by $\boldsymbol{\alpha} : \widetilde{KO}^n(X) \to \widetilde{KO}^{n-1}(X)$ the multiplication map by $\alpha \in KO^{-1}$. A fiber sequence $U/O \to BO \to BU$ gives a cofiber sequence $\Sigma KO \to KO \xrightarrow{\boldsymbol{c}} K$ of spectra. The following result is also known.

Proposition 1.7 *([4] Chap. III 5.18) There is a long exact sequence*

$$\cdots \to \widetilde{K}^{n-1}(X) \xrightarrow{\boldsymbol{r}B^{-1}} \widetilde{KO}^{n+1}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}^{n}(X) \xrightarrow{\boldsymbol{c}} \widetilde{K}^{n}(X) \xrightarrow{\boldsymbol{r}B^{-1}} \widetilde{KO}^{n+2}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}^{n+1}(X) \to \cdots .$$

Corollary 1.8 *Let* X *be a space such that* $K^1(X) = \{0\}$ *(*$X = \boldsymbol{C}P^l$ *or* $\boldsymbol{C}P^l \times \boldsymbol{C}P^m$*, for example). There is an exact sequence*

$$0 \to \widetilde{KO}^{2n+1}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}^{2n}(X) \xrightarrow{\boldsymbol{c}} \widetilde{K}^{2n}(X) \xrightarrow{\boldsymbol{r}B^{-1}} \widetilde{KO}^{2n+2}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}^{2n+1}(X) \to 0.$$

2 Real K-cohomology of complex projective spaces

Let us denote by η_l the canonical complex line bundle over $\boldsymbol{C}P^l$. Put $\mu = \eta_l - 1 \in \widetilde{K}^0(\boldsymbol{C}P^l)$. Then, $K^*(\boldsymbol{C}P^l) = K^*[\mu]/(\mu^{l+1})$ and $\Psi^{-1}(\mu) = (1+\mu)^{-1} - 1$. Hence it follows from (1.5) that $\boldsymbol{cr}(\mu) = \mu^2 - \mu^3 + \cdots + (-1)^l \mu^l$.

Remark 2.1 *Put* $\tilde{\mu} = \mu(1+\mu)^{-\frac{1}{2}} \in K^0(\boldsymbol{C}P^\infty)\widehat{\otimes}\boldsymbol{Q} = \boldsymbol{Q}[[\mu]]$. *Then* $\Psi^{-1}(\tilde{\mu}) = -\tilde{\mu}$. *Let us denote by* W_1 *(resp.* W_{-1}*) the eigen space of* $\Psi^{-1} : \boldsymbol{Q}[[\mu]] \to \boldsymbol{Q}[[\mu]]$ *corresponding to eigen value* 1 *(resp.* -1*). Then,* $\{\tilde{\mu}^{2i} | \, i = 0,1,2,\dots\}$ *(resp.* $\{\tilde{\mu}^{2i+1} | \, i = 0,1,2,\dots\}$*) generates* W_1 *(resp.* W_{-1}*) topologically.*

Consider the Atiyah-Hirzebruch spectral sequence $E_2^{p,q}(KO;\boldsymbol{C}P^l) \cong H^p(\boldsymbol{C}P^l;KO^q) \Rightarrow KO^{p+q}(\boldsymbol{C}P^l)$. Let us denote by u the generator of $E_2^{2,0}(KO;\boldsymbol{C}P^l) \cong H^2(\boldsymbol{C}P^l;KO^0)$, then

$$E_2^{*,*}(KO;\boldsymbol{C}P^l) = KO^*[u]/(u^{l+1}) = Z[\alpha, x, y, y^{-1}, u]/(2\alpha, \alpha^3, \alpha x, x^2 - 4y, u^{l+1}),$$

where $\alpha \in E_2^{0,-1}(KO;\boldsymbol{C}P^l)$, $x \in E_2^{0,-4}(KO;\boldsymbol{C}P^l)$, $y \in E_2^{0,-8}(KO;\boldsymbol{C}P^l)$.

Lemma 2.2 $d_2 : E_2^{p,q}(KO;\boldsymbol{C}P^l) \to E_2^{p+2,q-1}(KO;\boldsymbol{C}P^l)$ *is given by* $d_2(u^j) = j\alpha u^{j+1}$.

Proof. We first note that the p-skeleton $(\boldsymbol{C}P^l)^p$ is $\boldsymbol{C}P^{[\frac{p}{2}]}$ if $p \leqq 2l$. Hence $E_1^{p,q}(KO,\boldsymbol{C}P^l) = 0$ if p is odd and $E_2^{p,q}(KO;\boldsymbol{C}P^l) = E_1^{p,q}(KO;\boldsymbol{C}P^l) = \widetilde{KO}^{p+q}(\boldsymbol{C}P^{\frac{p}{2}}/\boldsymbol{C}P^{\frac{p}{2}-1})$ if p is positive and even. If p is even, $d_2 : E_2^{p,q}(KO;\boldsymbol{CP}^l) \to E_2^{p+2,q-1}(KO;\boldsymbol{CP}^l)$ coincides with the connecting homomorphism

$$\delta : \widetilde{KO}^{p+q}(\boldsymbol{CP}^{\frac{p}{2}}/\boldsymbol{CP}^{\frac{p}{2}-1}) \to \widetilde{KO}^{p+q+1}(\boldsymbol{CP}^{\frac{p}{2}+1}/\boldsymbol{CP}^{\frac{p}{2}})$$

of the long exact sequence associated with the cofibration

$$\boldsymbol{CP}^{\frac{p}{2}}/\boldsymbol{CP}^{\frac{p}{2}-1} \to \boldsymbol{CP}^{\frac{p}{2}+1}/\boldsymbol{CP}^{\frac{p}{2}-1} \to \boldsymbol{CP}^{\frac{p}{2}+1}/\boldsymbol{CP}^{\frac{p}{2}}.$$

Then, the result follows from (1.4). *Q.E.D.*

By the above result, $\alpha^2 u$, $2u$, u^2 and $xy^{-1}u$ are cocycles of the E_2-term. We denote by $u_0 \in E_3^{2,-2}(KO;\boldsymbol{CP}^l)$, $u_1 \in E_3^{2,0}(KO;\boldsymbol{CP}^l)$, $u_2 \in E_3^{4,0}(KO;\boldsymbol{CP}^l)$ and $u_3 \in E_3^{2,4}(KO;\boldsymbol{CP}^l)$ the elements of the E_3-term corresponding to $\alpha^2 u$, $2u$, u^2 and $xy^{-1}u$, respectively. Since u^l is also a cocycle if l is odd, we denote by $v_l \in E_3^{2l,0}(KO;\boldsymbol{CP}^l)$ the element corresponding to u^l. The following fact is a direct consequence of the definition of u_i, v_l and (2.2).

Proposition 2.3 *The following relations hold;* $2u_0 = xu_0 = \alpha u_0 = \alpha u_1 = \alpha u_2 = \alpha u_3 = 0$, $xu_3 = 2u_1$, $xu_1 = 2yu_3$, $u_0^2 = u_0u_1 = u_0u_3 = u_2^{[\frac{l}{2}]+1} = u_0u_2^{[\frac{l+1}{2}]} = u_1u_2^{[\frac{l+1}{2}]} = u_2^{[\frac{l+1}{2}]}u_3 = 0$, $u_1^2 = 4u_2$, $u_1u_3 = 2xy^{-1}u_2$, $u_3^2 = 4y^{-1}u_2$. *If l is odd,* $u_0v_l = u_1v_l = u_2u_l = u_3v_l = v_l^2 = 0$, $u_0u_2^{\frac{l-1}{2}} = \alpha^2 v_l$, $u_1u_2^{\frac{l-1}{2}} = 2v_l$, $u_2^{\frac{l-1}{2}}u_3 = xy^{-1}v_l$.

Proposition 2.4 *E_3-term is generated by the following set of elements over KO^*.*

1) If l is even, $\left\{ u_2^j u_k \;\middle|\; 0 \le j \le \frac{l}{2}-1,\ 0 \le k \le 3 \right\} \cup \{1\}$.

2) If l is odd, $\left\{ u_2^j u_k \;\middle|\; 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3 \right\} \cup \{1,\ v_l\}$.

Proof. By (2.2), the kernel of d_2 is generated over KO^* by αu^j $(j = 1,2,\dots,l)$, xu^j $(j = 1,2,\dots,l)$, $2u^{2j+1}$ $(j = 0,1,\dots,[\frac{l-1}{2}])$, u^{2j} $(j = 0,1,\dots,[\frac{l}{2}]$, and $\frac{l}{2}$ if l is odd.). The image of d_2 is generated over KO^* by α^2u^{2j} $(j = 1,2,\dots,[\frac{l}{2}])$. It follows that the E_3-term is generated over KO^* by u_2^j $(j = 0,1,\dots,[\frac{l}{2}])$, $u_2^j u_k$ $(k = 0,1,3,\ j = 0,1,\dots,[\frac{l-1}{2}])$ and, if l is odd, v_l. If l is odd, since $u_0u_2^{\frac{l-1}{2}} = \alpha^2 v_l$, $u_1u_2^{\frac{l-1}{2}} = 2v_l$, $u_2^{\frac{l-1}{2}}u_3 = xy^{-1}v_l$ by (2.3), $u_0u_2^{\frac{l-1}{2}}$, $u_1u_2^{\frac{l-1}{2}}$, $u_2^{\frac{l-1}{2}}u_3$ are not needed to generate the E_3-term. *Q.E.D.*

Corollary 2.5 $E_3^{*,*}(KO;\boldsymbol{CP}^l) = E_\infty^{*,*}(KO;\boldsymbol{CP}^l)$

Proof. Since $E_3^{p,q}(KO;\boldsymbol{CP}^l) = \{0\}$ if $p+q$ is odd and $0 < p < 2l$, there is no possibility of non-trivial differentials. *Q.E.D.*

We also consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q}(K;\boldsymbol{CP}^l) \cong H^p(\boldsymbol{CP}^l;K^q) \Rightarrow K^{p+q}(\boldsymbol{CP}^l).$$

The E_2-term is given by

$$E_2^{*,*}(K;\boldsymbol{CP}^l) = K^*[u]/(u^{l+1}) = \boldsymbol{Z}[t,t^{-1},u]/(u^{l+1})$$

and $tu \in E_2^{2,-2}(K;\boldsymbol{CP}^l)$ is the permanent cocycle corresponding to the generator $\mu \in K^0(\boldsymbol{CP}^l)$.

There are maps

$$\boldsymbol{r}_r : E_r^{p,q}(K;\boldsymbol{CP}^l) \to E_r^{p,q}(KO;\boldsymbol{CP}^l), \quad \boldsymbol{c}_r : E_r^{p,q}(KO;\boldsymbol{CP}^l) \to E_r^{p,q}(K;\boldsymbol{CP}^l)$$

of spectral sequences induced by $\boldsymbol{r} : K^*(\boldsymbol{CP}^l) \to KO^*(\boldsymbol{CP}^l)$ and $\boldsymbol{c} : KO^*(\boldsymbol{CP}^l) \to K^*(\boldsymbol{CP}^l)$. By 2) of (1.5), we have $\boldsymbol{r}_2(t^{4i}u^j) = 2y^iu^j$, $\boldsymbol{r}_2(t^{4i+1}u^j) = \alpha^2y^iu^j$, $\boldsymbol{r}_2(t^{4i+2}u^j) = xy^iu^j$ and $\boldsymbol{r}_2(t^{4i+3}u^j) = 0$.

If $l \geqq 2$, we define elements $\omega_i \in \widetilde{KO}^{2i}(\boldsymbol{CP}^l)$ for $i = 0, 1, 2, 3$ by $\omega_i = \boldsymbol{r}(t^{-i}\mu)$ as in [3].

Lemma 2.6 *$\alpha^2u \in E_2^{2,-2}(KO;\boldsymbol{CP}^l)$, $2u \in E_2^{2,0}(KO;\boldsymbol{CP}^l)$ and $xy^{-1}u \in E_2^{2,4}(KO;\boldsymbol{CP}^l)$ are permanent cocycles corresponding to ω_0, ω_1 and ω_3, respectively. Hence $\omega_0 \in F^{2,-2} - F^{3,-3}$, $\omega_1 \in F^{2,0} - F^{3,-1}$ and $\omega_3 \in F^{2,4} - F^{3,3}$.*

Proof. The assertion follows from $\boldsymbol{r}_2(tu) = \alpha^2u$, $\boldsymbol{r}_2(t^{-1}tu) = \boldsymbol{r}_2(u) = 2u$, $\boldsymbol{r}_2(t^{-3}tu) = \boldsymbol{r}_2(t^{-2}u) = xy^{-1}u$. *Q.E.D.*

Lemma 2.7 *$\boldsymbol{c} : KO^*(\boldsymbol{CP}^l) \to K^*(\boldsymbol{CP}^l)$ maps ω_j as follows.*

$$\boldsymbol{c}(\omega_{2i}) = t^{-2i}\mu(1 - (1+\mu)^{-1}), \quad \boldsymbol{c}(\omega_{2i+1}) = t^{-2i-1}\mu(1 + (1+\mu)^{-1}) \quad (i = 0, 1)$$

Proof. We note that $\Psi^{-1} : K^*(\boldsymbol{CP}^l) \to K^*(\boldsymbol{CP}^l)$ is a homomorphism of graded rings such that $\Psi^{-1}(t) = -t$ (1.6). Hence, by (1.5), $\boldsymbol{c}(\omega_{2i}) = \boldsymbol{cr}(t^{-2i}\mu) = t^{-2i}\mu + \Psi^{-1}(t^{-2i}\mu) = t^{-2i}\mu + t^{-2i}((1+\mu)^{-1} - 1) = t^{-2i}\mu(1 - (1+\mu)^{-1})$ for $i = 0, 1$. Similarly, $\boldsymbol{c}(\omega_{2i+1}) = \boldsymbol{cr}(t^{-2i-1}\mu) = t^{-2i-1}\mu + \Psi^{-1}(t^{-2i-1}\mu) = t^{-2i-1}\mu - t^{-2i-1}((1+\mu)^{-1} - 1) = t^{-2i-1}\mu(1 + (1+\mu)^{-1})$ for $i = 0, 1$. *Q.E.D.*

Lemma 2.8 *ω_2 belongs to the kernel $F^{4,0}$ of the map $KO^4(\boldsymbol{CP}^l) \to KO^4(\boldsymbol{CP}^1)$ induced by the inclusion map. On the other hand, ω_2 does not belong to the kernel $F^{5,-1}$ of the map $KO^4(\boldsymbol{CP}^l) \to KO^4(\boldsymbol{CP}^2)$.*

Proof. We observe that $\boldsymbol{r}_2 : E_2^{2,2}(K;\boldsymbol{CP}^1) \to E_2^{2,2}(KO;\boldsymbol{CP}^1)$ maps $t^{-1}u$ to zero. Since $E_2^{2,2}(K;\boldsymbol{CP}^1) = E_\infty^{2,2}(K;\boldsymbol{CP}^1)$, $E_2^{2,2}(KO;\boldsymbol{CP}^1) = E_\infty^{2,2}(KO;\boldsymbol{CP}^1)$ and $t^{-1}u$ is the permanent cocycle corresponding to $t^{-2}\mu \in K^4(\boldsymbol{CP}^1)$, we see

$$\boldsymbol{r}(t^{-2}\mu) \in F^{3,1} = \mathrm{Ker}(KO^4(\boldsymbol{CP}^1) \to KO^4(\boldsymbol{CP}^1)) = \{0\}.$$

By the commutativity of the following diagram, $t^{-2}\mu \in K^4(\boldsymbol{CP}^l)$ maps to the kernel $F^{4,0}$ of $KO^4(\boldsymbol{CP}^l) \to KO^4(\boldsymbol{CP}^1)$.

$$\begin{array}{ccc} K^4(\boldsymbol{CP}^l) & \longrightarrow & K^4(\boldsymbol{CP}^1) \\ \downarrow \boldsymbol{r} & & \downarrow \boldsymbol{r} \\ KO^4(\boldsymbol{CP}^l) & \longrightarrow & KO^4(\boldsymbol{CP}^1) \end{array}$$

By (2.7), $\boldsymbol{c} : KO^4(\boldsymbol{CP}^2) \to K^4(\boldsymbol{CP}^2)$ maps $\omega_2 \in KO^4(\boldsymbol{CP}^2)$ to non-zero element $t^{-2}\mu^2$ of $K^4(\boldsymbol{CP}^2)$. Hence ω_2 is not zero in $KO^*(\boldsymbol{CP}^2)$. *Q.E.D.*

Lemma 2.9 *$u^2 \in E_2^{4,0}(KO;\boldsymbol{CP}^l)$ is the permanent cocycle corresponding to ω_2.*

Proof. We first note that $E_2^{4,0}(KO;\boldsymbol{CP}^l)$ is isomorphic to $\boldsymbol{Z}$ generated by u^2. By (2.8), there exists a unique $k_l \in \boldsymbol{Z}$ such that k_lu^2 corresponds to $\omega_2 \in KO^4(\boldsymbol{CP}^l)$. $\boldsymbol{c}_2$:

$E_2^{4,0}(KO;\boldsymbol{C}P^2) \to E_2^{4,0}(K;\boldsymbol{C}P^2)$ maps k_2u^2 to k_2u^2 which is a permanent cocycle corresponding to $t^{-2}\mu^2$ by (2.7). On the other hand, the permanent cocycle in $E_2^{4,0}(K;\boldsymbol{C}P^2)$ corresponding to $t^{-2}\mu^2$ is u^2. Hence $k_2 = 1$. For $l \geq 2$, consider the map $i_l^{*,*} : E_r^{*,*}(KO;\boldsymbol{C}P^l) \to E_r^{*,*}(KO;\boldsymbol{C}P^2)$ of spectral sequences induced by the inclusion map $i_l : \boldsymbol{C}P^2 \to \boldsymbol{C}P^l$. Since $i_l^*(\omega_2) = \omega_2$, $i_l^{*,*}(k_lu^2) = k_lu^2$ is the permanent cocycle corresponding to $\omega_2 \in KO^4(\boldsymbol{C}P^2)$. Therefore we have $k_l = 1$. *Q.E.D.*

If l is odd, we denote by $\chi_l \in KO^{2l}(\boldsymbol{C}P^l)$ the element corresponding to

$$v_l \in E_3^{2l,0}(KO;\boldsymbol{C}P^l).$$

We note that, since $F^{2l+1,-1} = \{0\}$, $\chi_l \in F^{2l,0}$ is the unique element corresponding to v_l. Since $\boldsymbol{c}_2 : E_2^{2l,0}(KO;\boldsymbol{C}P^l) \to E_2^{2l,0}(K;\boldsymbol{C}P^l)$ maps u^l to u^l which corresponds to $t^{-l}\mu^l \in K^{2l}(\boldsymbol{C}P^l)$, we have the following.

Lemma 2.10 *$\boldsymbol{c} : KO^*(\boldsymbol{C}P^l) \to K^*(\boldsymbol{C}P^l)$ maps χ_l to $t^{-l}\mu^l$.*

It follows from (2.6) and (2.9), ω_i is the element corresponding to u_i for $i = 0,1,2,3$. Hence, by (2.4) and (2.5), we have the following result.

Theorem 2.11 *$KO^*(\boldsymbol{C}P^l)$ is generated by the following set of elements over KO^*.*

1) If l is even, $\left\{\omega_k\omega_2^j \,\middle|\, 0 \leq j \leq \frac{l}{2}-1,\ 0 \leq k \leq 3\right\} \cup \{1\}$.

2) If l is odd, $\left\{\omega_k\omega_2^j \,\middle|\, 0 \leq j \leq \frac{l-3}{2},\ 0 \leq k \leq 3\right\} \cup \{1,\ \chi_l\}$.

Theorem 2.12 *The following relations hold in $KO^*(\boldsymbol{C}P^l)$.*

$$x\omega_2 = 2\omega_0,\ x\omega_0 = 2y\omega_2,\ x\omega_3 = 2\omega_1,\ x\omega_1 = 2y\omega_3,\ \alpha\omega_0 = \alpha\omega_1 = \alpha\omega_2 = \alpha\omega_3 = 0,$$

$$\omega_0^2 = y\omega_2^2,\ \omega_0\omega_1 = y\omega_2\omega_3,\ \omega_0\omega_3 = \omega_1\omega_2,\ \omega_1^2 = 4\omega_2 + \omega_0\omega_2,\ \omega_1\omega_3 = 2xy^{-1}\omega_2 + \omega_2^2,$$

$$\omega_3^2 = 4y^{-1}\omega_2 + y^{-1}\omega_0\omega_2,\ \omega_2^{\left[\frac{l}{2}\right]+1} = \omega_0\omega_2^{\left[\frac{l+1}{2}\right]} = \omega_1\omega_2^{\left[\frac{l+1}{2}\right]} = \omega_2^{\left[\frac{l+1}{2}\right]}\omega_3 = 0.$$

If l is odd, $\omega_0\chi_l = \omega_1\chi_l = \omega_2\chi_l = \omega_3\chi_l = \chi_l^2 = 0$, $\omega_0\omega_2^{\frac{l-1}{2}} = \alpha^2\chi_l$, $\omega_1\omega_2^{\frac{l-1}{2}} = 2\chi_l$, $\omega_2^{\frac{l-1}{2}}\omega_3 = xy^{-1}\chi_l$.

Proof. Assume that l is even. By (2.11), $\widetilde{KO}^n(\boldsymbol{C}P^l) = \{0\}$ if n is odd. Hence $\alpha\omega_i = 0$ for $i = 0,1,2,3$ hold for dimensional reason. It follows from (1.8) that $\boldsymbol{c} : \widetilde{KO}^n(\boldsymbol{C}P^l) \to \widetilde{K}^n(\boldsymbol{C}P^l)$ is injective if n is even. It is easy to verify that $\boldsymbol{c}(x\omega_3 - 2\omega_1) = \boldsymbol{c}(x\omega_2 - 2\omega_0) = \boldsymbol{c}(x\omega_1 - 2y\omega_3) = \boldsymbol{c}(x\omega_0 - 2y\omega_2) = \boldsymbol{c}(\omega_0^2 - y\omega_2^2) = \boldsymbol{c}(\omega_0\omega_1 - y\omega_2\omega_3) = \boldsymbol{c}(\omega_0\omega_3 - \omega_1\omega_2) = \boldsymbol{c}(\omega_1^2 - 4\omega_2 - \omega_0\omega_2) = \boldsymbol{c}(\omega_1\omega_3 - 2xy^{-1}\omega_2 - \omega_2^2) = \boldsymbol{c}(\omega_3^2 - 4y^{-1}\omega_2 - y^{-1}\omega_0\omega_2) = 0$. Hence we have $x\omega_3 = 2\omega_1$, $x\omega_1 = 2y\omega_3$, $x\omega_0 = 2y\omega_2$, $\omega_0^2 = y\omega_2^2$, $\omega_0\omega_1 = y\omega_2\omega_3$, $\omega_0\omega_3 = \omega_1\omega_2$, $\omega_1^2 = 4\omega_2 + \omega_0\omega_2$, $\omega_1\omega_3 = 2xy^{-1}\omega_2 + \omega_2^2$, $\omega_3^2 = 4y^{-1}\omega_2 + y^{-1}\omega_0\omega_2$. Since $\omega_2^{\left[\frac{l}{2}\right]+1}$, $\omega_0\omega_2^{\left[\frac{l+1}{2}\right]}$, $\omega_1\omega_2^{\left[\frac{l+1}{2}\right]}$, $\omega_2^{\left[\frac{l+1}{2}\right]}\omega_3$ are contained in $F^{2l+1,s}$ for $s = 0,-2,4$ which are trivial groups, we see $\omega_2^{\left[\frac{l}{2}\right]+1} = \omega_0\omega_2^{\left[\frac{l+1}{2}\right]} = \omega_1\omega_2^{\left[\frac{l+1}{2}\right]} = \omega_2^{\left[\frac{l+1}{2}\right]}\omega_3 = 0$.

Assume that l is odd. Consider the map $\iota^* : KO^*(\boldsymbol{C}P^{l+1}) \to KO^*(\boldsymbol{C}P^l)$ induced by the inclusion map $\iota : \boldsymbol{C}P^l \to \boldsymbol{C}P^{l+1}$. Since $\iota^*(\omega_i) = \omega_i$ $(i = 0,1,2,3)$ and $l+1$ is even, we have $x\omega_3 = 2\omega_1$, $x\omega_1 = 2y\omega_3$, $x\omega_0 = 2y\omega_2$, $\alpha\omega_0 = \alpha\omega_1 = \alpha\omega_2 = \alpha\omega_3 = 0$, $\omega_0^2 = y\omega_2^2$, $\omega_0\omega_1 = y\omega_2\omega_3$, $\omega_0\omega_3 = \omega_1\omega_2$, $\omega_1^2 = 4\omega_2 + \omega_0\omega_2$, $\omega_1\omega_3 = 2xy^{-1}\omega_2 + \omega_2^2$, $\omega_3^2 = 4y^{-1}\omega_2 + y^{-1}\omega_0\omega_2$ in $KO^*(\boldsymbol{C}P^l)$. Since $\omega_2^{\left[\frac{l}{2}\right]+1}$, $\omega_0\omega_2^{\left[\frac{l+1}{2}\right]}$, $\omega_1\omega_2^{\left[\frac{l+1}{2}\right]}$, $\omega_2^{\left[\frac{l+1}{2}\right]}\omega_3$, $\omega_0\chi_l$, $\omega_1\chi_l$ $\omega_2\chi_l$, $\omega_3\chi_l$, χ_l^2,

$\omega_0\omega_2^{[\frac{l}{2}]} - \alpha^2\chi_l,\ \omega_1\omega_2^{[\frac{l}{2}]} - 2\chi_l,\ \omega_2^{[\frac{l}{2}]}\omega_3 - xy^{-1}\chi_l$ are contained in $F^{2l+1,s}$ for $s = 0, -2, 4$ which are trivial groups, we see $\omega_2^{[\frac{l}{2}]+1} = \omega_0\omega_2^{[\frac{l+1}{2}]} = \omega_1\omega_2^{[\frac{l+1}{2}]} = \omega_2^{[\frac{l+1}{2}]}\omega_3 = \omega_0\chi_l = \omega_1\chi_l = \omega_2\chi_l = \omega_3\chi_l = \chi_l^2 = \omega_0\omega_2^{\frac{l-1}{2}} - \alpha^2\chi_l = \omega_1\omega_2^{\frac{l-1}{2}} - 2\chi_l = \omega_2^{\frac{l-1}{2}}\omega_3 - xy^{-1}\chi_l = 0$. *Q.E.D.*

Let us denote by $\iota_l : \boldsymbol{C}P^l \to \boldsymbol{C}P^{l+1}$ the inclusion map. Clearly $\iota_l^* : KO^*(\boldsymbol{C}P^{l+1}) \to KO^*(\boldsymbol{C}P^l)$ maps ω_k to ω_k. Hence the inverse system $\left\{KO^*(\boldsymbol{C}P^{l+1}) \xrightarrow{\iota_l^*} KO^*(\boldsymbol{C}P^l)\right\}_{l\geq 1}$ satisfies the condition of Mittag-Leffler, in fact $\iota_{2m}^*\iota_{2m+1}^* : KO^*(\boldsymbol{C}P^{2m+2}) \to KO^*(\boldsymbol{C}P^{2m})$ is surjective. Therefore, the above result immediately implies the following.

Corollary 2.13 $KO^*(\boldsymbol{C}P^\infty)$ *is isomorphic to the quotient KO^*-algebra of the ring of formal power series $KO^*[\omega_0,\omega_1,\omega_3][[\omega_2]]$ over the polynomial algebra $KO^*[\omega_0,\omega_1,\omega_3]$ over KO^* by the ideal generated by the following elements.*

$$x\omega_2 - 2\omega_0,\ x\omega_0 - 2y\omega_2,\ x\omega_3 - 2\omega_1,\ x\omega_1 - 2y\omega_3,\ \alpha\omega_0,\ \alpha\omega_1,\ \alpha\omega_2,\ \alpha\omega_3,\ \omega_0^2 - y\omega_2^2,$$

$$\omega_0\omega_1 - y\omega_2\omega_3,\ \omega_0\omega_3 - \omega_1\omega_2,\ \omega_1^2 - 4\omega_2 - \omega_0\omega_2,\ \omega_1\omega_3 - 2xy^{-1}\omega_2 - \omega_2^2,\ \omega_3^2 - 4y^{-1}\omega_2 - y^{-1}\omega_0\omega_2$$

Let M_j^* (resp. N_j^*) $(0 \leq j \leq [\frac{l-2}{2}])$ be a submodule of $KO^*(\boldsymbol{C}P^l)$ generated by $\omega_0\omega_2^j$ and ω_2^{j+1} (resp. $\omega_1\omega_2^j$ and $\omega_3\omega_2^j$). By the above result, M_j^* and N_j^* are regarded as $KO^*/(\alpha)$-modules. Since $\boldsymbol{Z}[y,y^{-1}]$ is a subring of $KO^*/(\alpha)$, we also regard M_j^* and N_j^* as $\boldsymbol{Z}[y,y^{-1}]$-modules. Then, M_j^* (resp. N_j^*) is a free $\boldsymbol{Z}[y,y^{-1}]$-module with basis $\{\omega_0\omega_2^j,\ \omega_2^{j+1}\}$ (resp. $\{\omega_1\omega_2^j,\ \omega_3\omega_2^j\}$). Thus we have the following.

Proposition 2.14

$$KO^*(\boldsymbol{C}P^l) = \begin{cases} KO^* \oplus \bigoplus_{j=0}^{\frac{l}{2}-1} M_j^* \oplus \bigoplus_{j=0}^{\frac{l}{2}-1} N_j^* & l \textit{ is even} \\ KO^* \oplus \bigoplus_{j=0}^{\frac{l-3}{2}} M_j^* \oplus \bigoplus_{j=0}^{\frac{l-3}{2}} N_j^* \oplus KO^*\chi_l & l \textit{ is odd} \end{cases}$$

The following is a direct consequence of (2.11) and (2.12).

Proposition 2.15 $KO^0(\boldsymbol{C}P^l) = \begin{cases} \boldsymbol{Z}[\omega_0] \Big/ \left(\omega_0^{[\frac{l}{2}]+1}\right) & l \not\equiv 1 \textit{ modulo } 4 \\ \boldsymbol{Z}[\omega_0] \Big/ \left(2\omega_0^{[\frac{l}{2}]+1}, \omega_0^{[\frac{l}{2}]+2}\right) & l \equiv 1 \textit{ modulo } 4 \end{cases}$

3 Real K-cohomology of product of complex projective spaces Let l and m be positive integers such that $l + m > 2$. We consider the Atiyah-Hirzebruch spectral sequence $E_2^{p,q}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m) \cong H^p(\boldsymbol{C}P^l \times \boldsymbol{C}P^m; KO^q) \Rightarrow KO^{p+q}(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$. Let us denote by $p_1 : \boldsymbol{C}P^l \times \boldsymbol{C}P^m \to \boldsymbol{C}P^l$, $p_2 : \boldsymbol{C}P^l \times \boldsymbol{C}P^m \to \boldsymbol{C}P^m$ the projections. p_1 and p_2 induce the maps of spectral sequences

$$p_1^* : E_r^{p,q}(KO; \boldsymbol{C}P^l) \to E_r^{p,q}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m),$$

$$p_2^* : E_r^{p,q}(KO; \boldsymbol{C}P^m) \to E_r^{p,q}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m).$$

Put $p_1^*(u) = w_1$ and $p_2^*(u) = w_2$, then the E_2-term is given by

$$E_2^{*,*}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m) = KO^*[w_1, w_2]/(w_1^{l+1}, w_2^{m+1}).$$

It follows from (2.2) that $d_2(w_1) = \alpha w_1^2$, $d_2(w_2) = \alpha w_2^2$. Hence $\alpha^2 w_i$, $2w_i$, w_i^2, $xy^{-1}w_i$ are cocycles of $E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ for $i = 1, 2$. It is easy to verify that $\alpha^2 w_1 w_2$, $2w_1 w_2$, $w_1^2 w_2 + w_1 w_2^2$ are also cocycles of $E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$.

For $i = 1, 2$, let us denote by w_{i0}, w_{i1}, w_{i2}, w_{i3} the classes of $\alpha^2 w_i$, $2w_i$, w_i^2, $xy^{-1}w_i$ in $E_3^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$. We also denote by z_0, z_1, z_2, z_3 the classes of xw_1w_2, $\alpha^2 w_1 w_2$, $2w_1w_2$, $w_1^2w_2 + w_1w_2^2$ in $E_3^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$. Then, $p_i^*(u_j) = w_{ij}$ for $i = 1, 2$, $j = 0, 1, 2, 3$ and

$$w_{ij} \in E_3^{2,2j-2}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m) \quad \text{for} \quad j = 0, 1, 3, \quad w_{i2} \in E_3^{4,0}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m),$$

$$z_j \in E_3^{4,2j-4}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m) \quad \text{for} \quad j = 0, 1, 2, \quad z_3 \in E_3^{6,0}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m).$$

Since w_{ij}'s are the images of permanent cocycles, they are also permanent cocycles. If l is odd, let us denote by $v_{1l} \in E_3^{2l,0}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ the class of w_1^l. Similarly, if m is odd, $v_{2m} \in E_3^{2m,0}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ denotes the class of w_2^m.

We identify the complex $E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ with

$$E_2^{*,*}(KO; \boldsymbol{CP}^l) \otimes_{KO^*} E_2^{*,*}(KO; \boldsymbol{CP}^m)$$

and regard $E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ as the total complex of a bicomplex whose first and second differentials are given by $d'(w_1^i w_2^j) = i\alpha w_1^{i+1} w_2^j$ and $d''(w_1^i w_2^j) = j\alpha w_1^i w_2^{j+1}$. Consider the spectral sequence

$$E_2^{p,q} = H'_p H''_q(E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)) \Rightarrow E_3^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m)$$

associated with this bicomplex. Since the first factor $E_2^{*,*}(KO; \boldsymbol{CP}^l)$ is a free KO^*-module, we see that $H''_*(E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m))$ is isomorphic to

$$E_2^{*,*}(KO; \boldsymbol{CP}^l) \otimes_{KO^*} E_3^{*,*}(KO; \boldsymbol{CP}^m) = KO^*[u]/(u^{l+1}) \otimes_{KO^*} E_3^{*,*}(KO; \boldsymbol{CP}^m).$$

Let us denote by A_m^* a submodule of $E_3^{*,*}(KO; \boldsymbol{CP}^m)$ generated by

$$\left\{ u_2^j u_k \,\middle|\, 0 \leq j \leq \left[\tfrac{m}{2}\right] - 1,\ 0 \leq k \leq 3 \right\}.$$

If m is odd, B_m^* denotes a submodule of $E_3^{*,*}(KO; \boldsymbol{CP}^m)$ generated by v_l. We put $B_m^* = \{0\}$ if m is even. Then, $E_3^{*,*}(KO; \boldsymbol{CP}^m) = KO^* \oplus A_m^* \oplus B_m^*$, $\alpha A_m^* = \{0\}$ and $KO^* \oplus B_m^*$ is a free KO^*-module.

We observe that the differential $\tilde{d}$ of $H''_*(E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m))$ induced by the first differential maps $u^i \otimes u_2^{j+1}$, $u^i \otimes u_0 u_2^j$, $u^i \otimes u_1 u_2^j$, $u^i \otimes u_3 u_2^j$ to zero for $j \geq 0$. Hence $E_2^{*,*} = H'_* H''_*(E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m))$ is isomorphic to

$$E_3^{*,*}(KO; \boldsymbol{CP}^l) \otimes_{KO^*} (KO^* \oplus B_m^*) \oplus KO^*[u]/(u^{l+1}) \otimes_{KO^*} A_m^*.$$

This implies the following result.

Lemma 3.1 $E_2^{*,*} = H'_* H''_*(E_2^{*,*}(KO; \boldsymbol{CP}^l \times \boldsymbol{CP}^m))$ *is generated by the following set of elements over* KO^*.

1) If both l and m are even, $\left\{ u_2^j u_k \otimes 1 \,\middle|\, 0 \leq j \leq \frac{l}{2} - 1,\ 0 \leq k \leq 3 \right\} \cup \left\{ u^i \otimes u_2^j u_k \,\middle|\, 0 \leq i \leq l,\ 0 \leq j \leq \frac{m}{2} - 1,\ 0 \leq k \leq 3 \right\} \cup \{1 \otimes 1\}$.

2) If l is odd and m is even, $\left\{ u_2^j u_k \otimes 1 \,\middle|\, 0 \leq j \leq \frac{l-3}{2},\ 0 \leq k \leq 3 \right\} \cup \left\{ u^i \otimes u_2^j u_k \,\middle|\, 0 \leq i \leq l,\ 0 \leq j \leq \frac{m}{2} - 1,\ 0 \leq k \leq 3 \right\} \cup \{1 \otimes 1,\ v_l \otimes 1\}$.

3) If l is even and m is odd, $\left\{ u_2^j u_k \otimes v_m^s \,\middle|\, 0 \le j \le \frac{l}{2}-1,\ 0 \le k \le 3,\ s=0,1 \right\} \cup$ $\left\{ u^i \otimes u_2^j u_k \,\middle|\, 0 \le i \le l,\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3 \right\} \cup \{1 \otimes 1,\ 1 \otimes v_m\}$.

4) If both l and m are odd, $\left\{ u_2^j u_k \otimes v_m^s \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3,\ s=0,1 \right\} \cup$ $\left\{ u^i \otimes u_2^j u_k \,\middle|\, 0 \le i \le l,\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3 \right\} \cup \{v_l^t \otimes v_m^s | t,s=0,1\}$.

We remark that generators $u^{2i} \otimes u_2^j u_k$, $u_2^j u_k \otimes v_m^s$, $v_l^t \otimes v_m^s$ and $u^{2i+1} \otimes u_2^j u_k$ in the above lemma correspond to $w_{12}^i w_{2k} w_{22}^j$, $w_{1k} w_{12}^j v_{2m}^s$, $v_{1l}^t v_{2m}^s$ and $w_{12}^i w_{22}^j z_{k+1}$ (put $z_4 = y^{-1} z_0$), respectively. Thus the spectral sequence $E_2^{p,q} = H_p' H_q''(E_2^{*,*}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)) \Rightarrow E_3^{*,*}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ collapses and we have the following.

Proposition 3.2 *$E_3^{*,*}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ is generated by the following set of elements over KO^*.*

1) If both l and m are even, $\left\{ w_{1k} w_{12}^j \,\middle|\, 0 \le j \le \frac{l}{2}-1,\ 0 \le k \le 3 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j w_{2k} \,\middle|\, 0 \le i \le \frac{l}{2},\ 0 \le j \le \frac{m}{2}-1,\ 0 \le k \le 3 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j z_k \,\middle|\, 0 \le i \le \frac{l}{2}-1,\ 0 \le j \le \frac{m}{2}-1,\ 0 \le k \le 3 \right\} \cup \{1\}$.

2) If l is odd and m is even, $\left\{ w_{1k} w_{12}^j \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j w_{2k} \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m}{2}-1,\ 0 \le k \le 3 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j z_k \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m}{2}-1,\ 0 \le k \le 3 \right\} \cup \{1,\ v_{1l}\}$.

3) If l is even and m is odd, $\left\{ w_{1k} w_{12}^j v_{2m}^s \,\middle|\, 0 \le j \le \frac{l}{2}-1,\ 0 \le k \le 3,\ s=0,1 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j w_{2k} \,\middle|\, 0 \le i \le \frac{l}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j z_k \,\middle|\, 0 \le i \le \frac{l}{2}-1,\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3 \right\} \cup \{1, v_{2m}\}$.

4) If both l and m are odd, $\left\{ w_{1k} w_{12}^j v_{2m}^s \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3,\ s=0,1 \right\} \cup$
$\left\{ w_{12}^i w_{22}^j w_{2k} \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m-3}{2} \right\} \cup$
$\left\{ w_{12}^i w_{22}^j z_k \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3 \right\} \cup \left\{ v_{1l}^t v_{2m}^s | t,s=0,1 \right\}$.

Lemma 3.3 *The following relations hold in $E_3^{*,*}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$.*

$$2z_1 = xz_1 = \alpha z_0 = \alpha z_1 = \alpha z_2 = \alpha z_3 = 0,\ \ xz_2 = 2z_0,\ \ xz_0 = 2yz_2,\ \ z_0 z_1 = z_1^2 = z_1 z_2 = 0,$$

$$z_0^2 = 4yw_{12}w_{22},\ z_2^2 = 4w_{12}w_{22},\ z_0 z_2 = 2xw_{12}w_{22},\ z_0 z_3 = xw_{12}z_3 - yw_{12}^2 w_{23} + yw_{12}w_{22}w_{23},$$

$$z_1 z_3 = w_{12}^2 w_{20} + w_{12}w_{22}w_{20}, \qquad z_2 z_3 = 2w_{12}\zeta_3 - w_{12}^2 w_{22} + w_{12}w_{22}w_{21},$$

$$z_3^2 = w_{12}^2 w_{22} + w_{12}w_{22}^2 + w_{12}w_{22}z_2,\ w_{10}w_{20} = w_{11}w_{20} = w_{13}w_{20} = w_{10}w_{21} = 0,$$

$$w_{11}w_{21} = 2z_2,\ w_{13}w_{21} = 2y^{-1}z_0,\ w_{10}w_{22} = w_{12}w_{20}, w_{11}w_{22} = 2z_3 - w_{12}w_{21},$$

$$w_{13}w_{22} = xy^{-1}z_3 - w_{12}w_{23}, \quad w_{10}w_{23} = 0, \quad w_{11}w_{23} = 2y^{-1}z_0, \quad w_{13}w_{23} = 2y^{-1}z_2,$$

$$w_{10}z_0 = w_{10}z_1 = w_{10}z_2 = 0,\ w_{10}z_3 = w_{12}z_1,\ w_{20}z_0 = w_{20}z_1 = w_{20}z_2 = 0,\ w_{20}z_3 = w_{22}z_1,$$

$$w_{11}z_0 = xw_{12}w_{21}, \quad w_{11}z_1 = 0, \quad w_{11}z_2 = 2w_{12}w_{21}, \quad w_{11}z_3 = 2w_{12}w_{22} + w_{12}z_2,$$

$$w_{21}z_0 = 2xz_3 - xw_{12}w_{21}, \quad w_{21}z_1 = 0, \quad w_{21}z_2 = 4z_3 - 2w_{12}w_{21}, \quad w_{21}z_3 = 2w_{12}w_{22} + w_{22}z_2,$$

$$w_{13}z_0 = 2w_{12}w_{21}, \quad w_{13}z_1 = 0, \quad w_{13}z_2 = xy^{-1}w_{12}w_{21}, \quad w_{13}z_3 = xy^{-1}w_{12}w_{22} + y^{-1}w_{12}z_0,$$

$$w_{23}z_0 = 4z_3 - 2w_{12}w_{21},\ w_{23}z_1 = 0,\quad w_{23}z_2 = 2xy^{-1}z_3 - xy^{-1}w_{12}w_{21},$$
$$w_{23}z_3 = xy^{-1}w_{12}w_{22} + y^{-1}w_{22}z_0,$$

If l is odd, $z_0v_{1l} = z_1v_{1l} = z_2v_{1l} = z_3v_{1l} = 0$, $w_{20}v_{1l} = w_{12}^{\frac{l-1}{2}}z_1$, $w_{21}v_{1l} = w_{12}^{\frac{l-1}{2}}z_2$, $w_{22}v_{1l} = w_{12}^{\frac{l-1}{2}}z_3$, $w_{23}v_{1l} = y^{-1}w_{12}^{\frac{l-1}{2}}z_0$. If l is even, $w_{12}^{\frac{l}{2}}z_0 = w_{12}^{\frac{l}{2}}z_1 = w_{12}^{\frac{l}{2}}z_2 = w_{12}^{\frac{l}{2}}z_3 = 0$.
If m is odd, $z_0v_{2m} = z_1v_{2m} = z_2v_{2m} = z_3v_{2m} = 0$, $w_{10}v_{2m} = w_{22}^{\frac{m-1}{2}}z_1$, $w_{11}v_{2m} = w_{22}^{\frac{m-1}{2}}z_2$, $w_{12}v_{2m} = w_{22}^{\frac{m-1}{2}}z_3$, $w_{13}v_{2m} = y^{-1}w_{22}^{\frac{m-1}{2}}z_0$. If m is even, $w_{22}^{\frac{m}{2}}z_0 = w_{22}^{\frac{m}{2}}z_1 = w_{22}^{\frac{m}{2}}z_2 = w_{22}^{\frac{m}{2}}z_3 = 0$.

Proof. By the definition of z_3 and $d_2(w_1w_2) = \alpha(w_1^2w_2 + w_1w_2^2)$, we have $\alpha z_3 = 0$. Other relations follows from the definitions of w_{ij} and z_j. *Q.E.D.*

Proposition 3.4 $E_3^{*,*}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m) = E_\infty^{*,*}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$

Proof. Since w_{ij} is the image of a permanent cocycle u_j by p_i^*, it is also permanent cocycle. Similarly, if l (resp. m) is odd, v_{1l} (resp. v_{2m}) is a permanent cocycle. Suppose that both l and m are even. It follows from (3.2) and (3.3) that $E_3^{p,q}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m) = \{0\}$ if $p+q$ is odd and $p \neq 0$. Hence z_j's are permanent cocycles for $j = 0,1,2,3$. For general l and m, since z_j's in $E_3^{*,*}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ are the images of z_j's in $E_3^{*,*}(KO;\boldsymbol{CP}^{2l} \times \boldsymbol{CP}^{2m})$ by the map induced by the inclusion map $\boldsymbol{CP}^l \times \boldsymbol{CP}^m \to \boldsymbol{CP}^{2l} \times \boldsymbol{CP}^{2m}$, they are also permanent cocycles. Thus the assertion follows from (3.2). *Q.E.D.*

Put $\mu_i = p_i^*(\mu) \in K^0(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ for $i = 1,2$, then

$$K^*(\boldsymbol{CP}^l \times \boldsymbol{CP}^m) = K^*[\mu_1,\mu_2]/(\mu_1^{l+1},\mu_2^{m+1}).$$

We also put $\omega_{ij} = p_i^*(\omega_j) \in KO^{2j}(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ and $\zeta_j = \boldsymbol{r}(t^{-j}\mu_1\mu_2) \in KO^{2j}(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ for $i = 1,2$, $j = 0,1,2,3$. If l (resp. m) is odd, we put $\chi_{1l} = p_1^*(\chi_l)$ (resp. $\chi_{2m} = p_2^*(\chi_m)$). It is clear that α^2w_i, $2w_i$, w_i^2 and $xy^{-1}w_i$ are the permanent cocycles in $E_2^{*,*}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ corresponding to ω_{i0}, ω_{i1}, ω_{i2} and ω_{i3}, respectively. If l (resp. m) is odd, it is also clear that w_1^l (resp. w_2^m) is the permanent cocycle in $E_2^{2l,0}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ (resp. $E_2^{2m,0}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$) corresponding to χ_{1l} (resp. χ_{2m}).

Lemma 3.5 $\boldsymbol{c} : KO^*(\boldsymbol{CP}^l \times \boldsymbol{CP}^m) \to K^*(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ *maps ζ_{2i}, ζ_{2i+1} $(i = 0,1)$ as follows.*

$$\boldsymbol{c}(\zeta_{2i}) = t^{-2i}\mu_1\mu_2(1+(1+\mu_1)^{-1}(1+\mu_2)^{-1}),\quad \boldsymbol{c}(\zeta_{2i+1}) = t^{-2i-1}\mu_1\mu_2(1-(1+\mu_1)^{-1}(1+\mu_2)^{-1})$$

Proof. The result follows from (1.5), (1.6), $\Psi^{-1}(\mu_j) = (1+\mu_j)^{-1} - 1$ and the fact that Ψ^{-1} is a ring homomorphism. *Q.E.D.*

Lemma 3.6 *Cocycles $xw_1w_2 \in E_2^{4,-4}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$, $\alpha^2w_1w_2 \in E_2^{4,-2}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ and $2w_1w_2 \in E_2^{4,0}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ are permanent cocycles corresponding to ζ_0, ζ_1, ζ_2, respectively. Hence $\zeta_0 \in F^{4,-4} - F^{5,-5}$, $\zeta_1 \in F^{4,-2} - F^{5,-3}$, $\zeta_2 \in F^{4,0} - F^{5,-1}$.*

Proof. Consider the map $\boldsymbol{r}_r : E_r^{*,*}(K;\boldsymbol{CP}^l \times \boldsymbol{CP}^m) \to E_r^{*,*}(KO;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ induced by $\boldsymbol{r} : K \to KO$. Since $tw_i \in E_2^{2,-2}(K;\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ is the permanent cocycle corresponding to μ_i, the assertion follows from $\boldsymbol{r}_2(tw_1tw_2) = \boldsymbol{r}_2(t^2w_1w_2) = xw_1w_2$, $\boldsymbol{r}_2(t^{-1}tw_1tw_2) = \boldsymbol{r}_2(tw_1w_2) = \alpha^2w_1w_2$, $\boldsymbol{r}_2(t^{-2}tw_1tw_2) = \boldsymbol{r}_2(w_1w_2) = 2w_1w_2$. *Q.E.D.*

Lemma 3.7 *Let us denote by* $(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^k$ *the k-skeleton of* $\boldsymbol{C}P^l \times \boldsymbol{C}P^m$. ζ_3 *belongs to the kernel* $F^{6,0}$ *of the map*

$$KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to KO^6((\boldsymbol{C}P^1 \times \boldsymbol{C}P^m)^4)$$

induced by the inclusion map. On the other hand, ζ_3 *does not belong to the kernel* $F^{7,-1}$ *of the map* $KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to KO^6((\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^6)$.

Proof. Put

$$A = \begin{cases} * \times \boldsymbol{C}P^2 & l=1,\ m=2 \\ \boldsymbol{C}P^2 \times * & l=2,\ m=1 \\ \boldsymbol{C}P^2 \vee \boldsymbol{C}P^2 & l,m \geq 2 \end{cases} \text{ then, } A \cap (\boldsymbol{C}P^1 \times \boldsymbol{C}P^1) = \begin{cases} * \times \boldsymbol{C}P^1 & l=1,\ m=2 \\ \boldsymbol{C}P^1 \times * & l=2,\ m=1 \\ \boldsymbol{C}P^1 \vee \boldsymbol{C}P^1 & l,m \geq 2 \end{cases}$$

and $(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^4 = A \cup (\boldsymbol{C}P^1 \times \boldsymbol{C}P^1)$. Since $KO^5(A \cap (\boldsymbol{C}P^1 \times \boldsymbol{C}P^1)) = \{0\}$, the map $KO^6((\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^4) \to KO^6(A) \oplus KO^6(\boldsymbol{C}P^1 \times \boldsymbol{C}P^1)$ induced by the inclusion maps is injective. Let

$$\iota_4 : (\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^4 \to \boldsymbol{C}P^l \times \boldsymbol{C}P^m, \quad i : A \to \boldsymbol{C}P^l \times \boldsymbol{C}P^m, \quad j : \boldsymbol{C}P^1 \times \boldsymbol{C}P^1 \to \boldsymbol{C}P^l \times \boldsymbol{C}P^m$$

be the inclusion maps. Then, the kernel of $\iota_4^* : KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to KO^6((\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^4)$ coincides with the kernel of $(i^*, j^*) : KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to KO^6(A) \oplus KO^6(\boldsymbol{C}P^1 \times \boldsymbol{C}P^1)$. By the commutativity of the following square, it suffices to show that $\boldsymbol{r}i^*(t^{-3}\mu_1\mu_2) = 0$ and $\boldsymbol{r}j^*(t^{-3}\mu_1\mu_2) = 0$.

$$\begin{array}{ccc} K^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) & \xrightarrow{(i^*,j^*)} & K^6(A) \oplus K^6(\boldsymbol{C}P^1 \times \boldsymbol{C}P^1) \\ \downarrow \boldsymbol{r} & & \downarrow \boldsymbol{r} \oplus \boldsymbol{r} \\ KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) & \xrightarrow{(i^*,j^*)} & KO^6(A) \oplus KO^6(\boldsymbol{C}P^1 \times \boldsymbol{C}P^1) \end{array}$$

Let $i_1 : \boldsymbol{C}P^2 = \boldsymbol{C}P^2 \times * \to A$ and $i_2 : \boldsymbol{C}P^2 = * \times \boldsymbol{C}P^2 \to A$ be inclusion maps. We note that $p_2 i i_1 : \boldsymbol{C}P^2 \to \boldsymbol{C}P^m$ and $p_1 i i_2 : \boldsymbol{C}P^2 \to \boldsymbol{C}P^l$ are constant maps. Hence $i_s^* i^*(t^{-3}\mu_1\mu_2) = i_s^* i^*(t^{-3}p_1^*(\mu_1)p_2^*(\mu_2)) = i_s^* i^* p_1^*(\mu_1) i_s^* i^* p_2^*(\mu_2) = 0$ for $s = 1, 2$. This implies $i^*(t^{-3}\mu_1\mu_2) = 0$. Consider a map $\boldsymbol{r}_r : E_r^{p,q}(K; \boldsymbol{C}P^1 \times \boldsymbol{C}P^1) \to E_r^{p,q}(KO; \boldsymbol{C}P^1 \times \boldsymbol{C}P^1)$ of the Atiyah-Hirzebruch spectral sequences. $t^{-1}w_1w_2 \in E_2^{4,2}(K; \boldsymbol{C}P^1 \times \boldsymbol{C}P^1)$ is the permanent cocycle corresponding to $t^{-3}\mu_1\mu_2 \in K^6(\boldsymbol{C}P^1 \times \boldsymbol{C}P^1)$. Since $\boldsymbol{r}_2$ maps $t^{-1}w_1w_2$ to zero by (1.5), $\boldsymbol{r}(t^{-3}\mu_1\mu_2)$ is contained in $F^{5,1} = \mathrm{Ker}(KO^6(\boldsymbol{C}P^1 \times \boldsymbol{C}P^1) \to KO^6((\boldsymbol{C}P^1 \times \boldsymbol{C}P^1)^4)) = \{0\}$. Therefore $\boldsymbol{r}j^*(t^{-3}\mu_1\mu_2) = 0$.

Suppose that $l \geq 2$, then $\boldsymbol{C}P^2 \times \boldsymbol{C}P^1 \subset (\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^6$. It follows from (3.5) that $\boldsymbol{c}$ maps $\zeta_3 \in KO^6(\boldsymbol{C}P^2 \times \boldsymbol{C}P^1)$ to a non-zero element $t^{-3}\mu_1^2\mu_2$. Hence ζ_3 does not belong to the kernel of $KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to KO^6((\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^6)$. *Q.E.D.*

Lemma 3.8 $w_1^2w_2 + w_1w_2^2 \in E_2^{6,0}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ *is the permanent cocycle corresponding to* ζ_3.

Proof. We observe that the subgroup of $E_2^{6,0}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ consisting of cocycles is generated by $w_1^2w_2 + w_1w_2^2$ if $l, m \leq 2$ or $l, m \geq 4$. By (3.7), there exists a unique integer $k_{l,m}$ such that $k_{l,m}(w_1^2w_2 + w_1w_2^2)$ is the permanent cocycle corresponding to ζ_3 if $l, m \leq 2$ or $l, m \geq 4$.

Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q}(K; \boldsymbol{C}P^l \times \boldsymbol{C}P^m) \Rightarrow K^{p+q}(\boldsymbol{C}P^l \times \boldsymbol{C}P^m).$$

We also put $p_1^*(u) = w_1$ and $p_2^*(u) = w_2$ in $E_2^{*,*}(K; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$. We note that $w_1^2 w_2 + w_1 w_2^2 \in E_2^{6,0}(K; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ is the permanent cocycle corresponding to $t^{-3}(\mu_1^2\mu_2 + \mu_1\mu_2^2)$. Hence $t^{-3}(\mu_1^2\mu_2 + \mu_1\mu_2^2) \in F^{6,0} - F^{7,-1}$. On the other hand, it follows from (3.5) that

$$\boldsymbol{c}(\zeta_3) - t^{-3}(\mu_1^2\mu_2 + \mu_1\mu_2^2) \in F^{8,-2} = \mathrm{Ker}(K^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to K^6((\boldsymbol{C}P^l \times \boldsymbol{C}P^m)^6).$$

Thus both $\boldsymbol{c}(\zeta_3)$ and $t^{-3}(\mu_1^2\mu_2 + \mu_1\mu_2^2)$ are represented by the same permanent cocycle $w_1^2 w_2 + w_1 w_2^2$ of $E_2^{6,0}(K; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$. Consider the map

$$\boldsymbol{c}_r : E_r^{p,q}(KO; \boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to E_r^{p,q}(K; \boldsymbol{C}P^l \times \boldsymbol{C}P^m)$$

induced by $\boldsymbol{c} : KO \to K$. Since a permanent cocycle $\boldsymbol{c}_2(k_{l,m}(w_1^2 w_2 + w_1 w_2^2)) = k_{l,m}(w_1^2 w_2 + w_1 w_2^2)$ corresponds to both $\boldsymbol{c}(\zeta_3)$ and $t^{-3}(\mu_1^2\mu_2 + \mu_1\mu_2^2)$, we have $k_{l,m} = 1$ if $l, m \le 2$ or $l, m \ge 4$. If l or m is 3, consider the map $KO^6(\boldsymbol{C}P^{l+1} \times \boldsymbol{C}P^{m+1}) \to KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ induced by the inclusion map. Since $\zeta_3 \in KO^6(\boldsymbol{C}P^{l+1} \times \boldsymbol{C}P^{m+1})$ is mapped to $\zeta_3 \in KO^6(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ by this map, the assertion holds also in this case. *Q.E.D.*

By (3.2), (3.4), (3.6) and (3.8), we have the following result.

Theorem 3.9 *$KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ is generated by the following set of elements over KO^*.*

1) If both l and m are even, $\left\{\omega_{1k}\omega_{12}^j \,\middle|\, 0 \le j \le \frac{l}{2} - 1,\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l}{2},\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l}{2} - 1,\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\} \cup \{1\}$.

2) If l is odd and m is even, $\left\{\omega_{1k}\omega_{12}^j \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\} \cup \{1, \chi_{1l}\}$.

3) If l is even and m is odd, $\left\{\omega_{1k}\omega_{12}^j\chi_{2m}^s \,\middle|\, 0 \le j \le \frac{l}{2} - 1,\ 0 \le k \le 3,\ s = 0, 1\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l}{2} - 1,\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\} \cup \{1,\ \chi_{2m}\}$.

4) If both l and m are odd, $\left\{\omega_{1k}\omega_{12}^j\chi_{2m}^s \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3,\ s = 0, 1\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\} \cup \{\chi_{1l}^t\chi_{2m}^s | \, t, s = 0, 1\}$.

The following result is a direct consequence of (2.12).

Theorem 3.10 *The following relations hold in $KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$. Here $i = 1$ or 2.*

$$x\omega_{i2} = 2\omega_{i0},\ x\omega_{i0} = 2y\omega_{i2},\ x\omega_{i3} = 2\omega_{i1},\ x\omega_{i1} = 2y\omega_{i3},\ \alpha\omega_{i0} = \alpha\omega_{i1} = \alpha\omega_{i2} = \alpha\omega_{i3} = 0,$$

$$\omega_{i0}^2 = y\omega_{i2}^2, \qquad \omega_{i1}^2 = 4\omega_{i2} + \omega_{i0}\omega_{i2}, \qquad \omega_{i3}^2 = 4y^{-1}\omega_{i2} + y^{-1}\omega_{i0}\omega_{i2},$$

$$\omega_{i0}\omega_{i1} = y\omega_{i2}\omega_{i3}, \qquad \omega_{i0}\omega_{i3} = \omega_{i1}\omega_{i2}, \qquad \omega_{i1}\omega_{i3} = 2xy^{-1}\omega_{i2} + \omega_{i2}^2,$$

$$\omega_{12}^{[\frac{l}{2}]+1} = \omega_{10}\omega_{12}^{[\frac{l+1}{2}]} = \omega_{11}\omega_{12}^{[\frac{l+1}{2}]} = \omega_{12}^{[\frac{l+1}{2}]}\omega_{13} = \omega_{22}^{[\frac{m}{2}]+1} = \omega_{20}\omega_{22}^{[\frac{m+1}{2}]} = \omega_{21}\omega_{22}^{[\frac{m+1}{2}]} = \omega_{22}^{[\frac{m+1}{2}]}\omega_{23} = 0$$

If l is odd, $\omega_{10}\chi_{1l}=\omega_{11}\chi_{1l}=\omega_{12}\chi_{1l}=\omega_{13}\chi_{1l}=\chi_{1l}^2=0$, $\omega_{10}\omega_{12}^{\frac{l-1}{2}}=\alpha^2\chi_{1l}$, $\omega_{11}\omega_{12}^{\frac{l-1}{2}}=2\chi_{1l}$, $\omega_{12}^{\frac{l-1}{2}}\omega_{13}=xy^{-1}\chi_{1l}$.
If m is odd, $\omega_{20}\chi_{2m}=\omega_{21}\chi_{2m}=\omega_{22}\chi_{2m}=\omega_{23}\chi_{2m}=\chi_{2m}^2=0$, $\omega_{20}\omega_{22}^{\frac{m-1}{2}}=\alpha^2\chi_{2m}$, $\omega_{21}\omega_{22}^{\frac{m-1}{2}}=2\chi_{2m}$, $\omega_{22}^{\frac{m-1}{2}}\omega_{23}=xy^{-1}\chi_{2m}$.

The relations containing ζ_k's are given as follows.

Theorem 3.11 *The following relations hold in $KO^*(\boldsymbol{C}P^l\times\boldsymbol{C}P^m)$.*

$$\alpha\zeta_0=\alpha\zeta_1=\alpha\zeta_2=\alpha\zeta_3=0,\quad 2\zeta_1=x\zeta_3,\quad x\zeta_1=2y\zeta_3,\quad 2\zeta_0=x\zeta_2,\quad x\zeta_0=2y\zeta_2,$$

$$\zeta_0^2=4y\omega_{12}\omega_{22}+y\omega_{12}^2\omega_{20}+y\omega_{12}\omega_{20}\omega_{22}+y\omega_{12}\omega_{22}\zeta_0,$$

$$\zeta_0\zeta_1=x\omega_{12}\zeta_1-y\omega_{12}^2\omega_{21}+y\omega_{12}\omega_{22}\omega_{21}+y\omega_{12}\omega_{22}\zeta_1,$$

$$\zeta_0\zeta_2=2x\omega_{12}\omega_{22}+y\omega_{12}^2\omega_{22}+y\omega_{12}\omega_{22}^2+y\omega_{12}\omega_{22}\zeta_2,$$

$$\zeta_0\zeta_3=x\omega_{12}\zeta_3-y\omega_{12}^2\omega_{23}+y\omega_{12}\omega_{22}\omega_{23}+y\omega_{12}\omega_{22}\zeta_3,$$

$$\zeta_1^2=y\omega_{12}^2\omega_{22}+y\omega_{12}\omega_{22}^2+y\omega_{12}\omega_{22}\zeta_2,\quad \zeta_1\zeta_2=x\omega_{12}\zeta_3-y\omega_{12}^2\omega_{23}+y\omega_{12}\omega_{22}\omega_{23}+y\omega_{12}\omega_{22}\zeta_3,$$

$$\zeta_1\zeta_3=\omega_{12}^2\omega_{20}+\omega_{12}\omega_{22}\omega_{20}+\omega_{12}\omega_{22}\zeta_0,\quad \zeta_2^2=4\omega_{12}\omega_{22}+\omega_{12}^2\omega_{20}+\omega_{12}\omega_{22}\omega_{20}+\omega_{12}\omega_{22}\zeta_0,$$

$$\zeta_2\zeta_3=2\omega_{12}\zeta_3-\omega_{12}^2\omega_{21}+\omega_{12}\omega_{22}\omega_{21}+\omega_{12}\omega_{22}\zeta_1,\quad \zeta_3^2=\omega_{12}^2\omega_{22}+\omega_{12}\omega_{22}^2+\omega_{12}\omega_{22}\zeta_2$$

$$\omega_{10}\omega_{20}=y\omega_{11}\omega_{22},\ \omega_{11}\omega_{20}=x\zeta_3-y\omega_{12}\omega_{23},\ \omega_{13}\omega_{20}=2\zeta_3-\omega_{12}\omega_{21},\ \omega_{10}\omega_{21}=y\omega_{12}\omega_{23},$$

$$\omega_{11}\omega_{21}=2\zeta_2-\omega_{12}\omega_{20},\ \omega_{13}\omega_{21}=2y^{-1}\zeta_0-\omega_{12}\omega_{22},\ \omega_{10}\omega_{22}=\omega_{12}\omega_{20},\ \omega_{11}\omega_{22}=2\zeta_3-\omega_{12}\omega_{21},$$

$$\omega_{13}\omega_{22}=xy^{-1}\zeta_3-\omega_{12}\omega_{23},\quad \omega_{10}\omega_{23}=\omega_{12}\omega_{21},\quad \omega_{11}\omega_{23}=2y^{-1}\zeta_0-\omega_{12}\omega_{22},$$

$$\omega_{13}\omega_{23}=2y^{-1}\zeta_2-\omega_{12}\omega_{20},\ \omega_{10}\zeta_0=y\omega_{12}\zeta_2,\ \omega_{10}\zeta_1=y\omega_{12}\zeta_3,\ \omega_{10}\zeta_2=\omega_{12}\zeta_0,\ \omega_{10}\zeta_3=\omega_{12}\zeta_1,$$

$$\omega_{20}\zeta_0=y\omega_{22}\zeta_2,\ \omega_{20}\zeta_1=y\omega_{22}\zeta_3,\ \omega_{20}\zeta_2=\omega_{22}\zeta_0,\ \omega_{20}\zeta_3=\omega_{22}\zeta_1,\ \omega_{11}\zeta_0=x\omega_{12}\omega_{21}+y\omega_{12}\zeta_3,$$

$$\omega_{11}\zeta_1=x\omega_{12}\omega_{22}+\omega_{12}\zeta_0,\quad \omega_{11}\zeta_2=2\omega_{12}\omega_{21}+\omega_{12}\zeta_1,\quad \omega_{11}\zeta_3=2\omega_{12}\omega_{22}+\omega_{22}\zeta_2,$$

$$\omega_{21}\zeta_0=2x\zeta_3-x\omega_{12}\omega_{21}+y\omega_{22}\zeta_3,\quad \omega_{21}\zeta_1=x\omega_{12}\omega_{22}+\omega_{22}\zeta_0,\quad \omega_{21}\zeta_2=4\zeta_3-2\omega_{12}\omega_{21}+\omega_{22}\zeta_1,$$

$$\omega_{21}\zeta_3=2\omega_{12}\omega_{22}+\omega_{22}\zeta_2,\quad \omega_{13}\zeta_0=2\omega_{12}\omega_{21}+\omega_{12}\zeta_1,\quad \omega_{13}\zeta_1=2\omega_{12}\omega_{22}+\omega_{12}\zeta_2,$$

$$\omega_{13}\zeta_2=xy^{-1}\omega_{12}\omega_{21}+\omega_{12}\zeta_3,\ \omega_{13}\zeta_3=xy^{-1}\omega_{12}\omega_{22}+y^{-1}\omega_{12}\zeta_0,\ \omega_{23}\zeta_0=4\zeta_3-2\omega_{12}\omega_{21}+\omega_{22}\zeta_1,$$

$$\omega_{23}\zeta_1=2\omega_{12}\omega_{22}+\omega_{22}\zeta_2,\qquad \omega_{23}\zeta_2=2xy^{-1}\zeta_3-xy^{-1}\omega_{12}\omega_{21}+\omega_{22}\zeta_3,$$

$$\omega_{23}\zeta_3=xy^{-1}\omega_{12}\omega_{22}+y^{-1}\omega_{22}\zeta_0.$$

If l is odd, $\zeta_0\chi_{1l}=\zeta_1\chi_{1l}=\zeta_2\chi_{1l}=\zeta_3\chi_{1l}=0$, $\omega_{20}\chi_{1l}=\omega_{12}^{\frac{l-1}{2}}\zeta_1$, $\omega_{21}\chi_{1l}=\omega_{12}^{\frac{l-1}{2}}\zeta_2$, $\omega_{22}\chi_{1l}=\omega_{12}^{\frac{l-1}{2}}\zeta_3$, $\omega_{23}\chi_{1l}=y^{-1}\omega_{12}^{\frac{l-1}{2}}\zeta_0$. *If l is even,* $\omega_{12}^{\frac{l}{2}}\zeta_0=\omega_{12}^{\frac{l}{2}}\zeta_1=\omega_{12}^{\frac{l}{2}}\zeta_2=\omega_{12}^{\frac{l}{2}}\zeta_3=0$.
If m is odd, $\zeta_0\chi_{2m}=\zeta_1\chi_{2m}=\zeta_2\chi_{2m}=\zeta_3\chi_{2m}=0$, $\omega_{10}\chi_{2m}=\omega_{22}^{\frac{m-1}{2}}\zeta_1$, $\omega_{11}\chi_{2m}=\omega_{22}^{\frac{m-1}{2}}\zeta_2$, $\omega_{12}\chi_{2m}=\omega_{22}^{\frac{m-1}{2}}\zeta_3$, $\omega_{13}\chi_{2m}=y^{-1}\omega_{22}^{\frac{m-1}{2}}\zeta_0$. *If m is even,* $\omega_{22}^{\frac{m}{2}}\zeta_0=\omega_{22}^{\frac{m}{2}}\zeta_1=\omega_{22}^{\frac{m}{2}}\zeta_2=\omega_{22}^{\frac{m}{2}}\zeta_3=0$.

Proof. Relations between ω_{ij} and ζ_k are verified by the same method as in the proof of (2.12). For the proof of the relations involving χ_{1l} and χ_{2m}, we need some preparations. *Q.E.D.*

Let L^* be the submodule of $\widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ generated by $\{\chi_{1l}, \chi_{2m}, \chi_{1l}\chi_{2m}\}$, where we put $\chi_{1l} = 0$ (resp. $\chi_{2m} = 0$) if l (resp. m) is even. Note that L^* is a free KO^*-module. We also consider the submodule T^* of $\widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ generated by the following set of elements.

1) If both l and m are even, $\left\{\omega_{1k}\omega_{12}^j \,\middle|\, 0 \le j \le \frac{l}{2} - 1,\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l}{2},\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l}{2} - 1,\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\}$.

2) If l is odd and m is even, $\left\{\omega_{1k}\omega_{12}^j \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m}{2} - 1,\ 0 \le k \le 3\right\}$.

3) If l is even and m is odd, $\left\{\omega_{1k}\omega_{12}^j\chi_{2m}^s \,\middle|\, 0 \le j \le \frac{l}{2} - 1,\ 0 \le k \le 3,\ s = 0, 1\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l}{2} - 1,\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\}$.

4) If both l and m are odd, $\left\{\omega_{1k}\omega_{12}^j\chi_{2m}^s \,\middle|\, 0 \le j \le \frac{l-3}{2},\ 0 \le k \le 3, s = 0, 1\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\omega_{2k} \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\} \cup$
$\left\{\omega_{12}^i\omega_{22}^j\zeta_k \,\middle|\, 0 \le i \le \frac{l-1}{2},\ 0 \le j \le \frac{m-3}{2},\ 0 \le k \le 3\right\}$.

Since L^* is a free KO^*-module and $\alpha\omega_{ij} = \alpha\zeta_j = 0$, we have the following result by (3.9).

Lemma 3.12 *1)* $\widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) = T^* \oplus L^*$.
2) $\mathrm{Ker}(\boldsymbol{\alpha} : \widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to \widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)) = T^* \oplus \alpha^2 L^* \oplus xL^*$.
3) $\Im(\boldsymbol{\alpha} : \widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m) \to \widetilde{KO}^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)) = \alpha L^*$.

Note that αL^* is generated by $\{\alpha\chi_{1l}, \alpha^2\chi_{1l}, \alpha\chi_{2m}, \alpha^2\chi_{2m}, \alpha\chi_{1l}\chi_{2m}, \alpha^2\chi_{1l}\chi_{2m}\}$ over $\boldsymbol{Z}[y, y^{-1}]$.

Suppose that l is odd and m is even. Then, αL^* is generated by $\{\alpha^i y^j \chi_{1l} | \, i = 1, 2,\ y \in \boldsymbol{Z}\}$ over $\boldsymbol{Z}$. Since $\boldsymbol{c}(\zeta_k\chi_{1l}) = t^{-l}\mu_1^l c(\zeta_k) = 0$ by (2.10) and (3.5), it follows from (1.7) and (3.12) that $\zeta_k\chi_{1l} \in \alpha L^*$. Then, "$\zeta_k\chi_{1l} = 0$" or "$k = 3$ and $\zeta_3\chi_{1l} = c\alpha^2 y^{-1}\chi_{1l}$ for some $c \in \boldsymbol{Z}$". We observe that $\zeta_3\chi_{1l} \in F^{2l+6,0}$ and $\alpha^2 y^{-1}\chi_{1l} \in F^{2l,6} - F^{2l+1,5}$. This implies that $c = 0$, namely, $\zeta_3\chi_{1l} = 0$.

Similarly, since $\boldsymbol{c}(\omega_{2k}\chi_{1l} - \omega_{12}^{\frac{l-1}{2}}\zeta_{k+1}) = 0$, we have $\omega_{2k}\chi_{1l} - \omega_{12}^{\frac{l-1}{2}}\zeta_{k+1} \in \alpha L^*$. It follows "$\omega_{2k}\chi_{1l} = \omega_{12}^{\frac{l-1}{2}}\zeta_{k+1}$" or "$k = 3$ and $\omega_{23}\chi_{1l} - \omega_{12}^{\frac{l-1}{2}}\zeta_4 = c\alpha^2 y^{-1}\chi_{1l}$ for some $c \in \boldsymbol{Z}$". Note that $\omega_{23}\chi_{1l} - \omega_{12}^{\frac{l-1}{2}}\zeta_4 \in F^{2l+2,4}$ and $\alpha^2 y^{-1}\chi_{1l} \in F^{2l,6} - F^{2l+1,5}$. Thus we have $\omega_{2k}\chi_{1l} = \omega_{12}^{\frac{l-1}{2}}\zeta_{k+1}$.

If both l and m are odd, the map $KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^{m+1}) \to KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$ induced by the inclusion map maps the relations $\zeta_k\chi_{1l} = 0$ and $\omega_{2k}\chi_{1l} = \omega_{12}^{\frac{l-1}{2}}\zeta_{k+1}$ in $KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^{m+1})$ to those in $KO^*(\boldsymbol{C}P^l \times \boldsymbol{C}P^m)$.

Proof of $\zeta_k\chi_{2m} = 0$ and $\omega_{1k}\chi_{2m} = \omega_{22}^{\frac{m-1}{2}}\zeta_{k+1}$ for odd m is similar. This completes the proof of (3.11).

Let $\gamma : \boldsymbol{CP}^l \times \boldsymbol{CP}^m \to \boldsymbol{CP}^{l+m}$ be the map induced by the classifying map $\boldsymbol{CP}^\infty \times \boldsymbol{CP}^\infty \to \boldsymbol{CP}^\infty$ of the tensor product of the canonical line bundles.

Theorem 3.13 *$\gamma^* : KO^*(\boldsymbol{CP}^{l+m}) \to KO^*(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ maps ω_j to $\omega_{1j} + \omega_{2j} + \zeta_j$. Hence the image of γ^* is not contained in the image of the cross product $KO^*(\boldsymbol{CP}^l) \otimes KO^*(\boldsymbol{CP}^m) \to KO^*(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$.*

Proof. Recall that $\gamma^* : K^*(\boldsymbol{CP}^{l+m}) \to K^*(\boldsymbol{CP}^l \times \boldsymbol{CP}^m)$ maps μ to $\mu_1 + \mu_2 + \mu_1\mu_2$ ([2]). By the naturality of $\boldsymbol{r} : K^*(X) \to KO^*(X)$, the assertion follows from the definition of ω_j, ω_{ij} and ζ_j. *Q.E.D.*

The above result shows that the classifying map $\boldsymbol{CP}^\infty \times \boldsymbol{CP}^\infty \to \boldsymbol{CP}^\infty$ does not give a formal group structure on $KO^*(\boldsymbol{CP}^\infty)$.

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