A RELAXATION OF THE HOMOGENEITIES IMPOSED ON THE RELATING FUNCTIONS IN THE EXTREMAL PROBLEM FOR THE DERIVATION OF CONSERVED QUANTITIES

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Abstract. In contrast with Noether theorem, we built up a new operative procedure for the derivation of conserved quantities and then applied it to the extremal problem for the integration under constraints in the space of state and control variables. In the problem, conserved quantities were constructed by imposing the homogeneities with respect to the state variables through control variables on relating functions. The purpose of this paper is to release the control variables from the homogeneities and then construct such a conserved quantity in a similar procedure. The quantity is illustratively constructed in a economic model, with the aid of which optimal paths can be determined completely.

Introduction. Noether theorem (Noether [10]) concerning with symmetries of the action integral or its generalization (Bessel-Hagan [2]) with those up to divergence plays an effective role for discovering conserved quantity from the Lagrangian or the Hamiltonian structures of considering problem. In contrast with Noether theorem, a new operative procedure for the derivation of conserved quantity was established without using either Lagrangian or Hamiltonian structures (Mimura and Nôno [6]). It was discussed for a second-order differential system which was supposed later to be of the Euler-Lagrange system, and also for higher order system (Mimura, Fujiwara and Nôno [8]). And the results were applied to various economic growth models (Mimura, Fujiwara and Nôno [7], [9]; Fujiwara, Mimura and Nôno [3], [4], [5]) to discover new economic conserved quantities including non-Noether ones.

In the applications, the Euler-Lagrange system was given in the extremal (maxmizing or minimizing) problem for the integration over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$\int_0^T e^{-\rho t} U(x, u) dt, \tag{1}$$

under constraints

$$\dot{x}^\mu = F^\mu(x, u), \tag{2}$$

where $x = (x^\mu(t))$ $(\mu = 1, \cdots, k)$ and $u = (u^\sigma(t))$ $(\sigma = 1, \cdots, \ell)$ are the state and control variables respectively, and ρ $(\rho \geq 0)$ is a constant. There exists a conserved quantity Ω, whenever $F^\mu(x, u)$ and $U(x, u)$ are homogeneous functions of degree one and r (r may not

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be an integer) with respect to the state variables $x^1, \cdots, x^k$ through the control variables $u^1, \cdots, u^\ell$, i.e., they satisfy

$$F^\mu(\alpha x, \alpha u) = \alpha F^\mu(x, u), \quad U(\alpha x, \alpha u) = \alpha^r U(x, u)$$

for arbitrary constant α ($\alpha \neq 0$), respectively.

Within the state variables $x^1 = p$, $x^2 = h$ and the control variables $u^1 = s_p$, $u^2 = s_h$, Mankiw gave the maximizing problem with the function $U(x, u)$ of the form

$$\frac{(1 - s_p - s_h)^{1-\sigma} f(p, h)^{1-\sigma}}{1-\sigma} \quad (\sigma\text{: const.})$$

under some growth processes relating to $f(p, h)$, where $f(p, h)$ is a homogeneous production function of degree one with respect to p and h (e.g., Askenazy [1]). The homogeneity of the function is guaranteed with respect to the state variables p and h, but it is not with respect to the state and control variables $p,\ h$ and $s_p,\ s_h$. This fact stimulates us to release the control variables $u^1, \cdots, u^\ell$ from the homogeneities. So, in this paper, we show that there exists such a conserved quantity of the above Ω even if $F^\mu(x, u)$ and $U(x, u)$ are homogeneous functions of degree one and r with respect to the state variables $x^1, \cdots, x^k$ respectively (Theorem 1). The theorem 1 gives an anothor approach to the conserved quantities obtained by Noether theorem (while it gives also non-Noether ones [3]), for example, that of the economic growth model of Mankiw. In the model, with the aid of obtained conserved quantity, the optimal path can be determined completely.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

1 New derivation of conserved quantity. In the papers ([3],[4],[7],[9]), we have discussed the extremal (maximizing or minimizing) problem for the integration (1) under constraints (2) and the results were carried out into various economic growth models. In the multiplier technique to the problem, the Lagrangian is given by (π_μ are the multipliers):

$$L = e^{-\rho t} U + \pi_\mu (\dot{x}^\mu - F^\mu), \tag{3}$$

whose Euler-Lagrange equations consist of (2) and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 : \quad \dot{\pi}_\mu + \frac{\partial F^\nu}{\partial x^\mu}\pi_\nu = e^{-\rho t}\frac{\partial U}{\partial x^\mu}, \tag{4a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\sigma}\right) - \frac{\partial L}{\partial u^\sigma} = 0 : \quad \frac{\partial F^\mu}{\partial u^\sigma}\pi_\mu = e^{-\rho t}\frac{\partial U}{\partial u^\sigma}. \tag{4b}$$

A conserved quantity (first integral) for the extremal problem is a quantity Ω of the variables $\dot{\pi}_\mu$, $\dot{x}^\mu$, $\dot{u}^\sigma$, π_μ, x^μ, u^σ and t whose total time derivative vanishes ($\dot{\Omega} = 0$: conservation law) on the optimal paths, i.e., on solutions to the relating Euler-Lagrange equations (2), (4a) and (4b). To develop the discussion, we recall the following procedure for the derivation of conserved quantity for the extremal problem ([9], Theorem 1):

For the Lagrangian L of (3), *let the functions* $(\xi_1^\alpha) = (\eta_\mu^1, \varphi_1^\mu, \tau_1^\sigma)$ *and* $(\xi_2^\alpha) = (\eta_\mu^2, \varphi_2^\mu, \tau_2^\sigma)$ *of the variables* $\dot{\pi}_\mu$, $\dot{x}^\mu$, $\dot{u}^\sigma$, π_μ, x^μ, u^σ *and* t *satisfy the equations*

$$\frac{d\varphi^\mu}{dt} = \frac{\partial F^\mu}{\partial x^\nu}\varphi^\nu + \frac{\partial F^\mu}{\partial u^\sigma}\tau^\sigma, \tag{5a}$$

$$\frac{d\eta_\mu}{dt}+\frac{\partial F^\nu}{\partial x^\mu}\eta_\nu+\pi_\nu\Big(\frac{\partial^2 F^\nu}{\partial x^\kappa\partial x^\mu}\varphi^\kappa+\frac{\partial^2 F^\nu}{\partial u^\sigma\partial x^\mu}\tau_\sigma\Big)=e^{-\rho t}\Big(\frac{\partial^2 U}{\partial x^\nu\partial x^\mu}\varphi^\nu+\frac{\partial^2 U}{\partial u^\sigma\partial x^\mu}\tau^\sigma\Big), \tag{5b}$$

$$\frac{\partial F^\mu}{\partial u^\sigma}\eta_\mu+\pi_\mu\left(\frac{\partial^2 F^\mu}{\partial x^\nu\partial u^\sigma}\varphi^\nu+\frac{\partial^2 F^\mu}{\partial u^\omega\partial u^\sigma}\tau^\omega\right)=e^{-\rho t}\left(\frac{\partial^2 U}{\partial x^\mu\partial u^\sigma}\varphi^\mu+\frac{\partial^2 U}{\partial u^\omega\partial u^\sigma}\tau^\omega\right), \tag{5c}$$

on the optimal paths for the extremal problem of (1) *under the constraints* (2). *Then the following conserved quantity* Ω *is constructed*:

$$\Omega=\eta^2_\mu\varphi^\mu_1-\eta^1_\mu\varphi^\mu_2. \tag{6}$$

Moreover, by substituting $\eta^1_\mu=\dot\pi_\mu+\rho\pi_\mu$, $\varphi^\mu_1=\dot x^\mu$ *in the solution* $(\xi^\alpha_1)=(\eta^1_\mu,\ \varphi^\mu_1,\ \tau^\sigma_1)=(\dot\pi_\mu{+}\rho\pi_\mu,\ \dot x^\mu,\ \dot u^\sigma)$ *of* (5a), (5b) *and* (5c) *for* Ω *of* (6), *while* (ξ^α_2) *is left as* $(\xi^\alpha_2)=(\eta^2_\mu,\ \varphi^\mu_2,\ \tau^\sigma_2)=(\eta_\mu,\ \varphi^\mu,\ \tau^\sigma)$, *the conserved quantity* Ω *of* (6) *is reduced to*

$$\Omega=\dot x^\mu\eta_\mu-(\dot\pi_\mu+\rho\pi_\mu)\varphi^\mu. \tag{7}$$

We assume that $F^\mu(x,u)$ and $U(x,u)$ are homogeneous functions of degree one and r with respect to the state variables $x^1,\cdots,x^\kappa$ respectively (control variables $u^1,\cdots,u^\ell$ are not counted in the homogeneities), i.e.,

$$\frac{\partial F^\mu}{\partial x^\nu}x^\nu=F^\mu, \tag{8}$$

$$\frac{\partial U}{\partial x^\mu}x^\mu=rU. \tag{9}$$

Then the relation (8) is combined with (2) to see

$$\dot x^\mu=\frac{\partial F^\mu}{\partial x^\nu}x^\nu,$$

which gurantees that $(\varphi^\mu,\ \tau^\sigma)=(x^\mu,\ 0)$ is a solution of (5a). Together with the solution, the following differentiations of (8) with respect to x^κ or u^σ:

$$\frac{\partial^2 F^\mu}{\partial x^\kappa\partial x^\nu}x^\nu=0,\quad \frac{\partial^2 F^\mu}{\partial u^\sigma\partial x^\nu}x^\nu=\frac{\partial F^\mu}{\partial u^\sigma};$$

and also of (9) with respect to x^ν or u^σ:

$$\frac{\partial^2 U}{\partial x^\nu\partial x^\mu}x^\mu=(r-1)\frac{\partial U}{\partial x^\nu},\quad \frac{\partial^2 U}{\partial u^\sigma\partial x^\mu}x^\mu=r\frac{\partial U}{\partial u^\sigma};$$

are substituted for (5b) and (5c). Then the resulting equations of (5b) and (5c) are written respectively as

$$\frac{d\eta_\mu}{dt}+\frac{\partial F^\nu}{\partial x^\mu}\eta_\nu=(r-1)e^{-\rho t}\frac{\partial U}{\partial x^\mu},$$

$$\frac{\partial F^\mu}{\partial u^\sigma}\eta_\mu+\pi_\mu\frac{\partial F^\mu}{\partial u^\sigma}=re^{-\rho t}\frac{\partial U}{\partial u^\sigma},$$

whose solution $\eta_\mu=(r-1)\pi_\mu$ can be found immediately by (4a) and (4b). Finally the solution $(\xi^\alpha)=(\eta_\mu,\ \varphi^\mu,\ \tau^\sigma)=((r-1)\pi_\mu,\ x^\mu,\ 0)$ is substituted for (7) to have the following theorem:

Theorem 1 *For the extremal problem of* (1) *under the constraints* (2), *let* $F^{\mu}(x, u)$ *and* $U(x, u)$ *be homogeneous functions of degree one and* r *respectively with respect to the state variables* $x^1, \cdots, x^k$. *Then there exists the following conserved quantity* Ω*:*

$$\Omega = (r-1)\pi_{\mu}\dot{x}^{\mu} - (\dot{\pi}_{\mu} + \rho\pi_{\mu})x^{\mu}. \tag{10}$$

Multiply (4a) by x^{μ} and then summing up for the indices μ. Then, through the homogeneities of U and F^{μ}, it follows that

$$re^{-\rho t}U = \dot{\pi}_{\mu}x^{\mu} + \pi_{\mu}F^{\mu} = \dot{\pi}_{\mu}x^{\mu} + \pi_{\mu}\dot{x}^{\mu},$$

which is used to eliminate $\dot{\pi}_{\mu}x^{\mu}$ in Ω. Consequently Ω of (10) is written as

$$\Omega = -re^{-\rho t}U + \pi_{\mu}(r\dot{x}^{\mu} - \rho x^{\mu}). \tag{11}$$

2 An application to economic model.

2.1 Derivation of conserved quantity. For the derivation of conserved quantity, our theorem 1 in the previous section can be applied effectively to the following two-sector economic model, while Askenazy used his modified Noether theorem ([1], Theorem 1). The discussion of (Mankiw *et.al.*, [1]) begins with the setting of variables: the physical capital $x^1 = p$, the human capital $x^2 = h$, the saving rates or the shares $u^1 = s_p$ and $u^2 = s_h$ of production devoted to physical investments and education or training, respectively. So, consider the maximizing problem for the integration over an infinite period of time:

$$\int_0^{\infty} e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma}dt \quad (\rho,\, \sigma\text{: const.;}\ \rho \geq 0) \tag{12}$$

under the constraints

$$\begin{aligned} \dot{p} &= s_p f(p, h) - (\delta_p + n)p, \\ \dot{h} &= s_h f(p, h) - (\delta_h + n)h, \end{aligned} \tag{13}$$

where $f(p, h)$ is homogeneous production function of degree one with respect to the state variables p and h, $c = (1 - s_p - s_h)f(p, h)$ $(c \neq 0)$ is the amount of consumption goods, n is the growth rate of the population, δ_p and δ_h are the rates of depreciation of physical and human capitals respectively. The relating Lagrangian with the multipliers π_1 and π_2 is given as

$$L = e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma} + \pi_1(\dot{p} - s_p f(p, h) + (\delta_p + n)p) + \pi_2(\dot{h} - s_h f(p, h) + (\delta_h + n)h),$$

whose Euler-Lagrange equations consist of (13) and

$$\dot{\pi}_1 - (\delta_p + n)\pi_1 + (s_p\pi_1 + s_h\pi_2)\frac{\partial f}{\partial p} = e^{-\rho t}c^{-\sigma}\frac{\partial c}{\partial p}, \tag{14a}$$

$$\dot{\pi}_2 - (\delta_h + n)\pi_2 + (s_p\pi_1 + s_h\pi_2)\frac{\partial f}{\partial h} = e^{-\rho t}c^{-\sigma}\frac{\partial c}{\partial h}, \tag{14b}$$

$$\pi_1 = \pi_2 = -e^{-\rho t}c^{-\sigma}. \tag{14c}$$

Then, (11) leads to the same conserved quantity as in (eq.(46) in [1]):

$$\Omega = -e^{-\rho t}c^{1-\sigma} + (1-\sigma)(\pi_1\dot{p} + \pi_2\dot{h}) - \rho(\pi_1 p + \pi_2 h). \tag{15}$$

Corollary. *Let $f(p, h)$ be homogeneous production function of degree one with respect to the physical capital p and the human capital h. Then, in the maximizing problem of* (12) *for the amount of consumption goods $c = (1 - s_p - s_h)f(p, h)$ under the constraints* (13), *the theorem 1 gives rise to the conserved quantity* (15) *which is derivable without using the Noether theorem.*

2.2 Determination of optimal paths. First note that optimal paths in the infinite horizon have to satisfy the transversality condition $\lim_{t\to\infty}(\pi_1 p + \pi_2 h) = 0$. Besides the condition, we place

$$\lim_{t\to\infty} e^{-\rho t}c^{1-\sigma} = 0, \quad \lim_{t\to\infty}(\pi_1\dot{p} + \pi_2\dot{h}) = 0, \tag{16}$$

so as to be $\lim_{t\to\infty}\Omega = 0$. Therefore, since a conserved quantity is constant on the optimal paths, the quantity Ω of (15) is zero on the optimal paths:

$$-e^{-\rho t}c^{1-\sigma} + (1-\sigma)(\pi_1\dot{p} + \pi_2\dot{h}) - \rho(\pi_1 p + \pi_2 h) = 0,$$

for which (14c) and $c = (1 - s_p - s_h)f$ are substituted to see

$$(1 - s_p - s_h)f + (1-\sigma)(\dot{p} + \dot{h}) - \rho(p + h) = 0. \tag{17}$$

The constraints of (13) are added to get

$$s_p + s_h = \frac{\dot{p} + \dot{h} + (\delta_p + n)p + (\delta_h + n)h}{f}, \tag{18}$$

which is substituted for (17) to obtain (cf. eq.(47) in [1])

$$f - (\delta_p + n)p - (\delta_h + n)h - \rho(p + h) - \sigma(\dot{p} + \dot{h}) = 0. \tag{19}$$

In view of (14c) and $c = (1 - s_p - s_h)f$, the difference of (14a) and (14b) yields

$$\frac{\partial f}{\partial p} - \frac{\partial f}{\partial h} = \delta_p - \delta_h. \tag{20}$$

The homogeneity of f:

$$\frac{\partial f}{\partial p}p + \frac{\partial f}{\partial h}h = f$$

and (20) are used to write $(\delta_p + n)p$ as

$$\begin{aligned}
(\delta_p + n)p &= \left(\delta_h + \frac{\partial f}{\partial p} - \frac{\partial f}{\partial h}\right)p + np \\
&= (\delta_h + n)p + \left(f - \frac{\partial f}{\partial h}h - \frac{\partial f}{\partial h}p\right) \\
&= (\delta_h + n)p + f - (p + h)\frac{\partial f}{\partial h},
\end{aligned}$$

which is substituted for (19) to derive

$$\frac{\dot{p}+\dot{h}}{p+h}=\frac{1}{\sigma}\left(\frac{\partial f}{\partial h}-(\delta_h+n+\rho)\right). \tag{21}$$

The equality $\pi_1=\pi_2$ in (14c) and $c=(1-s_p-s_h)f$ are substituted for (14b) to have

$$\frac{\dot{\pi}_2}{\pi_2}=\delta_h+n-\frac{\partial f}{\partial h}. \tag{22}$$

Therefore, it follows from (21) and (22) that

$$\sigma\frac{\dot{p}+\dot{h}}{p+h}=-\frac{\dot{\pi}_2}{\pi_2}-\rho. \tag{23}$$

For the integration of (23), $\pi_2=-e^{-\rho t}c^{-\sigma}$ in (14c) is substituted to see

$$k(p+h)=c=(1-s_p-s_h)f \quad (k\text{: const., } k>0), \tag{24}$$

which is substituted for (17) to have

$$\frac{\dot{p}+\dot{h}}{p+h}=\frac{\rho-k}{1-\sigma}, \tag{25}$$

so that

$$p+h=C_1e^{\frac{\rho-k}{1-\sigma}t} \quad (C_1\text{: const}). \tag{26}$$

The equations (21) and (25) yield the relation

$$\frac{\partial f}{\partial h}=\delta_h+n+\rho+\frac{\sigma(\rho-k)}{1-\sigma}, \tag{27}$$

which is used to write (22) as

$$\frac{\dot{\pi}_2}{\pi_2}=\frac{\sigma k-\rho}{1-\sigma},$$

so that

$$\pi_2=C_2e^{\frac{\sigma k-\rho}{1-\sigma}t} \quad (C_2\text{: const}). \tag{28}$$

By (26) and (28), the term $\pi_1p+\pi_2h$ in the transversality condition, the terms $e^{-\rho t}c^{1-\sigma}$ and $\pi_1\dot{p}+\pi_2\dot{h}$ in the condition (16) are written respectively as

$$\pi_1p+\pi_2h=C_1C_2e^{-kt}, \quad e^{-\rho t}c^{1-\sigma}=(-C_2)^{\frac{\sigma-1}{\sigma}}e^{-kt}, \quad \pi_1\dot{p}+\pi_2\dot{h}=\frac{\rho-k}{1-\sigma}C_1C_2e^{-kt},$$

each term of which goes to zero as $t\to 0$.

Here, the constant k in (24):

$$k=\frac{(1-s_p-s_h)f}{p+h}$$

is written by (18) as

$$k=\frac{f-(\delta_p+n)p-(\delta_h+n)h}{p+h}-\frac{\dot{p}+\dot{h}}{p+h},$$

for which (25) is substituted to have

$$k = \frac{\rho}{\sigma} - \frac{1-\sigma}{\sigma}\frac{f(p,\, h) - (\delta_p + n)p - (\delta_h + n)h}{p + h}.$$

Therefore, the constant k is determined by the initial values $p(0) = p_0$ and $h(0) = h_0$ as

$$k = \frac{\rho}{\sigma} - \frac{1-\sigma}{\sigma}\frac{f(p_0,\, h_0) - (\delta_p + n)p_0 - (\delta_h + n)h_0}{p_0 + h_0},$$

which is substituted to the right handside of (27) to have

$$\delta_h + n + \rho + \frac{\sigma(\rho - k)}{1-\sigma} = \frac{f(p_0,\, h_0) - (\delta_p - \delta_h)p_0}{p_0 + h_0}. \tag{29}$$

Since $f(p,\, h)$ is homogeneous function of degree one, it can be arranged as

$$f(p,\, h) = pf(1,\, X),$$

where $X = h/p$ $(X > 0)$. Then, in view of (29) and $\partial f(p, h)/\partial h = df(1,\, X)/dX$, the equation (27) is written as

$$\frac{df(1,\, X)}{dX} = \frac{f(p_0,\, h_0) - (\delta_p - \delta_h)p_0}{p_0 + h_0}. \tag{30}$$

Whenever (30) has a solution

$$X = \frac{h}{p} = A \qquad (A\text{: const.}), \tag{31}$$

there exist by (26) the following optimal paths

$$p(t) = \frac{C_1}{1+A}e^{\frac{\rho-k}{1-\sigma}t}, \qquad h(t) = \frac{AC_1}{1+A}e^{\frac{\rho-k}{1-\sigma}t}, \tag{32}$$

which together with (13) conclude:

$$s_p(t) = \left(\frac{\rho-k}{1-\sigma} + \delta_p + n\right)\frac{1}{f(1,\, A)}, \qquad s_h(t) = \left(\frac{\rho-k}{1-\sigma} + \delta_h + n\right)\frac{A}{f(1,\, A)}. \tag{33}$$

Theorem 2 *Let $f(p,\, h)$ be homogeneous production function of degree one with respect to the physical capital p and the human capital h. Then, in the maximizing problem of* (12) *for the amount of consumption goods $c = (1 - s_p - s_h)f(p,\, h)$ under the constraints* (13), *there exist the optimal paths $p(t)$, $h(t)$ of* (32) *and $s_p(t)$, $s_h(t)$ of* (33), *whenever the equation* (30) *for the given initial values p_0 and h_0 has a solution $X = h/p = A$.*

Cobb-Douglus production function. Let the homogeneous production function $f(p,\, h)$ of degree one be of the form

$$f(p,\, h) = p^{1-\beta}h^{\beta} \qquad (\beta\text{: const.; } 0 < \beta < 1).$$

Then the left handside $Y \equiv df(1, X)/dX$ of (30) reduces to $Y = \beta X^{\beta-1}$, so that

$$\frac{dY}{dX} = \beta(\beta-1)X^{\beta-2} < 0, \quad \lim_{X\to+0} Y = \infty, \quad \lim_{X\to\infty} Y = 0.$$

Hence $Y > 0$. Here, the right handside of (30) reduces to

$$\frac{f(p_0, h_0) - (\delta_p - \delta_h)p_0}{p_0 + h_0} = \frac{p_0^{1-\beta}h_0^{\beta} - (\delta_p - \delta_h)p_0}{p_0 + h_0} = \frac{(h_0/p_0)^{\beta} - (\delta_p - \delta_h)}{1 + h_0/p_0},$$

which becomes positive if

$$\left(\frac{h_0}{p_0}\right)^{\beta} > \delta_p - \delta_h. \tag{34}$$

Therefore, for the initial values p_0 and h_0 satisfying (34), the equation (30) has the unique solution. It is subustituted for (32) and (33) with $f(1, A) = A^{\beta}$ to complete the optimal paths.

CES production function. Let the homogeneous production function $f(p, h)$ of degree one be of the form

$$f(p, h) = (ap^{\kappa} + bh^{\kappa})^{\frac{1}{\kappa}} \qquad (a,\, b,\, \kappa\text{: const.}; \ a > 0,\, b > 0,\, \kappa > 0; \ \kappa \neq 1).$$

Then the left handside of (30) $Y \equiv df(1, X)/dX$ reduces to $Y = bX^{\kappa-1}(a + bX^{\kappa})^{\frac{1}{\kappa}-1}$, so that

$$\frac{dY}{dX} = ab(\kappa-1)X^{\kappa-2}(a + bX^{\kappa})^{\frac{1}{\kappa}-2}, \quad \lim_{X\to\infty} Y = \lim_{X\to\infty} \frac{b}{(a/X^{\kappa} + b)^{\frac{\kappa-1}{\kappa}}} = b^{\frac{1}{\kappa}}.$$

The right handside of (30) reduces to

$$\frac{f(p_0, h_0) - (\delta_p - \delta_h)p_0}{p_0 + h_0} = \frac{(ap_0^{\kappa} + bh_0^{\kappa})^{\frac{1}{\kappa}} - (\delta_p - \delta_h)p_0}{p_0 + h_0} = \frac{(a + b\,(h_0/p_0)^{\kappa})^{\frac{1}{\kappa}} - (\delta_p - \delta_h)}{1 + h_0/p_0}.$$

(i) The case of $\kappa > 1$. Since $dY/dX > 0$ and $Y(0) = 0$, it follows that $0 \leq Y < b^{\frac{1}{\kappa}}$. Here, the initial values p_0 and h_0 can be given such that the equation (30) has a solution, i.e., they can be given so as to satisfy

$$0 \leq \frac{(a + b\,(h_0/p_0)^{\kappa})^{\frac{1}{\kappa}} - (\delta_p - \delta_h)}{1 + h_0/p_0} < b^{\frac{1}{\kappa}}, \ \text{ i.e., } \ 0 \leq \left(\frac{a}{b} + \left(\frac{h_0}{p_0}\right)^{\kappa}\right)^{\frac{1}{\kappa}} + \frac{\delta_h - \delta_p}{b^{\frac{1}{\kappa}}} < 1 + \frac{h_0}{p_0}.$$

In fact, in view for the derivative of the monotone increasing functions $g_1(X) \equiv (a/b + X^{\kappa})^{1/\kappa} + (\delta_h - \delta_p)/b^{\frac{1}{\kappa}}$ and $g_2(X) \equiv 1 + X$:

$$g_1'(X) = X^{\kappa-1}\left(\frac{a}{b} + X^{\kappa}\right)^{\frac{1-\kappa}{\kappa}} = \left(\frac{1}{X^{\kappa}}\frac{a}{b} + 1\right)^{\frac{1-\kappa}{\kappa}} < 1 = g_2'(X),$$

there exist a constant X_0 for which

$$0 \leq \left(\frac{a}{b} + X^{\kappa}\right)^{\frac{1}{\kappa}} + \frac{\delta_h - \delta_p}{b^{\frac{1}{\kappa}}} < 1 + X \quad \text{ if } \ X > X_0.$$

Therefore the initial values p_0 and h_0 are given to satisfy $h_0/p_0 > X_0$. Then the equation (30) has the unique solution $X = h/p = A$ which is subustituted for (32) and (33) with $f(1, A) = (1 + bA^\kappa)$ to complete the optimal paths.

(ii) The case of $0 < \kappa < 1$. Since $dY/dX < 0$ and $\lim_{x\to+0} Y = \infty$, it follows that $Y > b^{\frac{1}{\kappa}}$. Similarly, it can be seen that the initial values p_0 and h_0 can be given such that the equation (30) has a solution $X = h/p = A$, i.e., they can be given so as to satisfy

$$\frac{\left(a + b\,(h_0/p_0)^\kappa\right)^{\frac{1}{\kappa}} - (\delta_p - \delta_h)}{1 + h_0/p_0} > b^{\frac{1}{\kappa}}, \quad \text{i.e.,} \quad \left(\frac{a}{b} + \left(\frac{h_0}{p_0}\right)^\kappa\right)^{\frac{1}{\kappa}} + \frac{\delta_h - \delta_p}{b^{\frac{1}{\kappa}}} > 1 + \frac{h_0}{p_0}.$$

The solution is used to complete the optimal paths (32) and (33) with $f(1, A) = (1 + bA^\kappa)$.

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