WEAK AND STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF NONEXPANSIVE MAPPINGS

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ABSTRACT. The purpose of this paper is to study weak and strong convergence of a new implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in a uniformly convex Banach space. The results obtained in this paper extend and improve the corresponding results of [C. E. Chidume, N. Shahzad, Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings, Nonlinear Anal. 62 (2005) 1149-1156; H.K. Xu, R. Ori, An implicit iterative process for nonexpansive mappings, Numer. Funct. Anal. Optim. 22 (2001) 767-773].

1. **Introduction**

Let X be a normed space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be nonexpansive on C if for all $x, y \in C$ the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\|.$$

Convergence theorems for nonexpansive mappings have been established by a number of authors(e.g., [6], [12], [13], [17] and the references therein).The convergence problems of an implicit iteration process have been studied by Browder [1, 2], Xu and Yin [19], Takahashi and Kim [16], and Jung and Kim [7], respectively. In 2001, Xu and Ori [18] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in J\}$ (here $J = \{1, 2, \ldots , N\}$) with $\{\alpha_n\}$ is a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_{N+1} x_{N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad n \geqslant 1, \tag{1.1}$$

where $T_n = T_{n(mod\ N)}$ (here the *mod N* function takes values in J). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings in a Hilbert space.

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Recently, Chidume and Shahzad [4] proved that Xu and Ori's iteration process converges strongly to a common fixed point for a finite family of nonexpansive mappings if one of the mappings is semi-compact. Inspired and motivated by these facts, we introduce and study an implicit iterative scheme for a finite family of nonexpansive mappings in Banach spaces. The scheme is defined as follows:

Let X be a normed linear space, let C be a nonempty convex subset of X, and let $\{T_i : i \in J\}$ be a finite family of nonexpansive self-mappings of C. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$. Then for an arbitrary $x_0 \in C$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_0 + (1 - \alpha_1 - \beta_1) T_1 x_1, \\ x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_1 + (1 - \alpha_2 - \beta_2) T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + \beta_N T_N x_{N-1} + (1 - \alpha_N - \beta_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_{N+1} x_N + (1 - \alpha_{N+1} - \beta_{N+1}) T_{N+1} x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1 - \alpha_n - \beta_n) T_n x_n, \ \ \forall n \geq 1, \tag{1.2}$$

where $T_n = T_{n(mod\,N)}$ (here the $mod\,N$ function takes values in J).

We note that Xu and Ori's iteration is a special case of the above implicit iterative scheme. If $\beta_n \equiv 0$, then (1.2) reduces to Xu and Ori's iteration [18].

The purpose of this paper is to establish strong and weak convergence theorems of the implicit iterative scheme (1.2) for a finite family of nonexpansive mappings. More precisely, we prove weak convergence of the implicit iteration process in a uniformly convex Banach space X such that its dual X^* has the Kadec-Klee property. The results presented in this paper extend and improve the corresponding ones announced by Xu and Ori [18], Chidume and Shahzad [4], and many others.

Now, we recall the well known concepts and results.

A mapping $T : C \to C$ is called *demi-closed* with respect to $y \in X$ if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$. A Banach space X is said to satisfy *Opial's condition* [10] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in C$ with $x \neq y$. A Banach space X is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$. A family $\{T_i : i \in J\}$ of N self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \varnothing$ is said to satisfy *condition* (B) on C [4] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\max_{1 \leqslant l \leqslant N} \{\|x - T_l x\|\} \geqslant f(d(x, F))$$

for all $x \in C$; see ([14], p.377) for an example of nonexpansive mappings satisfying *condition* (B).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 (Lemma 1, [17]). *Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ \ \forall n = 1, 2, ...$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(1) $\lim_{n\to\infty} a_n$ *exists.*

(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 1.2 (Lemma 1.4, [5]). *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. *Then there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \to [0, \infty), g(0) = 0$ *such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, *and all* $\lambda, \beta, \gamma \in [0, 1]$ *with* $\lambda + \beta + \gamma = 1$.

Lemma 1.3 (Lemma 2.7, [15]). *Let* X *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* X . *Let* u, v *be two elements of* X *such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{n_j}\}$ *are subsequences of* $\{x_n\}$ *which converge weakly to* u *and* v, *respectively, then* $u = v$.

Lemma 1.4 (Kaczor [8]). *Let* X *be a real reflexive Banach space such that its dual* X^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* X *and* $x^*, y^* \in \omega_w(x_n)$; *here* $\omega_w(x_n)$ *denote the set of all weak subsequential limits of* $\{x_n\}$. *Suppose* $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ *exists for all* $t \in [0, 1]$. *Then* $x^* = y^*$.

Lemma 1.5 (Browder [1]). *Let* X *be a uniformly convex Banach space, let* C *be a nonempty closed convex subset of* X *and let* $T : C \to X$ *be a nonexpansive mapping. Then* $I - T$ *is demi-closed at zero.*

We denote by Γ the set of strictly increasing, continuous convex function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a convex subset of the Banach space X. A mapping $T : C \to C$ is said to be type (γ) if $\gamma \in \Gamma$ and $0 \leqslant \alpha \leqslant 1$,

$$\gamma(\|\alpha Tx + (1-\alpha)Ty - T(\alpha x + (1-\alpha)y)\|) \leqslant \|x - y\| - \|Tx - Ty\|$$

for all x, y in C.

Lemma 1.6 (Bruck [3] and Oka [9]). *Let* X *be a uniformly convex Banach space and* C *a convex subset of* X. *Then there exists* $\gamma \in \Gamma$ *such that for each mapping* $S : C \to C$ *with* Lipschitz constant L,

$$\|\alpha Sx + (1-\alpha)Sy - S(\alpha x + (1-\alpha)y)\| \leqslant L\gamma^{-1}(\|x - y\| - \frac{1}{L}\|Sx - Sy\|)$$

for all $x, y \in C$ *and* $0 < \alpha < 1$.

2. **Main Results**

In this section, we prove weak and strong convergence of the implicit iteration process (1.2) to a common fixed point for a finite family of nonexpansive mappings in a uniformly convex Banach space.

Lemma 2.1. *Let* X *be a uniformly convex Banach space and let* C *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in J\}$ *be* N *nonexpansive self-mappings of* C *with* $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ *(here* $F(T_i)$ *denotes the set of fixed points of* T_i*). Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n$ *is in* $[0, 1]$ *for all* $n \geq 1$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. *From an arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by (1.2).*

(i) If $x^* \in F$, *then* $\lim_{n\to\infty} \|x_n - x^*\|$ *exists.*

(ii) For all $l \in J$, $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$.

Proof. Let $x^* \in F$. (i) For each $n \geq 1$, we have

$$\begin{aligned}\|x_n - x^*\| &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n - x^*\| \\ &\leq \alpha_n\|x_{n-1}-x^*\| + \beta_n\|T_n x_{n-1}-x^*\| + (1-\alpha_n-\beta_n)\|T_n x_n - x^*\| \\ &\leq \alpha_n\|x_{n-1}-x^*\| + \beta_n\|x_{n-1}-x^*\| + (1-\alpha_n-\beta_n)\|x_n - x^*\| \\ &= (\alpha_n+\beta_n)\|x_{n-1}-x^*\| + (1-\alpha_n-\beta_n)\|x_n - x^*\|.\end{aligned}$$

This implies that

$$\|x_n - x^*\| \leq \|x_{n-1} - x^*\|.$$

It implies by Lemma 1.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.
(ii) We shall show that $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. Using Lemma 1.2, we have

$$\begin{aligned}\|x_n - x^*\|^2 &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n - x^*\|^2 \\ &= \|\alpha_n(x_{n-1}-x^*) + \beta_n(T_n x_{n-1}-x^*) + (1-\alpha_n-\beta_n)(T_n x_n - x^*)\|^2 \\ &\leq \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|T_n x_{n-1}-x^*\|^2 + (1-\alpha_n-\beta_n)\|T_n x_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|x_{n-1}-T_n x_n\|) \\ &\leq \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|x_{n-1}-x^*\|^2 + (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|x_{n-1}-T_n x_n\|).\end{aligned}$$

Hence

$$\begin{aligned}\alpha_n(1-\alpha_n-\beta_n)g(\|x_{n-1}-T_n x_n\|) &\leq \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|x_{n-1}-x^*\|^2 \\ &\quad + (1-\alpha_n-\beta_n)\|x_n-x^*\|^2 - \|x_n-x^*\|^2 \\ &\leq \alpha_n\|x_{n-1}-x^*\|^2 + \beta_n\|x_{n-1}-x^*\|^2 \\ &\quad + (1-\alpha_n-\beta_n)\|x_{n-1}-x^*\|^2 - \|x_n-x^*\|^2 \\ &= \|x_{n-1}-x^*\|^2 - \|x_n-x^*\|^2.\end{aligned}$$

If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, then there exists a positive integer n_0 and $\eta, \eta^{'} \in (0,1)$ such that $0 < \eta < \alpha_n$ and $\alpha_n + \beta_n < \eta^{'} < 1$, $\forall n \geq n_0$. Hence

$$\eta(1-\eta^{'})g(\|x_{n-1}-T_n x_n\|) \leq \|x_{n-1}-x^*\|^2 - \|x_n-x^*\|^2, \quad \forall n \geq n_0.$$

It follows that for $m \geq n_0$,

$$\sum_{n=n_0}^{m} g(\|x_{n-1}-T_n x_n\|) \leq \frac{1}{\eta(1-\eta')}\sum_{n=n_0}^{m}(\|x_{n-1}-x^*\|^2 - \|x_n-x^*\|^2).$$

We get $\sum_{n=n_0}^{\infty} g(\|x_{n-1}-T_n x_n\|) < \infty$ as $m \to \infty$. This implies that $\lim_{n\to\infty} g(\|x_{n-1} - T_n x_n\|) = 0$. Since g is strictly increasing, continuous and $g(0) = 0$, we have $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. Since T_n is nonexpansive, we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n - x_{n-1}\| \\ &= \|\alpha_n(x_{n-1} - x_{n-1}) + \beta_n(T_n x_{n-1} - x_{n-1}) \\ &\quad + (1-\alpha_n-\beta_n)(T_n x_n - x_{n-1})\| \\ &= \|\beta_n(T_n x_{n-1} - T_n x_n + T_n x_n - x_{n-1}) \\ &\quad + (1-\alpha_n-\beta_n)(T_n x_n - x_{n-1})\| \\ &\leq \beta_n\|T_n x_{n-1} - T_n x_n\| + \beta_n\|T_n x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n-\beta_n)\|T_n x_n - x_{n-1}\| \\ &\leq \beta_n\|x_{n-1} - x_n\| + (1-\alpha_n)\|T_n x_n - x_{n-1}\| \\ &\leq \beta_n\|x_{n-1} - x_n\| + \|T_n x_n - x_{n-1}\| \\ &= \beta_n\|x_{n-1} - x_n\| + \|x_{n-1} - T_n x_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - x_{n-1}\| \leq \|x_{n-1} - T_n x_n\|.$$

By $\limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, there exists a positive integer n_0 and $\beta \in (0,1)$ such that $\beta_n \leq \alpha_n + \beta_n < \beta$, $\forall n \geq n_0$. Hence, we have

$$(1-\beta)\|x_n - x_{n-1}\| \leq \|x_{n-1} - T_n x_n\|.$$

Let $n \to \infty$. It follows that $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$. Also $\lim_{n\to\infty}\|x_n - x_{n+l}\| = 0$ for all $l \in J$. Since $\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|$, we have $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Now since for all $l \in J$

$$\begin{aligned}\|x_n - T_{n+l}x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|x_{n+l} - x_n\|,\end{aligned}$$

we have that $\lim_{n\to\infty}\|x_n - T_{n+l}x_n\| = 0$ for all $l \in J$. Since for each $l \in J$, $\{\|x_n - T_l x_n\|\}$ is a subset of $\cup_{l=1}^N\{\|x_n - T_{n+l}x_n\|\}$, we have $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l \in J$. This completes the proof. □

Theorem 2.2. *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that $\{T_i : i \in J\}$ satisfies condition(B). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.2). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Proof. Let $x^* \in F$. By Lemma 2.1 (i), we have that $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - x^*\|$ exists and $\|x_n - x^*\| \leq \|x_{n-1} - x^*\|$ for all $n \geq 1$. This implies that $d(x_n, F) \leq d(x_{n-1}, F)$, so $\lim_{n\to\infty} d(x_n, F)$ exists. Also, by Lemma 2.1 (ii), $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l \in J$. Since $\{T_i : i \in J\}$ satisfies *condition*(B), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, for any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\epsilon}{2}$ for all $n \geq n_0$. So we can find $y^* \in F$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{2}$. For all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - y^*\| + \|x_n - y^*\| \\ &\le \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = z^*$. Then $z^* \in C$. It remains to show that $z^* \in F$. Let $\epsilon^{'} > 0$ be given. Then there exists $n_1 \in \mathbb{N}$ such that $\|x_n - z^*\| < \frac{\epsilon^{'}}{4}$, $\forall n \ge n_1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists $n_2 \in \mathbb{N}$ and $n_2 \ge n_1$ such that for all $n \ge n_2$ we have $d(x_n, F) < \frac{\epsilon^{'}}{4}$ and in particular we have $d(x_{n_2}, F) < \frac{\epsilon^{'}}{4}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_2} - w^*\| < \frac{\epsilon^{'}}{4}$. For any $i \in J$ and $n \ge n_2$, we have

$$\begin{aligned}\|T_i z^* - z^*\| &\le \|T_i z^* - w^*\| + \|w^* - z^*\| \\ &\le 2\|w^* - z^*\| \\ &\le 2(\|w^* - x_{n_2}\| + \|x_{n_2} - z^*\|) \\ &< 2(\frac{\epsilon^{'}}{4} + \frac{\epsilon^{'}}{4}) = \epsilon^{'}.\end{aligned}$$

This implies that $T_i z^* = z^*$. Hence $z^* \in F(T_i)$ for all $i \in J$ and so $z^* \in F$. This completes the proof. ☐

We recall that a mapping $T : C \to C$ is called semi-compact(or hemicompact) if any sequence $\{x_n\}$ in C satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 2.3. *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that one of the mappings in $\{T_i : i \in J\}$ is semi-compact. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n$ is in $[0, 1]$ for all $n \ge 1$ and $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.2). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in J$. By Lemma 2.1 (ii), we have $\lim_{n\to\infty} \|x_n - T_{i_0} x_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$ as $j \to \infty$. Now Lemma 2.1 (ii) guarantees that $\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l \in J$ and so $\|x^* - T_l x^*\| = 0$ for all $l \in J$. This implies that $x^* \in F$. By Lemma 2.1 (i), $\lim_{n\to\infty} \|x_n - x^*\|$ exists and then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \|x_{n_j} - x^*\| = 0.$$

This completes the proof. ☐

For $\beta_n \equiv 0$, the iterative scheme (1.2) reduces to that of (1.1) and the following results are directly obtained by Theorem 2.2 and Theorem 2.3, respectively.

Theorem 2.4. (Theorem 3.2, [4]) *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \varnothing$. Suppose that $\{T_i : i \in J\}$ satisfies condition (B). Let*

$\{\alpha_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *From an arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geqslant 1.$$

Then $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in J\}$.

Theorem 2.5. (Theorem 3.3, [4]) *Let* X *be a uniformly convex Banach space and let* C *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in J\}$ *be* N *nonexpansive self-mappings of* C *with* $F = \cap_{i=1}^{N} F(T_i) \neq \varnothing$. *Suppose that one of the mappings in* $\{T_i : i \in J\}$ *is semi-compact. Let* $\{\alpha_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *From an arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geqslant 1.$$

Then $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in J\}$.

In the next results, we prove the weak convergence of the sequence $\{x_n\}$ defined by (1.2) in a uniformly convex Banach space satisfying *Opial's condition*.

Lemma 2.6. *Let* X *be a uniformly convex Banach space which satisfies Opial's condition and let* C *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in J\}$ *be* N *nonexpansive self-mappings of* C *with* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n$ *is in* $[0,1]$ *for all* $n \geq 1$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. *From an arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by (1.2). Then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i : i \in J\}$.

Proof. It follows from Lemma 2.1(ii) that $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for all $l \in J$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to x^*$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.5, we have $x^* \in F(T_i)$ for all $i \in J$. Hence $x^* \in F$. Suppose that there exist subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to y^* and z^*, respectively. By Lemma 1.5, $y^*, z^* \in F$. By Lemma 2.1 (i), we have $\lim_{n\to\infty} \|x_n - y^*\|$ and $\lim_{n\to\infty} \|x_n - z^*\|$ exist. It follows from Lemma 1.3 that $y^* = z^*$. Therefore $\{x_n\}$ converges weakly to a common fixed point x^* in F. □

Finally, we will prove weak convergence of the sequence $\{x_n\}$ defined by (1.2) in a uniformly convex Banach space X whose its dual X^* has the Kadec-Klee property.

Theorem 2.7. *Let* X *be a uniformly convex Banach space and let* C *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in J\}$ *be* N *nonexpansive self-mappings of* C *with* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n$ *is in* $[0,1]$ *for all* $n \geq 1$. *From an arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by (1.2). Then for all* $y^*, z^* \in F$, *the limit* $\lim_{n\to\infty} \|tx_n + (1-t)y^* - z^*\|$ *exists for all* $t \in [0,1]$.

Proof. It follows from Lemma 2.1 (i) that the sequence $\{x_n\}$ is bounded. Then there exists $R > 0$ such that $\{x_n\} \subset B_R \cap C$. Let $a_n(t) = \|tx_n + (1-t)y^* - z^*\|$, where $t \in (0,1)$. Then $\lim_{n\to\infty} a_n(0) = \|y^* - z^*\|$ and by Lemma 2.1 (i), $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - z^*\|$ exists. So we let $\lim_{n\to\infty} \|x_n - z^*\| = r$ for some positive number r. Let $x \in C$. We note that for all $i = 1, 2, \cdots, N, N+1$, the mappings

$$S_{x,i-1} := \alpha_i x + \beta_i T_i x + (1-\alpha_i-\beta_i)T_i$$

are contractions. It follows from the Banach contraction principle that there exists a unique fixed point $y_{x,i-1}$ of $S_{x,i-1}$ for each i. Hence, we can define $G_n : C \to C$ by

$$G_n x = y_{x,n}, \quad \forall x \in C, \ n \geq 0;$$

see [11]. Using G_n, we can be written the following compact form:

$$G_n x = \alpha_{n+1}x + \beta_{n+1}T_{n+1}x + (1-\alpha_{n+1}-\beta_{n+1})T_{n+1}G_n x,$$

where $T_n = T_{n(mod\ N)}$. By the definition of G_n, it easy to see that $\|G_n w - G_n z\| \leq \|w - z\|$ for each $w, z \in C$. This implies that G_n is a nonexpansive mapping for all $n \geq 0$. Moreover, we have

$$\begin{aligned}
\|G_n x_n - x_{n+1}\| &= \|\alpha_{n+1}x_n + \beta_{n+1}T_{n+1}x_n + (1-\alpha_{n+1}-\beta_{n+1})T_{n+1}G_n x_n - x_{n+1}\| \\
&= \|(1-\alpha_{n+1}-\beta_{n+1})T_{n+1}G_n x_n - (1-\alpha_{n+1}-\beta_{n+1})T_{n+1}x_{n+1}\| \\
&\leq (1-\alpha_{n+1}-\beta_{n+1})\|G_n x_n - x_{n+1}\|.
\end{aligned}$$

This implies that $G_n x_n = x_{n+1}$ for all $n \geq 0$. Similarly, $G_n x^* = x^*$ for all $x^* \in F$ for all $n \geq 0$. Set $H_{n,m} := G_{n+m-1}G_{n+m-2}\cdots G_n$, $n, m \geq 1$ and $b_{n,m} = \|H_{n,m}(tx_n + (1-t)y^*) - (tH_{n,m}x_n + (1-t)y^*)\|$, where $0 \leq t \leq 1$. It is easy to see that $H_{n,m}x_n = x_{n+m}$ and $H_{n,m}x^* = x^*$ for all $x^* \in F$. It follows from Lemma 1.6 that

$$\begin{aligned}
b_{n,m} &= \|H_{n,m}(tx_n + (1-t)y^*) - (tH_{n,m}x_n + (1-t)y^*)\| \\
&\leq \gamma^{-1}(\|x_n - y^*\| - \|H_{n,m}x_n - H_{n,m}y^*\|) \\
&= \gamma^{-1}(\|x_n - y^*\| - \|x_{n+m} - y^*\|).
\end{aligned}$$

Hence $\gamma(b_{n,m}) \leq \|x_n - y^*\| - \|x_{n+m} - y^*\|$. This implies that $\lim_{n,m\to\infty} \gamma(b_{n,m}) = 0$. By the property of γ, we obtain that $\lim_{n,m\to\infty} b_{n,m} = 0$. Observe that

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)y^* - z^*\| \\
&\leq \|H_{n,m}(tx_n + (1-t)y^*) - (tH_{n,m}x_n + (1-t)y^*)\| \\
&+ \|H_{n,m}(tx_n + (1-t)y^*) - z^*\| \\
&\leq b_{n,m} + \|tx_n + (1-t)y^* - z^*\| = b_{n,m} + a_n(t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\limsup_{m\to\infty} a_m(t) &= \limsup_{m\to\infty} a_{n+m}(t) \\
&\leq \limsup_{m\to\infty} (b_{n,m} + a_n(t)) \\
&\leq \gamma^{-1}(\|x_n - y^*\| - \lim_{m\to\infty} \|x_m - y^*\|) + a_n(t)
\end{aligned}$$

and

$$\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t).$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$. This completes the proof. □

Theorem 2.8. *Let X be a uniformly convex Banach space such that its dual X^* has the Kadec-Klee property and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be N nonexpansive self-mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ is in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by (1.2). Then $\{x_n\}$ converges weakly to some common fixed point of $\{T_i : i \in J\}$.*

Proof. It follows from Lemma 2.1 (i) that the sequence $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to a point $z^* \in C$. By Lemma 2.1 (ii), we have $\lim_{k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0$. Now using Lemma 1.5, we have $(I - T_l)z^* = 0$, that is $T_l z^* = z^*$ for all $l \in J$. Thus $z^* \in F$. Next we prove that $\{x_n\}$ converges weakly to z^*. Suppose that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $y^* \in C$ and so $z^*, y^* \in \omega_w(x_n) \cap F$. By Theorem 2.7, $\lim_{n\to\infty} \|tx_n + (1-t)y^* - z^*\|$ exists for all $t \in [0,1]$. It follows from Lemma 1.4, we have $z^* = y^*$. As a result, $\omega_w(x_n)$ is a singleton, and so $\{x_n\}$ converges weakly to some fixed point in F. $\square$

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