CORRELATED EQUILIBRIA IN N-PLAYER STOPPING GAMES

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Abstract. This paper extends the concept of correlated stopping times in 2-player games to n-player games. Communication between the players may be used to select an equilibrium according to a criterion of choice. Examples of 3-player games based on the job search problem are presented. One example shows that different correlations can be used to satisfy a given criterion of choice.

1 Introduction The concept of correlated stopping times in 2-player stopping games was introduced by Ramsey and Szajowski [19],[20]. This article is devoted to extending the definition of correlated stopping times to n-player stopping games. In order to illustrate this concept, examples of correlated equilibria in a 3-player stopping game based on the job search problem are presented.

The theory of stopping games was initiated by Dynkin [4], who considered a zero-sum game in which two players observe a sequence of N objects whose values $X_1, X_2, \dots, X_N$ are i.i.d. random variables. The i-th object is observed at moment i ($i = 1, 2, \dots, N$). Each player may obtain at most one object and an object can be accepted only at the moment of its appearance. Dynkin assumed that one player could stop at odd moments and the other at even moments. The class of pure stopping times is broad enough to define equilibria in such a game. The theory of such games has been extended to models of non-zero-sum games with a special payoff structure in which players simultaneously observe the sequence. For example, Ohtsubo [17] considers a problem in which the players observe the same objects, but have individual utility functions and Ferenstein [6] considers a model in which each player observes his own sequence. The structure of the payoff functions has a strong influence on the existence and form of solutions.

When players jointly observe a sequence of observations, it is necessary to resolve the problem of which player obtains an object when more than one player wishes to accept it. This led to discussion of the concept of the priority of a player (see Fushimi [9] and Szajowski [26]). The set of pure stopping times is not rich enough to describe all the Nash equilibria in such stopping games. For this reason Yasuda [29] introduced the notion of randomized strategies in stopping games. Ramsey and Szajowski [18] presented a model of a 3-player game with random priority and the concept of priority used in this article is a direct extension of their concept. Such models assume that objects cannot be shared.

Sakaguchi [22] and Sakaguchi and Mazalov [23] consider a different mechanism. If both players wish to accept an object, the game ends and the players obtain a payoff based on the value of this object. If only one player wishes to stop, then a lottery is used to decide whether the players both stop or both continue. One obvious interpretation of such models is that the players are searching for an object to share, but may differ on their valuations of the objects. Ben Abdelaziz and Krichen [2] extend this model to a model in which the priority of a player depends on the history of the process. If only one player wishes to accept an object, but the lottery determines that the players should continue, the player who wished to stop is later given a higher priority. Such approaches have been extended to n-player games. Ferguson [7] and Szajowski and Yasuda [27] present a different approach to n-player stopping games, in which all the players stop when an agreed number of players r, where $1 \leq r \leq n$, wish to stop. In the case $r = 1$ such games are quitting games. Kurano et al. [12], Nakagami et al. [14] and Yasuda et al.

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[28] use a similar approach. These models differ from the one presented here, since the players "share" the one object chosen. In the model considered here, each individual can obtain at most one object.

The models mentioned above assume that there is no communication between players in deciding upon what strategy should be played and only Nash equilibria are considered. Any conflict resulting from the decisions made are resolved using a lottery. For a review of models of stopping games and types of equilibria see Neumann et al. [16] and Sakaguchi [21].

In many stopping games the problem of equilibrium selection exists. Hence, communication between players may be beneficial. Ramsey and Szajowski [19], [20] introduced the concept of correlated stopping times and correlated equilibria in 2-player stopping games. The concept of correlated equilibria in matrix games was introduced by Aumann [1]. Players can use communication and signals (e.g. the result of a coin toss) to correlate their actions (i.e. the lottery is used to determine the actions taken, rather than resolve any conflicts after the decisions have been made). Aumann showed that the set of Nash equilibria is a subset of the set of correlated equilibria. For example, if there is a unique Nash equilibrium, this is the unique correlated equilibrium. Also, a randomization over Nash equilibria is a correlated equilibrium (see Moulin [13]). Hence, the problem of the selection of a correlated equilibrium exists whenever the problem of the selection of a Nash equilibrium exists. However, the concept of correlated equilibria assumes that communication between the players is possible. Hence, the problem of equilibrium selection can be solved by the players adopting some criterion for choosing an equilibrium. It is assumed that players may communicate before the game is played to agree on the correlated strategy to be used. At each stage of the game the players may observe a signal from an arbitrator. The arbitrator chooses the actions that should be taken by each player from the appropriate joint distribution (corresponding to the strategy used). The arbitrator then tells each player individually which action he should take. Certain correlated equilibria may be obtained by the players jointly observing a signal at each stage (this is the case when each player knows the actions recommended to the other players given the signal he received). The game is treated as a non-cooperative game (i.e. payoffs are not transferrable).

Greenwald and Hall [10] proposed various criteria for choosing correlated equilibria. Herings and Peeters [11] also consider selection of equilibria in stochastic games. Their approach is completely different, being based on Bayesian rationality. They assume that the state space is finite and that there is no communication between players (the strategies of the players "converge" on a Nash equilibrium). Such an approach seems reasonable for games with a small number of states. In general, the state space in stopping games is complex and often uncountable.

Solan and Vohra [24] considered correlated equilibria in quitting games (games which finish when at least one player wishes to stop). Although 2-player stopping games can be formulated as quitting games, n-player stopping games intrinsically differ from quitting games (see the considerations made regarding the extension of correlated stopping times from 2-player games to n-player games in Section 2). For an overview of correlated strategies in stochastic games see Solan and Vieille [25].

Enns and Ferenstein [5] consider a model of an n-player stopping game with deterministic priority. This model is specific, due to the fact that the problem of equilibrium selection does not exist (the value of the game is uniquely defined). However, in games with random priority there may be a multitude of equilibria. Szajowski [26] presents examples of such equilibria in a 2-player stopping game. The introduction of the concept of correlated equilibria in stopping times has given a firmer mathematical base for the concept of equilibrium selection.

2 Correlated equilibria in n-player stopping games We consider the following n-person stopping game. n players simultaneously observe a sequence of objects of non-negative values $X_1, X_2, \ldots, X_N$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with state space $(\mathtt{E}, \mathcal{B})$. Define $\mathcal{F}_i = \sigma(X_1, X_2, \ldots, X_i)$ to be the σ-field generated by the random variables $X_1, X_2, \ldots, X_i$. The i-th object appears at moment i. A player can obtain at most one object and an object can be obtained only at the moment of its appearance.

If only one player wishes to accept an object (denoted as the action s), then that player obtains the object and stops observing the sequence. The remaining players are free to continue observing the sequence. Suppose all the players are still searching and all of them wish to accept the object appearing at moment i. A random lottery is used to assign this object to one of the players and the other players continue observing the sequence. In this case it is assumed that Player m obtains the object with probability $p_{m,i}$. $p_{m,i}$ is referred to as the priority of Player m at moment i. Obviously, $\sum_{m=1}^{n} p_{m,i} = 1$, $1 \leq i \leq N$. Whenever a set $\mathcal{K}$ of players wish to accept an object at moment i, then Player m, $m \in \mathcal{K}$, obtains the object with probability proportional to $p_{m,i}$. That is to say that if $\sum_{m\in\mathcal{K}} p_{m,i} > 0$, then the probability of Player m obtaining the object is $p_{m,i,\mathcal{K}}$, where $p_{m,i,\mathcal{K}} = \frac{p_{m,i}}{\sum_{m\in\mathcal{K}} p_{m,i}}$.

If $p_{m,i} = 0$, $\forall m \in \mathcal{K}$, then the priorities of the players should be explicitly defined. A similar approach was used by Ramsey and Szajowski [18]. For example, consider a 3-player game where Player 1 always has priority over Players 2 and 3 and Player 2 always has priority over Player 3. The following definitions are sufficient to define the priorities of the players: $p_{1,i} = 1$ and $p_{2,i,\{2,3\}} = 1$, $1 \leq i \leq N$.

The strategy of Player m is a set of stopping times $\{\lambda_m(\mathcal{K})\}_{\mathcal{K}\subseteq\{1,2,\dots n\}:m\in\mathcal{K}}$, which are Markov times with respect to $\mathcal{F}_i$. The stopping time $\lambda_m(\mathcal{K})$ determines the time at which Player m wishes to stop given that the set of players $\mathcal{K}$ is still observing the sequence. The restricted stopping time $\lambda_m^i(\mathcal{K})$ defines the time at which Player m wishes to stop given that the subset of players $\mathcal{K}$ are still searching after observing the first i objects. A player may not be able to stop when he wishes and thus his actual stopping time must be defined recursively by considering the stopping rules of the other players, the definition of the lottery and the resulting restricted stopping times according to which player first obtained an object. A strict mathematical definition of such a stopping time is somewhat unwieldy and is thus omitted. Szajowski [26] gives the appropriate definition for such stopping times in a 2-player game.

The actual stopping time of Player m, i.e. the time at which Player m obtains an object in the n-player game, is defined to be μ_m, where $\mu_m = N + 1$ when Player m does not obtain an object. The payoff of Player m is assumed to be $g_m(X_{\mu_m})$, where $g_m(0) = 0$, g_m is non-decreasing and has bounded expected value. It is assumed that $X_{N+1} = 0$. We now recall the definition of a pair of correlated stopping times in a 2-player stopping game given by Ramsey and Szajowski [20].

Definition 2.1 *A random sequence $\hat{q} = \{(q_i^1, q_i^2, q_i^3)\}$ such that, for each i,*

$$i)\ q_i^j \text{ is adapted to } \mathcal{F}_i \text{ for } j = 1, 2, 3; \qquad ii)\ 0 \leq q_i^1 \leq q_i^2 \leq q_i^3 \leq 1 \text{ a.s.}$$

is called a correlated stopping strategy in a 2-player stopping game. The set of all such sequences will be denoted by $\hat{\mathcal{Q}}^N$.

Let $A_1, A_2, \dots, A_N$ be a sequence of i.i.d. r.v. with uniform distribution on $[0, 1]$ and independent of the Markov process $(X_i, \mathcal{F}_i, \mathbf{P}_x)_{i=0}^N$. Denote $\vec{q}_i = (q_i^1, q_i^2, q_i^3)$. The correlated stopping times associated with $\hat{q}$ are a pair $(\lambda^1(\hat{q}), \lambda^2(\hat{q}))$ of Markov times with respect to the σ-fields $\mathcal{H}_i = \sigma\{\mathcal{F}_i,\ \{A_1, A_2, ..., A_i\}\}$ defined as follows:

$$\lambda^1(\hat{q}) = \inf\{0 \leq i \leq N : A_i \leq q_i^2\} \tag{1}$$

$$\lambda^2(\hat{q}) = \inf\{0 \leq i \leq N : A_i \leq q_i^1 \text{ or } q_i^2 < A_i \leq q_i^3\}. \tag{2}$$

The strategy $\hat{q}$ will be called the correlation profile. It can be seen that at moment i the probabilities of the players taking the pair of actions (s, s), (s, c), (c, s) and (c, c) are $q_i^1, q_i^2 - q_i^1, q_i^3 - q_i^2$ and $1 - q_i^3$, respectively.

This definition relies on the fact that given at least one of the players wishes to accept an object, then the expected payoffs of the players can be defined by considering the optimal payoff of each player when observing the sequence alone. In the n-player version of such a game these payoffs cannot be derived directly, it is necessary to inductively derive the payoffs in all the subgames in which $2, 3, \dots, n-1$ players are still observing the sequence. Hence, to define a correlated stopping strategy in an n-player stopping game, we must define an appropriate correlated strategy to be used in each of these subgames. It follows that a correlated strategy in an n-player stopping game must define an appropriate strategy in each of the possible $2^n - n - 1$ subgames in which between 2 and n players are observing the sequence.

Let $\mathcal{M} = \{1, 2, \ldots, n\}$ be the set of players. For $\mathcal{C} \subseteq \mathcal{M}$, define $\mathcal{S}_{\mathcal{C}}$ to be the set of subsets of $\mathcal{C}$ with cardinality greater or equal to 2 (including $\mathcal{C}$ itself).

Definition 2.2 *Let $\vec{q_i}(\mathcal{C}) = (q_i^0(\mathcal{C}), q_i^1(\mathcal{C}), \ldots, q_i^{2^{\#(\mathcal{C})}-1}(\mathcal{C}))$, where $\mathcal{C} \in \mathcal{S}_{\mathcal{M}}$, $\vec{q_i}(\mathcal{C})$ is adapted to $\mathcal{F}_i$ for $i = 1, 2, \ldots N$, $\#$ denotes cardinality, $q_i^j(\mathcal{C}) \geq 0$ for $j = 0, 1, \ldots, 2^{\#(\mathcal{C})} - 1$ and $\sum_{j=0}^{2^{\#(\mathcal{C})}-1} q_i^j(\mathcal{C}) = 1$. Let $\hat{q}(\mathcal{C}) = \{\vec{q_i}(\mathcal{C})\}_{i=1}^N$.*

Define $(b_{1,j}^l b_{2,j}^l \ldots b_{l,j}^l)_2$ to be the binary expansion of the integer j in decimal form with length l [i.e. $0 \leq j < 2^l$ and $j_{10} = (b_{1,j}^l b_{2,j}^l \ldots b_{l,j}^l)_2$] and $\mathcal{C} = \{n_1(\mathcal{C}), n_2(\mathcal{C}), \ldots, n_{\#(\mathcal{C})}(\mathcal{C})\}$, where $n_1(\mathcal{C}) < n_2(\mathcal{C}) < \ldots < n_{\#(\mathcal{C})}(\mathcal{C})$. We assume j represents the combination of actions taken by the players in $\mathcal{C}$ as follows: if $b_{m,j}^{\#(\mathcal{C})} = 1$, then Player $n_m(\mathcal{C})$ wishes to stop and if $b_{m,j}^{\#(\mathcal{C})} = 0$, then Player $n_m(\mathcal{C})$ continues. $q_i^j(\mathcal{C})$ is interpreted as the probability that the combination of actions corresponding to this binary expansion of j is made on the appearance of the i-th object when the subset $\mathcal{C}$ of players is still observing the sequence.

A correlated strategy $\hat{q}$ in an n-player stopping game is defined to be a set of $2^n - n - 1$ sequences, $\{\hat{q}(\mathcal{C})\}_{\mathcal{C} \in \mathcal{S}_{\mathcal{M}}}$, where each $\vec{q_i}(\mathcal{C})$ satisfies the conditions given directly above. The set of all such strategies will be denoted $\hat{\mathcal{Q}}_n^N$.

The sequence $\hat{q}(\mathcal{C})$ will be referred to as the correlated strategy used in the subgame in which the set $\mathcal{C}$ of players is still observing the sequence. The set of restricted strategies $\hat{q}^m(\mathcal{C}) = \{\vec{q_i}(\mathcal{C})\}_{i=m}^N$ defines the correlated strategy used from moment m onwards in the subgame with the set of players $\mathcal{C}$.

It should be noted that in the case of a 2-player game this definition is equivalent to the definition given by Ramsey and Szajowski [20], although it is given in a slightly different form (this new definition explicitly gives all the appropriate probabilities for each pair of actions).

It is assumed that a correlated strategy may be achieved by a judge carrying out the appropriate randomizations and suggesting to each player individually the action he should take at each moment. Suppose the subset of players $\mathcal{C} = \{n_1(\mathcal{C}), n_2(\mathcal{C}), \ldots, n_{\#(\mathcal{C})}(\mathcal{C})\}$ are searching at moment i and the judge generates β from the uniform distribution on $[0, 1]$. Suppose $\sum_{k=0}^{j-1} q_i^k(\mathcal{C}) < \beta \leq \sum_{k=0}^{j} q_i^k(\mathcal{C})$. If $b_{m,j}^{\#(\mathcal{C})} = 1$ the judge informs Player $n_m(\mathcal{C})$ to accept the object, otherwise Player $n_m(\mathcal{C})$ is informed to reject the object. This process is referred to as the correlating device. Each player is free to ignore such a recommendation. In certain cases (see Equilibrium 1 in Example 3.1) the appropriate correlation can be achieved by the players jointly observing the result of an appropriate randomization device. The following remark illustrates how this definition should be interpreted in a 3-player game.

Remark 2.1 *Suppose that all three players are searching at moment i. Then*

$$\vec{q_i}(\{1,2,3\}) = (1,0,0,0,0,0,0,0); \quad \vec{q_i}(\{1,2,3\}) = (0,0,0,0,0,0,0,1); \quad \vec{q_i}(\{1,2,3\}) = (0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, 0)$$

represent the action triplets i) all three players continue, ii) all three players wish to stop, iii) one randomly chosen player stops (each player is chosen with equal probability). The correlated strategy may be described by the following four sequences:

$$\{\vec{q_i}(\{1,2,3\})\}_{i=1}^N, \quad \{\vec{q_i}(\{1,2\})\}_{i=2}^N, \quad \{\vec{q_i}(\{1,3\})\}_{i=2}^N, \quad \{\vec{q_i}(\{2,3\})\}_{i=2}^N.$$

The sequence $\{\vec{q_i}(\{j,m\})\}_{i=2}^N$ describes the correlated strategy to be used in the subgame in which Players j, m are still observing the sequence. Define $\{A_i\}$ to be a sequence of i.i.d. random variables from the $U[0,1]$ distribution and $s_{i,j} = \sum_{m=0}^{j} q_i^m(\{1,2,3\})$ for $m = 0, 1, \ldots 7$. Note $s_{i,7} = 1, \forall i$. This sequence is used to define the recommendations made by the arbitrator in the following way:

$$\begin{aligned}
\lambda_1(\hat{q}) &= \min_i \{A_i > s_{i,3}\} \qquad \lambda_2(\hat{q}) = \min_i \{A_i \in (s_{i,1}, s_{i,3}] \cup (s_{i,5}, 1]\} \\
\lambda_3(\hat{q}) &= \min_i \{A_i \in (s_{i,0}, s_{i,1}] \cup (s_{i,2}, s_{i,3}] \cup (s_{i,4}, s_{i,5}] \cup (s_{i,6}, 1]\}.
\end{aligned}$$

The distribution of the triplet of actions taken when the three players are all still searching is illustrated in the tables below. The probability of each triplet of actions is given, together with the values of A_i for which a triplet is played.

	Player 2 - s	Player 2 - c
Player 1 - s	$q_i^7(\{1,2,3\})$, $A_i \in (s_{i,6}, 1]$	$q_i^5(\{1,2,3\})$, $A_i \in (s_{i,4}, s_{i,5}]$
Player 1 - c	$q_i^3(\{1,2,3\})$, $A_i \in (s_{i,2}, s_{i,3}]$	$q_i^1(\{1,2,3\})$, $A_i \in (s_{i,0}, s_{i,1}]$

Table 1: Player 3 plays s

	Player 2 - s	Player 2 - c
Player 1 - s	$q_i^6(\{1,2,3\})$, $A_i \in (s_{i,5}, s_{i,6}]$	$q_i^4(\{1,2,3\})$, $A_i \in (s_{i,3}, s_{i,4}]$
Player 1 - c	$q_i^2(\{1,2,3\})$, $A_i \in (s_{i,1}, s_{i,2}]$	$q_i^0(\{1,2,3\})$, $A_i \in (0, s_{i,0}]$

Table 2: Player 3 plays c

$\lambda_m(\hat{q})$ defines the first moment at which Player m wishes to stop given that none of the other players have taken an object. Let $k = \min_{m\in\{1,2,3\}} \lambda_m(\hat{q})$. Suppose this minimum is achieved for some unique m, say m_0. Then Player m_0 obtains the object appearing at moment k and the remaining two players play the appropriate 2-player restricted stopping game starting at moment $k+1$ as described in Ramsey and Szajowski [19], [20]. If this minimum is achieved for more than one value of m, then a lottery is carried out to assign the k-th object to one of the players wishing to accept it. The two players who do not obtain this object play a 2-player restricted subgame as described above.

Given the correlation strategy used in an n-player game, the future expected payoffs of each player at each moment can be found using the following procedure:

(i) Solve the optimal choice problems faced by each of the players when they are searching alone. This can be done using recursion (with respect to the sequence of observations).

(ii) The payoff matrices for any of the possible 2-player subgames at moment i (in which the players have two pure strategies c-continue and s-stop) can be defined recursively, together with the future expected payoffs of the players at each moment.

(iii) Given the future expected payoffs of the players in each k-player subgame at each moment, the corresponding payoffs in all the possible $k+1$-player games can be defined recursively. It follows that the future expected payoffs of each player in the n-player game can be calculated.

Let $\mathcal{C} = \{n_1(\mathcal{C}), n_2(\mathcal{C}), \dots n_{\#(\mathcal{C})}(\mathcal{C})\}$. Define $u_m^j(i, x, \{\hat{q}^{i+1}(\mathcal{L})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}})$ to be the future expected payoff of Player $n_m(\mathcal{C})$ at moment i when the value of the object observed is x, the set $\mathcal{C}$ of players has not yet obtained an object, $\{\hat{q}^{i+1}(\mathcal{L})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}}$ describes the restricted correlated strategies used in all of the possible subgames and the actions taken by the players at moment i are given by the binary expansion of j. The payoff matrix of the subgame played by the set of Players $\mathcal{C}$ at moment i is defined by the following set of payoffs: $u_m^j(i, x_i, \{\hat{q}^{i+1}(\mathcal{L})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}})$ for $1 \le m \le k, 0 \le j \le 2^k - 1$.

Suppose at moment i Player $n_m(\mathcal{C})$ departs from the action suggested to him by the correlating device. The distribution of the actions of the other players is the conditional correlated profile given the suggestion made to Player $n_m(\mathcal{C})$, $\vec{q}_{i,-m}(\mathcal{C})$. Suppose he uses the strategy "play c with probability p, otherwise play s". According to the definition of a correlated equilibrium, Player $n_m(\mathcal{C})$ cannot gain by ignoring the suggestion. From the linearity of his payoffs in p at a correlated equilibrium, he has no incentive to change his action from the one suggested. We hence obtain the following definition of a correlated equilibrium in an n-player stopping game.

Definition 2.3 *A correlated equilibrium $\hat{q}^*$ in an n-player stopping game is a set of $2^n - n - 1$ sequences, $\{\hat{q}^*(\mathcal{C})\}_{\mathcal{C}\in\mathcal{S}_\mathcal{M}}$, which all satisfy the following inequalities for all $1 \le i \le N$, $\mathcal{C} \in \mathcal{S}_\mathcal{M}$ and $1 \le m \le \#(\mathcal{C})$.*

$$\sum_{0\le j\le 2^{\#(\mathcal{C})}-1,\, b^{\#(\mathcal{C})}_{m,j}=1} u^j_m(i, x_i, \{\hat{q}^{i+1}(\mathcal{C})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}}) q^j_i(\mathcal{C}) \ \ge \sum_{0\le j\le 2^{\#(\mathcal{C})}-1,\, b^{\#(\mathcal{C})}_{m,j}=1} u^{j-2^{\#(\mathcal{C})-m}}_m(i, x_i, \{\hat{q}^{i+1}(\mathcal{C})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}}) q^j_i(\mathcal{C}) \tag{3}$$

$$\sum_{0\le j\le 2^{\#(\mathcal{C})}-1,\, b^{\#(\mathcal{C})}_{m,j}=0} u^j_m(i, x_i, \{\hat{q}^{i+1}(\mathcal{C})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}}) q^j_i(\mathcal{C}) \ \ge \sum_{0\le j\le 2^{\#(\mathcal{C})}-1,\, b^{\#(\mathcal{C})}_{m,j}=0} u^{j+2^{\#(\mathcal{C})-m}}_m(i, x_i, \{\hat{q}^{i+1}(\mathcal{C})\}_{\mathcal{L}\in\mathcal{S}_\mathcal{C}}) q^j_i(\mathcal{C}) \tag{4}$$

Remark 2.2 *Suppose $\sum_{0\le j\le 2^{\#(\mathcal{C})}-1,\, b^{\#(\mathcal{C})}_{m,j}=1} q^j_i(\mathcal{C}) > 0$. Player $n_m(\mathcal{C})$ is advised to play s iff $b^{\#(\mathcal{C})}_{m,j} = 1$. It follows that the left-hand side of Inequality (3) is the expected payoff of Player $n_m(\mathcal{C})$ given that he follows this suggestion multiplied by the probability that he is advised to play s. From the definition of a correlated strategy, when Player $n_m(\mathcal{C})$ unilaterally changes his action from s to c, the index defining the combination of actions changes from j to $j - 2^{\#(\mathcal{C})-m}$ (the m-th element of the corresponding binary expansion changes from 1 to 0). It follows that the right-hand side of this inequality is the expected payoff of Player $n_m(\mathcal{C})$ given that he does not follow this suggestion multiplied by the probability that he is advised to play s. Hence, this inequality states that Player $n_m(\mathcal{C})$ cannot gain by ignoring a suggestion to stop. Analogously, Inequality (4) states that Player $n_m(\mathcal{C})$ cannot gain by ignoring a suggestion to continue.*

Theorem 2.1 *A correlated equilibrium exists in such an n-player stopping game.*

This follows from the fact that a correlated equilibrium is defined to be a sequence of correlated equilibria in all of the appropriately defined matrix subgames played by any subset of at least 2 of the n players. It follows from Nash [15] that a Nash equilibrium exists in each of these games. Since a Nash equilibria is a correlated equilibrium, the result is proven.

3 Examples I present two examples of 3-player games based on the job search problem. It is assumed that the values $X_1, X_2, \ldots, X_N$ are i.i.d. random variables from the uniform distribution on [0,1]. The following criterion of choice is used: at each moment in each of the possible subgames the equilibrium chosen maximizes the minimum expected payoff of the players still searching. This is an extension of the stepwise egalitarian equilibrium in the 2-player game considered by Ramsey and Szajowski [20]. The first example shows two ways of obtaining such an equilibrium in a symmetric game. The first equilibrium can be realized by, when required, jointly observing a signal which is the result of an appropriately defined random experiment. No signal is needed to achieve the second equilibrium. The second example is based on an asymmetric 3-player game. In this case it may be necessary for the players to individually observe signals. This example illustrates a more general approach to such a problem.

It is necessary to derive the optimal strategy of an individual, who is searching alone. Let h_i be his optimal expected reward when i objects have yet to appear. It follows that $h_0 = 0$ and

$$h_{i+1} = E[\max\{X_{N-i}, h_i\}].$$

Hence, $h_{i+1} = \frac{1+h_i^2}{2}$. An individual should accept the $N-i$-th object iff $X_{N-i} \ge h_i$. These thresholds can be calculated by induction.

3.1 Example 1 Consider the 3-player game in which the players have equal priority i.e. $p_{m,l} = \frac{1}{3}$, $\forall l \in \{1, 2, \ldots, N\}, m \in \{1, 2, 3\}$. It is clear that such dan equilibrium must equalize the expected payoffs of the players still searching at each moment in each of the subgames. At the first equilibrium, whenever there exists more than one correlated equilibrium, a randomization is carried out over the pure Nash equilibria. At the second correlated equilibrium, the value of the object is used to assign it to one of the players. Hence, no randomization device is required.

3.1.1 Equilibrium 1 Let u_i and v_i denote the expected reward of the players in the 2-player subgames and the 3-player game, respectively, when i objects have yet to appear. The value of the game is (v_N, v_N, v_N). First, we solve the 2-player subgames. These subgames are of the same form. In the 2-player subgames Player 1 and Player 2 will refer to the players who have not yet accepted an object with the smallest and largest index, respectively. The normal form of the "sub-subgame" played at moment $N-i$, $i \in \{0, 1, \ldots, N-1\}$ given the value of the object is x is given by the matrix $A_i(x)$, where

$$A_i(x) = \begin{array}{cc} & \begin{array}{cc} \text{s} & \text{c} \end{array} \\ \begin{array}{c} \text{s} \\ \text{c} \end{array} & \left(\begin{array}{cc} (\frac{x+h_i}{2}, \frac{x+h_i}{2}) & (x, h_i) \\ (h_i, x) & (u_i, u_i) \end{array} \right) \end{array}.$$

Whenever at least one of the players accepts the object, the sum of the expected payoffs is $x + h_i$. It is clear that $h_i \geq u_i \geq v_i$ (this can be shown by induction). When $x > h_i$, (s, s) is the unique Nash equilibrium and thus the only correlated equilibrium. Similarly, (c, c) is the only correlated equilibrium when $x < u_i$. Both (c, s) and (s, c) are Nash equilibria when $u_i \leq x \leq h_i$. In this case $x + h_i > 2u_i$, hence these Nash equilibria maximize the sum of the players' expected payoffs. Thus the minimum expected payoff of the players can be maximized by choosing each of these equilibria with probability $\frac{1}{2}$ (any randomization over the set of Nash equilibria is a correlated equilibrium). This equilibrium can be achieved by the players jointly observing the result of a coin toss. Other correlated equilibria which place a positive mass of probability on the action pair (s, s) satisfy the criterion. However, they cannot be achieved by the joint observation of a randomizing device. u_i can be calculated by induction. We have

$$u_{i+1} = \int_0^{u_i} u_i dx + \int_{u_i}^1 \frac{x + h_i}{2} dx = \frac{1 + 2h_i(1 - u_i) + 3u_i^2}{4}. \tag{5}$$

Now we consider the 3-player game. The normal form of the subgame played at moment $N - i$ when an object of value x appears is given by the pair of matrices $B_{s,i}(x)$ and $B_{c,i}(x)$ describing the payoffs of the players when Player 3 stops and continues, respectively. We have

$$\begin{aligned} B_{s,i}(x) &= \begin{array}{cc} & \begin{array}{cc} \text{s} & \text{c} \end{array} \\ \begin{array}{c} \text{s} \\ \text{c} \end{array} & \left(\begin{array}{cc} (\frac{x+2u_i}{3}, \frac{x+2u_i}{3}, \frac{x+2u_i}{3}) & (\frac{x+u_i}{2}, u_i, \frac{x+u_i}{2}) \\ (u_i, \frac{x+u_i}{2}, \frac{x+u_i}{2}) & (u_i, u_i, x) \end{array} \right) \end{array} \\ B_{c,i}(x) &= \begin{array}{cc} & \begin{array}{cc} \text{s} & \text{c} \end{array} \\ \begin{array}{c} \text{s} \\ \text{c} \end{array} & \left(\begin{array}{cc} (\frac{x+u_i}{2}, \frac{x+u_i}{2}, u_i) & (x, u_i, u_i) \\ (u_i, x, u_i) & (v_i, v_i, v_i) \end{array} \right) \end{array}. \end{aligned}$$

(s, s, s) is the only correlated equilibrium when $x > u_i$ and (c, c, c) is the only correlated equilibrium when $x < v_i$. (s, c, c), (c, s, c) and (c, c, s) are Nash equilibria when $v_i \leq x \leq u_i$. Arguing as above, the minimum expected payoff is maximized by choosing each of these Nash equilibria with probability $\frac{1}{3}$.

When recommendations are given to players individually and a player is told to reject an object, he does not know the exact actions recommended to the other two players. However, he knows that exactly one player has been told to accept the object. By symmetry, it follows that such a correlated strategy may be achieved by jointly observing the result of e.g. a dice roll. v_i can be calculated by induction.

$$v_{i+1} = \int_0^{v_i} v_i dx + \int_{v_i}^1 \frac{x + 2u_i}{3} dx = \frac{1 + 5v_i^2 + 4u_i(1 - v_i)}{6}. \tag{6}$$

Table 6 gives these values for $N \leq 5$.

3.1.2 Equilibrium 2 When there are multiple equilibria, the correlation is based on the value of the object. First we consider the 2-player subgames. At moment $N - i$ the correlation is based on the realisation x of X_i as given in the following table. Note that $a_i \in (u_i, h_i)$.

	Player 2 - s	Player 2 - c
Player 1 - s	$x \geq h_i$	$a_i < x < h_i$
Player 1 - c	$u_i < x \leq a_i$	$x \leq u_i$

Table 3: Correlated strategy in the 2-player subgames

Clearly, the value of the game must be equal to the value of the game at Equilibrium 1. Hence, considering Player 2's payoff at Equilibrium 2

$$u_{i+1} = \int_0^{u_i} u_i dx + \int_{u_i}^{a_i} x dx + \int_{a_i}^{h_i} h_i dx + \int_{h_i}^1 \frac{x+h_i}{2} dx = \frac{1+2h_i(1-u_i)+3u_i^2}{4}.$$

This leads to the following quadratic equation for a_i:

$$\frac{a_i^2}{2} - a_i h_i + \frac{h_i^2}{4} + \frac{h_i u_i}{2} - \frac{u_i^2}{4} = 0.$$

The appropriate solution of this equation ($a_i \in [u_i, h_i]$) is

$$a_i = h_i - \frac{\sqrt{2}(h_i - u_i)}{2}.$$

The values of a_i for $i \leq 5$ are given in Table 6. Given there are multiple equilibria, the probability that Player 1 accepts the object is $\frac{\sqrt{2}}{2} \approx 0.7071$. The player who continues searching has a higher expected payoff than the player who accepts the object. Hence, Player 2 is compensated for being required to accept lower value objects by being the player more likely to continue searching alone.

Now we consider a correlation in the 3-player game of the form given below, where $v_i < b_i < c_i < u_i$:

	Player 2 - s	Player 2 - c
Player 1 - s	$x \geq u_i$	Never Played
Player 1 - c	Never Played	$v_i < x \leq b_i$

Table 4: Correlated strategy in the 3-player game: Player 3 plays s

	Player 2 - s	Player 2 - c
Player 1 - s	Never Played	$c_i < x < u_i$
Player 1 - c	$b_i < x \leq c_i$	$x \leq v_i$

Table 5: Correlated strategy in the 3-player game: Player 3 plays c

Considering the payoffs of Player 1 and Player 3, respectively, we obtain

$$\begin{aligned} v_{i+1} &= \int_0^{v_i} v_i dx + \int_{v_i}^{c_i} u_i dx + \int_{c_i}^{u_i} x dx + \int_{u_i}^1 \frac{x+2u_i}{3} dx = \frac{1+5v_i^2+4u_i(1-v_i)}{6} \\ v_{i+1} &= \int_0^{v_i} v_i dx + \int_{v_i}^{b_i} x dx + \int_{b_i}^{u_i} u_i dx + \int_{u_i}^1 \frac{x+2u_i}{3} dx = \frac{1+5v_i^2+4u_i(1-v_i)}{6}. \end{aligned}$$

N	h_N	u_N	v_N	a_N	b_N	c_N
1	0.5000	0.2500	0.1667	0.3232	0.1820	0.2019
2	0.6250	0.4844	0.3287	0.5256	0.3573	0.3945
3	0.6953	0.5871	0.4735	0.6188	0.4943	0.5215
4	0.7417	0.6521	0.5596	0.6783	0.5765	0.5987
5	0.7751	0.6979	0.6191	0.7205	0.6335	0.6524

Table 6: Value functions and thresholds in the symmetric 3-player game.

These equations lead to the following quadratic equations for b_i and c_i:

$$\frac{c_i^2}{2} - u_i c_i - \frac{v_i^2}{6} + \frac{u_i v_i}{3} + \frac{u_i^2}{3} = 0; \qquad \frac{b_i^2}{2} - u_i b_i - \frac{v_i^2}{3} - \frac{2u_i v_i}{3} + \frac{u_i^2}{6} = 0.$$

The appropriate solutions for these equations are

$$c_i = u_i - \frac{\sqrt{3}(u_i - v_i)}{3}; \qquad b_i = u_i - \frac{\sqrt{6}(u_i - v_i)}{3}.$$

Numerical values for b_i and c_i are given in Table 6. Given that there are multiple equilibria, the probabilities of Player 1, Player 2 and Player 3 obtaining the object are $\frac{\sqrt{3}}{3} \approx 0.5774$, $\frac{\sqrt{6}-\sqrt{3}}{3} \approx 0.2391$ and $1 - \frac{\sqrt{6}}{3} \approx 0.1835$, respectively. It should be noted that similar egalitarian equilibria can be defined which do not require randomization. However, this is the simplest such procedure.

3.2 Example 2 We now solve the 3-player game in which $p_{1,l} = \frac{1}{2}, p_{2,l} = \frac{1}{3}, p_{3,l} = \frac{1}{6}$, for all $l \in \{1, 2, \dots, N\}$. Consider the subgames. The priorities of the players in the subgames are: $\alpha_{1,2} = p_{1,l,\{1,2\}} = \frac{3}{5}$, $\alpha_{1,3} = p_{1,l,\{1,3\}} = \frac{3}{4}$, $\alpha_{2,3} = p_{2,l,\{2,3\}} = \frac{2}{3}$. When Players j and k ($j < k$) are still searching, i objects are yet to appear and the value of the object is x, the payoff matrix is

$$A_i^{j,k}(x) = \begin{array}{c} \\ \text{s} \\ \text{c} \end{array} \begin{array}{c} \begin{array}{cc} \text{s} & \text{c} \end{array} \\ \begin{pmatrix} (\alpha_{j,k}x + [1-\alpha_{j,k}]h_i, [1-\alpha_{j,k}]x + \alpha_{j,k}h_i) & (x, h_i) \\ (h_i, x) & (u_i^{j,k}, u_i^{k,j}) \end{pmatrix} \end{array},$$

where $u_i^{j,k}$ is the value of the restricted stopping game to Player j when both players continue and $u_i^{k,j}$ is the value of the same game to Player k. Intuitively, at an egalitarian equilibrium

$$\begin{aligned} u_i^{3,1} &\leq u_i^{3,2} \leq u_i^{2,1} \leq u_i^{1,2} \leq u_i^{2,3} \leq u_i^{1,3} \leq h_i \\ v_i^j &\leq u_i^{j,k}; \qquad v_i^3 \leq v_i^2 \leq v_i^1, \end{aligned}$$

where $1 \leq j, k \leq 3$. These relations can be proven by induction.

When $x > h_i$, the only correlated equilibrium is (s, s), when $u_i^{k,j} < x < u_i^{j,k}$ ($j < k$), the only correlated equilibrium is (c, s) and when $x < u_i^{k,j}$ the only correlated equilibrium is (c, c). Both (s, c) and (c, s) are pure Nash equilibria when $u_i^{j,k} \leq x \leq h_i$. In this case (s, c) is the strategy pair which maximizes the payoff of Player k.

Suppose that whenever there are multiple equilibria, the strategy pair (s, c) is played. If the expected payoff of Player k is not greater than the expected payoff of Player j, then this is the strategy pair that should be chosen at an egalitarian equilibrium. Given this is true, then the form of the egalitarian equilibrium in these subgames is

Numerical calculations show that for $1 \leq i \leq 10$ Player k obtains a lower expected payoff than Player j in each of these subgames. Hence, for $1 \leq j < k \leq 3$, $1 \leq i \leq 10$ the values of the subgames are given

	Player k - s	Player k - c
Player j - s	$x \geq h_i$	$u_i^{j,k} < x < h_i$
Player j - c	$u_i^{k,j} < x \leq u_i^{j,k}$	$x \leq u_i^{k,j}$

Table 7: Correlated strategy in the 2-player subgames

N	$u_N^{1,2}$	$u_N^{2,1}$	$u_N^{1,3}$	$u_N^{3,1}$	$u_N^{2,3}$	$u_N^{3,2}$	$t_{N,1}$	$t_{N,2}$
1	0.3000	0.2000	0.3750	0.1250	0.3333	0.1667	0.3500	0.3000
2	0.5150	0.4650	0.5703	0.4219	0.5417	0.4444	0.5519	0.5225
3	0.6100	0.5720	0.6532	0.5378	0.6314	0.5554	0.6388	0.6165
4	0.6707	0.6394	0.7066	0.6107	0.6887	0.6253	0.6946	0.6746
5	0.7138	0.6869	0.7447	0.6621	0.7294	0.6747	0.7344	0.7188

Table 8: Value functions in the subgames and thresholds for the asymmetric 3-player game.

by the following induction formulae:

$$\begin{aligned} u_{i+1}^{j,k} &= \int_0^{u_i^{k,j}} u_i^{j,k} dx + \int_{u_i^{k,j}}^{u_i^{j,k}} h_i dx + \int_{u_i^{j,k}}^{h_i} x dx + \int_{h_i}^1 [\alpha_{j,k} x + (1-\alpha_{j,k}) h_i] dx \\ u_{i+1}^{k,j} &= \int_0^{u_i^{k,j}} u_i^{k,j} dx + \int_{u_i^{k,j}}^{u_i^{j,k}} x dx + \int_{u_i^{j,k}}^{h_i} h_i dx + \int_{h_i}^1 [\alpha_{j,k} h_i + (1-\alpha_{j,k}) x] dx. \end{aligned}$$

These equations lead to

$$\begin{aligned} u_{i+1}^{j,k} &= u_i^{j,k} u_i^{k,j} + h_i(1-\alpha_{j,k} + u_i^{j,k} - u_i^{k,j}) + \frac{\alpha_{j,k} - (u_i^{j,k})^2 - h_i^2(1-\alpha_{j,k})}{2} \\ u_{i+1}^{j,k} &= h_i(\alpha_{j,k} - u_i^{j,k}) + \frac{(1-\alpha_{j,k})(1+h_i^2) + (u_i^{j,k})^2 + (u_i^{k,j})^2}{2}. \end{aligned}$$

The values of these subgames for $1 \leq N \leq 5$ are given in Table 8.

Now we consider the 3-player game. The matrices for the subgame played at moment $N-i$ when an object of value x appears are given by

$$B_{s,i}(x) \quad = \quad \begin{array}{c} \\ \text{s} \\ \text{c} \end{array} \begin{array}{c} \begin{array}{cc} \text{s} & \text{c} \end{array} \\ \begin{pmatrix} \left(\frac{3x+2u_i^{1,3}+u_i^{1,2}}{6}, \frac{2x+3u_i^{2,3}+u_i^{2,1}}{6}, \frac{x+3u_i^{3,2}+2u_i^{3,1}}{6}\right) & \left(\frac{3x+u_i^{1,2}}{4}, \frac{3u_i^{2,3}+u_i^{2,1}}{4}, \frac{x+3u_i^{3,2}}{4}\right) \\ \left(\frac{2u_i^{1,3}+u_i^{1,2}}{3}, \frac{2x+u_i^{2,1}}{3}, \frac{x+2u_i^{3,1}}{3}\right) & (u_i^{1,2}, u_i^{2,1}, x) \end{pmatrix} \end{array}$$

$$B_{c,i}(x) \quad = \quad \begin{array}{c} \\ \text{s} \\ \text{c} \end{array} \begin{array}{c} \begin{array}{cc} \text{s} & \text{c} \end{array} \\ \begin{pmatrix} \left(\frac{3x+2u_i^{1,3}}{5}, \frac{2x+3u_i^{2,3}}{5}, \frac{3u_i^{3,2}+2u_i^{3,1}}{5}\right) & (x, u_i^{2,3}, u_i^{3,2}) \\ (u_i^{1,3}, x, u_i^{3,1}) & (v_i^1, v_i^2, v_i^3) \end{pmatrix} \end{array},$$

where $B_{s,i}(x)$ is the payoff matrix when Player 3 wishes to stop, $B_{c,i}(x)$ the payoff matrix when Player 3 wishes to continue and v_i^j the value to Player j of the restricted game in which i objects have yet to appear. Hence the value of the game is given by (v_N^1, v_N^2, v_N^3).

Tables 9 and 10 give the values of x for which each triplet of actions is a Nash equilibrium, where

$$t_{i,1} = \frac{2u_i^{1,3} + u_i^{1,2}}{3}; \qquad t_{i,2} = \frac{3u_i^{2,3} + u_i^{2,1}}{4}.$$

	Player 2 - s	Player 2 - c
Player 1 - s	$x \geq t_{i,1}$	$u_i^{1,2} \leq x \leq t_{i,2}$
Player 1 - c	$u_i^{2,1} \leq x \leq t_{i,1}$	$v_i^3 \leq x \leq u_i^{2,1}$

Table 9: Nash equilibria in the asymmetric 3-player game: Player 3 plays s

	Player 2 - s	Player 2 - c
Player 1 - s	Never an equilibrium	$v_i^1 \leq x \leq u_i^{3,2}$
Player 1 - c	$v_i^2 \leq x \leq u_i^{3,1}$	$x \leq v_i^3$

Table 10: Nash equilibria in the asymmetric 3-player game: Player 3 plays c

Numerical values of $t_{i,1}$ and $t_{i,2}$ are given for $1 \leq i \leq 5$ in Table 8. It should be noted that $t_{i,1} > t_{i,2}$.

Now we derive the value of the 3-player game by recursion. Clearly, given that no player has previously accepted an object, each player should accept the final object. It follows that $(v_1^1, v_1^2, v_1^3) = (\frac{1}{4}, \frac{1}{6}, \frac{1}{12})$.

The values of x for which a triplet of actions constitutes a Nash equilibrium at the penultimate moment are given in the tables below.

	Player 2 - s	Player 2 - c
Player 1 - s	$x \geq 0.35$	$x = 0.3$
Player 1 - c	$0.2 \leq x \leq 0.35$	$0.0833 \leq x \leq 0.2$

Table 11: Nash equilibria at the penultimate moment: Player 3 plays s

Hence, equilibrium selection is required on a set of measure zero. We have

$$\begin{aligned}
v_2^1 &= \int_0^{\frac{1}{12}} v_1^1 dx + \int_{\frac{1}{12}}^{\frac{1}{5}} u_1^{1,2} dx + \int_{\frac{1}{5}}^{\frac{7}{20}} \frac{2u_1^{1,3} + u_1^{1,2}}{3} dx + \int_{\frac{7}{20}}^{1} \frac{3x + 2u_1^{1,3} + u_1^{1,2}}{6} dx \\
v_2^2 &= \int_0^{\frac{1}{12}} v_1^2 dx + \int_{\frac{1}{12}}^{\frac{1}{5}} u_1^{2,1} dx + \int_{\frac{1}{5}}^{\frac{7}{20}} \frac{2x + u_1^{2,1}}{3} dx + \int_{\frac{7}{20}}^{1} \frac{2x + 3u_1^{2,3} + u_1^{2,1}}{6} dx \\
v_2^3 &= \int_0^{\frac{1}{12}} v_1^3 dx + \int_{\frac{1}{12}}^{\frac{1}{5}} x dx + \int_{\frac{1}{5}}^{\frac{7}{20}} \frac{x + 2u_1^{3,1}}{3} dx + \int_{\frac{7}{20}}^{1} \frac{x + 3u_1^{3,2} + 2u_1^{3,1}}{6} dx.
\end{aligned}$$

These equations lead to $v_2^1 = \frac{2119}{4800} \approx 0.4415$, $v_2^2 = \frac{2527}{7200} \approx 0.3510$, $v_2^3 = \frac{2939}{14400} \approx 0.2041$.

The values of x for which a triplet of actions constitutes a Nash equilibrium at the third last moment are given in Tables 13 and 14.

It can be seen that equilibrium selection is required on the following intervals:

(i) $x \in (0.3510, 0.4219)$ - both (c, s, c) and (c, c, s) are Nash equilibria,

(ii) $x \in (0.4415, 0.4444)$ - both (s, c, c) and (c, c, s) are Nash equilibria,

(iii) $x \in (0.5150, 0.5225)$ - both (c, s, s) and (s, c, s) are Nash equilibria.

The correlated equilibrium selected maximizes the expected payoff of Player 3 given that this payoff does not exceed the payoff of the other players. Let $p_{l,i}(x)$ denote the probability that the action triplet

	Player 2 - s	Player 2 - c
Player 1 - s	Never an equilibrium	Never an equilibrium
Player 1 - c	Never an equilibrium	$x \leq 0.0833$

Table 12: Nash equilibria at the penultimate moment: Player 3 plays c

	Player 2 - s	Player 2 - c
Player 1 - s	$x \geq t_{2,1} \approx 0.5519$	$u_2^{1,2} = 0.5150 \leq x \leq t_{2,2} = 0.5225$
Player 1 - c	$u_2^{2,1} = 0.4650 \leq x \leq t_{2,1} \approx 0.5519$	$v_2^3 \approx 0.2041 \leq x \leq u_2^{2,1} = 0.4650$

Table 13: Nash equilibria at the third last moment: Player 3 plays s

corresponding to the binary expansion of l $(0 \leq l \leq 7)$ is taken under a correlated strategy when i objects are yet to appear. Define $w_{l,i}^m(x)$ to be the corresponding expected payoff of Player m when at future moments the players play according to an egalitarian equilibrium. The problem of maximizing Player 3's expected payoff can be expressed as the following linear programming problem:

$$\max\{\sum_{l=0}^{7} p_{l,i}(x) w_{l,i}^3(x)\},$$

subject to the condition that the set of probabilities $\{p_{l,i}(x)\}_{l=0}^7$ defines a correlated equilibrium in the subgame defined by payoff matrices $B_{s,i}(x)$ and $B_{c,i}(x)$.

When $x \in (0.3510, 0.4219)$, Player 1's action c dominates his action s. Hence, no correlated equilibrium places any probability mass on action triplets where Player 1 plays s. When Player 1 plays c, Player 3's expected payoff is maximized at the triplet (c, s, c). Since this is a Nash equilibrium, it is also a correlated equilibrium. Similarly, when $x \in (0.4415, 0.4444)$, Player 3's expected payoff is maximized when the action triplet (s, c, c) is taken.

When $x \in (0.5150, 0.5225)$, Player 3's action s dominates his action c. Hence, no correlated equilibrium can place any probability mass on action triplets where Player 3 rejects such an object. The action triplet which maximizes the payoff of Player 3 is (c, c, s). However, this is not a Nash equilibrium and is thus not a correlated equilibrium. A correlated equilibrium must satisfy the following conditions

$$p_{1,2}(x) + p_{3,2}(x) + p_{5,2}(x) + p_{7,2}(x) = 1 \tag{7}$$

$$p_{7,2}(x)[w_{7,2}^1(x) - w_{3,2}^1(x)] + p_{5,2}(x)[w_{5,2}^1(x) - w_{1,2}^1(x)] \geq 0 \tag{8}$$

$$p_{3,2}(x)[w_{3,2}^1(x) - w_{7,2}^1(x)] + p_{1,2}(x)[w_{1,2}^1(x) - w_{5,2}^1(x)] \geq 0 \tag{9}$$

$$p_{7,2}(x)[w_{7,2}^2(x) - w_{5,2}^2(x)] + p_{3,2}(x)[w_{3,2}^2(x) - w_{1,2}^2(x)] \geq 0 \tag{10}$$

$$p_{5,2}(x)[w_{5,2}^2(x) - w_{7,2}^2(x)] + p_{1,2}(x)[w_{1,2}^2(x) - w_{3,2}^2(x)] \geq 0. \tag{11}$$

	Player 2 - s	Player 2 - c
Player 1 - s	Never an equilibrium	$v_2^1 \approx 0.4415 \le x \le u_2^{3,2} \approx 0.4444$
Player 1 - c	$v_2^2 \approx 0.3510 \le x \le u_2^{3,1} \approx 0.4219$	$x \le v_2^3 \approx 0.2041$

Table 14: Nash equilibria at the third last moment: Player 3 plays c

On this interval

$$\begin{aligned}
w_{7,2}^1(x) &= \frac{3x + 2u_2^{1,3} + u_2^{1,2}}{6} \le w_{3,2}^1(x) = \frac{2u_2^{1,3} + u_2^{1,2}}{3} \\
w_{5,2}^1(x) &= \frac{3x + u_2^{1,2}}{4} \ge w_{1,2}^1(x) = u_2^{1,2} \\
w_{7,2}^2(x) &= \frac{2x + 3u_2^{2,3} + u_2^{2,1}}{6} \le w_{5,2}^2(x) = \frac{3u_2^{2,3} + u_2^{2,1}}{4} \\
w_{3,2}^2(x) &= \frac{2x + u_2^{2,1}}{3} \ge w_{1,2}^1(x) = u_2^{2,1} \\
w_{1,2}^3(x) &= x \ge w_{5,2}^3(x) = \frac{3u_2^{3,2} + x}{4} \ge w_{3,2}^3(x) = \frac{2u_2^{3,1} + x}{3} \ge w_{7,1}^3(x) = \frac{x + 3u_2^{3,2} + 2u_2^{3,1}}{6}
\end{aligned}$$

Since the solution of the linear programming problem must occur at one of the apexes of the feasible set, it suffices to consider the following cases (for more information on the geometry of the set of correlated equilibria see Forges [8] and Calvó-Armengol [3]):

(i) Pure Nash equilibria. These are the action triplets (s, c, s) and (c, s, s). The payoff of Player 3 is maximized at (s, c, s). Correlated equilibria with two probability masses are randomizations over the two pure Nash equilibria. At such an equilibrium Player 3 cannot gain a greater payoff than at (s, c, s) (such correlated equilibria never correspond to an apex of the feasible region).

(ii) Correlated equilibria with three probability masses. From the equilibrium conditions there are two possible types: there is no probability mass at either (c, c, s) or (s, s, s). At the first type, Player 3 obtains a lower expected payoff than at (s, c, s). At the apex of the feasible region corresponding to the second type conditions (9) and (11) become equalities. It follows that

$$p_{1,2}(x) = \frac{1}{1 + a(x) + b(x)}; \qquad p_{3,2}(x) = \frac{a(x)}{1 + a(x) + b(x)}; \qquad p_{5,2}(x) = \frac{b(x)}{1 + a(x) + b(x)},$$

where

$$a(x) = \frac{9(x - u_2^{1,2})}{4u_2^{1,3} + 2u_2^{1,2} - 6x}; \qquad b(x) = \frac{8(x - u_2^{2,1})}{3u_2^{2,3} + u_2^{2,1} - 4x}.$$

Given $x \in (0.5150, 0.5225)$, the expected payoff of Player 3 at such a correlated equilibrium is

$$x p_{1,2}(x) + p_{3,2}(x) \left(\frac{x + 2u_2^{3,1}}{3} \right) + p_{5,2}(x) \left(\frac{x + 3u_2^{3,2}}{4} \right).$$

Numerical calculations show that Player 3 obtains a greater payoff at this equilibrium than at the action triplet (s, c, s). In order to achieve such an equilibrium, Players 1 and 2 must individually observe the appropriately defined random signal.

(iii) Correlated equilibria defined by four positive probability masses. At the apex of the feasible region corresponding to such a correlated equilibrium, each of the equilibrium conditions (7)-(11) becomes an equality. This equilibrium is the mixed Nash equilibrium. It follows that

$$p_{1,2}(x) = \frac{1}{1+a(x)+b(x)+a(x)b(x)}; \qquad p_{3,2}(x) = \frac{a(x)}{1+a(x)+b(x)+a(x)b(x)};$$
$$p_{5,2}(x) = \frac{b(x)}{1+a(x)+b(x)+a(x)b(x)}; \qquad p_{7,2}(x) = \frac{a(x)b(x)}{1+a(x)+b(x)+a(x)b(x)},$$

where $a(x), b(x)$ are defined as before. The ratio between the probabilities $p_{1,2}(x), p_{3,2}(x)$ and $p_{5,2}(x)$ are the same as at the correlated equilibrium defined directly before. Since Player 3 obtains a lower payoff at the action triplet (s, s, s) than at the previous correlated equilibrium, he obtains a lower expected reward at this mixed equilibrium than at that correlated equilibrium.

The correlated equilibrium that maximizes the expected payoff of Player 3 at moment $N-3$ is given by:

(i) (s, s, s) when $x \geq t_{2,1} \approx 0.5519$,

(ii) (c, s, s) when $t_{2,2} = 0.5225 \leq x < t_{2,1} \approx 0.5519$,

(iii) The correlated strategy which plays (c, c, s), (c, s, s) and (s, c, s) with probabilities $\frac{1}{1+a(x)+b(x)}$, $\frac{a(x)}{1+a(x)+b(x)}$ and $\frac{b(x)}{1+a(x)+b(x)}$, respectively, when $u_2^{1,2} = 0.515 < x < t_{2,2} = 0.5225$,

(iv) (c, s, s) when $u_2^{2,1} = 0.465 \leq x \leq u_2^{1,2} = 0.515$,

(v) (c, c, s) when $u_2^{3,2} \approx 0.4444 \leq x < u_2^{2,1} = 0.465$,

(vi) (s, c, c) when $v_2^1 \approx 0.4415 \leq x < u_2^{3,2} \approx 0.4444$,

(vii) (c, c, s) when $u_2^{3,1} \approx 0.4219 \leq x < v_2^1 \approx 0.4415$,

(viii) (c, s, c) when $v_2^2 \approx 0.3510 \leq x < u_2^{3,1} \approx 0.4219$,

(ix) (s, c, c) when $v_2^3 \approx 0.2041 \leq x < v_2^2 \approx 0.3510$,

(x) (c, c, c) when $x < v_2^3 \approx 0.2041$.

Define the value of the matrix subgame described by $B_{s,i}(x)$ and $B_{c,i}(x)$ to Player m at the egalitarian equilibrium by $r_i^m(x)$. The expected reward of Player m when all the players are searching on the appearance of the $N-i$-th object is given by $v_{i+1}^m = \int_0^1 r_i^m(x)dx$.

Since at the equilibrium described above the expected payoff of Player 3 is less than the expected payoff of the other players, this equilibrium describes the stepwise egalitarian correlated equilibrium at moment $N-3$. Numerical calculations show that when i objects are yet to appear ($4 \leq i \leq 10$) the egalitarian correlated equilibrium is of an analogous form to the one given above, except that $v_i^1 > u_i^{3,2}$. It follows that (s, c, c) is never a Nash equilibrium and (c, c, s) is the unique equilibrium on the interval $(u_i^{3,1}, u_i^{2,1})$. Table 15 gives numerical results for the values of the game for $1 \leq N \leq 5$.

N	v_N^1	v_N^2	v_N^3
1	0.2500	0.1667	0.0833
2	0.4415	0.3510	0.2041
3	0.5734	0.5025	0.3912
4	0.6420	0.5851	0.4926
5	0.6893	0.6413	0.5616

Table 15: Value functions in the subgames of the asymmetric 3-player game and thresholds in the 3-player game.

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