THREE-MEMBER COMMITTEE WHERE ODD-MAN'S JUDGEMENT IS PAID REGARD

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Abstract. A three-member committee wants to employ one specialist among n applicants. The committee interviews applicants sequentially one-by-one. Facing each applicant each member chooses either A(=accept) or R(=reject). If choices are different, odd-man's judgement is not neglected and he can make some arbitration for deciding the committee's A or R. Let (X_j, Y_j, Z_j) be the evaluations of the j-th applicant's ability by the committee members, where X_j, Y_j, Z_j are *i.i.d.* with $U_{[0,1]}$ distribution. Each member of the committee wants to maximize the expected value u_n of the applicant accepted by the committee. This three-player two-choice multistage game is formulated and is given a solution, as a function of $p \in [0, \frac{1}{2}]$ *i.e.*, odd-man's power of arbitration. It is shown that $u_n \uparrow u_\infty(p)$ and $u_\infty(p)$ decreases as $p \in [0, \frac{1}{2}]$ increases.

1 Statement and Formulation of the Problem.

A 3-player(=member) committee has players I, II, III (sometimes written by 1, 2, 3) observe $(X_j, Y_j, Z_j), j = 1, 2, \cdots n$, *iid* with $U_{[0,1]\times[0,1]\times[0,1]}$ distribution sequentially one-by-one, and each player chooses either one of R(=reject) or A(=accept). $X_j(Y_j, Z_j)$ is I's (II's, III's) evaluation of the j-th applicant's ability of some specific talent.

If all players choose A, the committee chooses A. If all players choose R, committee's choice is R, and the $j+1$ st applicant is interviewed. If players choose different choices, then the odd-man forces the committee to take the same choices as the odd-man's (even-man's) with probability $p(\overline{p}/2$, each), where $0 \le p \le \frac{1}{2}$. When $p = 0(\frac{1}{3})$, the game is under simple-majority (equal-priority) rule. When $p = \frac{1}{2}$, majority and minority have the equal priorities. Each member of the committee wants to maximize the expected value $u_n(p)$ of the ability of the applicant accepted by the committee.

Define the state (n, x, y, z) to mean that the committee evaluates the present applicant at $x(y, z)$ by I (II, III) and $n-1$ uninterviewed applicants remain if the present applicant is rejected by the committee.

Let EQV(=eq. value) for the n-stage game be (u_n, v_n, w_n). Then the Optimality Equation is

$$(1) \qquad (u_n, v_n, w_n) = E_{x,y,z}[\text{EQV of } \mathbf{M}_n(x,y,z)], \quad \left(n \ge 1, u_1 = v_1 = w_1 = \frac{1}{2}\right),$$

where the payoff matrix $\mathbf{M}_n(x, y, z)$ in state (n, x, y, z) is represented by

$$(2) \qquad \mathbf{M}_n(x,y,z) \begin{cases} \xrightarrow{\text{R by I}} \mathbf{M}_{n,R}(x,y,z) \\ \xrightarrow{\text{A by I}} \mathbf{M}_{n,A}(x,y,z) \end{cases}$$

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(3) $\mathbf{M}_{n,R}(x,y,z) =$

	III's R	III's A
II's R	$u, \quad v, \quad w$	$p(x,y,z)+\overline{p}(u,v,w)$
II's A	$p(x,y,z)+\overline{p}(u,v,w)$	$p(u,v,w)+\overline{p}(x,y,z)$

(4) $\mathbf{M}_{n,A}(x,y,z) =$

$p(x,y,z)+\overline{p}(u,v,w)$	$p(u,v,w)+\overline{p}(x,y,z)$
$p(u,v,w)+\overline{p}(x,y,z)$	$x, \quad y, \quad z$

(In each cell, the subscript $n-1$ of $u_{n-1}, v_{n-1}, w_{n-1}$ is omitted. We use this convention hereafter too, if needed.)

2 Solution to the Problem.

Lemma 1 *The bimatrix games played by II and III in state (n,x,y,z) have the solutions*

	$z < w$	$z > w$
$y < v$	*R-R* $v, \quad w$	*R-A* $py+\overline{p}v, \quad pz+\overline{p}w$
$y > v$	*A-R* $py+\overline{p}v, \quad pz+\overline{p}w$	*A-A* $pv+\overline{p}y, \quad pw+\overline{p}z$

in $\mathbf{M}_{nR}(x,y,z)$

	$z < w$	$z > w$
$y < v$	*R-R* $py+\overline{p}v, \quad pz+\overline{p}w$	*R-A* $pv+\overline{p}y, \quad pw+\overline{p}z$
$y > v$	*A-R* $pv+\overline{p}y, \quad pw+\overline{p}z$	*A-A* $y, \quad z$

in $\mathbf{M}_{nA}(x,y,z)$

where, in each cell, the pure EQ (EQV) is written in the upper (lower) part.

Proof is easy, since $0 \le p \le \frac{1}{2} \le \overline{p} \le 1$. □

Lemma 2 *For I, the choice* R (A) *dominates the choice* A (R) *if* $x < (>) u$.

Proof. I's payoff matrices are

$$\begin{array}{ccc} & \text{III's R} & \text{III's A} \\ \begin{array}{c}\text{II's R}\\ \text{II's A}\end{array} & \left(\begin{array}{c} u \\ px+\overline{p}u \end{array}\right. & \left.\begin{array}{c} px+\overline{p}u \\ pu+\overline{p}x \end{array}\right) \end{array} \quad \text{and} \quad \begin{pmatrix} px+\overline{p}u & pu+\overline{p}x \\ pu+\overline{p}x & x \end{pmatrix}$$

in $\mathbf{M}_{nR}(x,y,z)$ and $\mathbf{M}_{nA}(x,y,z)$, *resp.* Since $0 < p < \frac{1}{2} < \overline{p}$, both of $u-(px+\overline{p}u) = (pu+\overline{p}x)-x = p(u-x)$, and $(px+\overline{p}u)-(pu+\overline{p}x) = (\overline{p}-p)(u-x)$ are $> (<) 0$, if $x < (>) u$. So, the lemma follows. □

Lemma 3 *If we assume that $u_n \to u, v_n \to v, w_n \to w$, then the recurrence relation for player I*

$$u_n = \left[(3p-1)(u^4-2u^3) + \left(4p-\frac{1}{2}\right)u^2 - pu + \frac{1}{2}\right]_{u=u_{n-1}} \qquad (n \ge 1, u_0 = 0) \tag{5}$$

holds.

Proof. From Lemmas 1, 2 and Eqs.(2)∼(4), the equilibrium payoff for player I is the sum of $2^3 = 8$ terms :

$$(6)\qquad uI(x<u,y<v,z<w)+(px+\overline{p}u)I(x<u,y<v,z>w)$$
$$+(px+\overline{p}u)I(x<u,y>v,z<w)+(pu+\overline{p}x)I(x<u,y>v,z>w)$$
$$+(px+\overline{p}u)I(x>u,y<v,z<w)+(pu+\overline{p}x)I(x>u,y<v,z>w)$$
$$+(pu+\overline{p}x)I(x>u,y>v,z<w)+xI(x>u,y>v,z>w)$$

Taking $E_{x,y,z}$ of the r.v.'s, we get

$$\begin{aligned}(7)\quad E_{x,y,z}[Eq.(6)] &= u^2vw+\left\{(p/2)u^2v\overline{w}+\overline{p}u^2v\overline{w}\right\}\\ &\quad+\left\{(p/2)u^2\overline{v}w+\overline{p}u^2\overline{v}w\right\}+\left\{pu^2\overline{v}\,\overline{w}+(\overline{p}/2)u^2\overline{v}\,\overline{w}\right\}\\ &\quad+\left\{(p/2)(1-u^2)vw+\overline{p}u\overline{u}vw\right\}+\left\{pu\overline{u}v\overline{w}+(\overline{p}/2)(1-u^2)v\overline{w}\right\}\\ &\quad+\left\{pu\overline{u}\,\overline{v}w+(\overline{p}/2)(1-u^2)\overline{v}w\right\}+\frac{1}{2}(1-u^2)\overline{v}\,\overline{w}\\ &= u^2vw+u^2v\overline{w}\cdot\frac{1}{2}(1+\overline{p})+u^2\overline{v}w\cdot\frac{1}{2}(1+\overline{p})+u^2\overline{v}\,\overline{w}\cdot\frac{1}{2}(1+p)\\ &\quad+\left\{(p/2)(1-u^2)+\overline{p}u\overline{u}\right\}vw+\left\{(\overline{p}/2)(1-u^2)+pu\overline{u}\right\}v\overline{w}\\ &\quad+\left\{(\overline{p}/2)(1-u^2)+pu\overline{u}\right\}\overline{v}w+\frac{1}{2}(1-u^2)\overline{v}\,\overline{w}\\ &= \frac{1}{2}\left(p+2\overline{p}u+pu^2\right)vw+\frac{1}{2}\left\{\overline{p}+2pu+(\overline{p}-p)u^2\right\}(v\overline{w}+\overline{v}w)\\ &\quad+\frac{1}{2}(1+pu^2)\overline{v}\,\overline{w}\end{aligned}$$

There exists symmetry in the roles of players. Whoever cannot be the odd-man, even if he wants to become it. We can consider that u_n, v_n, w_n have the same limit u. Then Eq.(7) becomes

$$(8)\qquad \frac{1}{2}\left(p+2\overline{p}u+pu^2\right)u^2+\left\{\overline{p}+2pu+(\overline{p}-p)u^2\right\}u\overline{u}+\frac{1}{2}(1+pu^2)\overline{u}^2.$$

After a bit of algebra, this becomes

$$(9)\qquad (3p-1)u^4+(2-6p)u^3+\left(4p-\frac{1}{2}\right)u^2-pu+\frac{1}{2},$$

which is the r.h.s. of Eq.(5). □

Lemma 4 *The sequence $\{u_n\}$ defined by Eq.(5) satisfies $u_n \uparrow u_\infty$, and $u_\infty(= u$ say) is a unique root in $\left(\frac{1}{2},1\right)$ of the cubic equation*

$$(10)\qquad (3p-1)(u^3-u^2)+\left(p+\frac{1}{2}\right)u-\frac{1}{2}=0,$$

if $p \neq 1/3$. If $p = 1/3$, then $u = 3/5$.

Proof. Let $p \neq 1/3$. We have from (5),

$$\begin{aligned}(11)\qquad u_n-u_{n-1} &= \left[(3p-1)(u^4-2u^3)+\left(4p-\frac{1}{2}\right)u^2-(1+p)u+\frac{1}{2}\right]_{u=u_{n-1}}\\ &= (3p-1)[(u-1)f(u)]_{u=u_{n-1}}\end{aligned}$$

where $f(u) \equiv u^3-u^2+(3p-1)^{-1}\left\{\left(p+\frac{1}{2}\right)u-\frac{1}{2}\right\}$.

It is easy to show that

(a) For $1/3 < p < 1/2, f(u)$ is increasing, concave-convex with a point of inflection $u = 1/3$, since $f'(u) = 3u^2 - 2u + \frac{p+1/2}{3p-1} = 0$ has no real root and $f''(u) = 6\left(u - \frac{1}{3}\right)$. Also, sine $f\left(\frac{1}{2}\right) = -\frac{\overline{p}}{8(3p-1)} < 0 < f(1) = \frac{p}{3p-1}$, the equation $f(u) = 0$ has a unique root $u = u_\infty$ in $\left(\frac{1}{2}, 1\right)$.

(b) For $0 < p < 1/3, f(u)$ is again concave-convex with a point of inflection $u = 1/3$. $f(u)$ can have a minimal point in $(1/2, 1)$ at $u = \frac{1}{3}\left(1 + \sqrt{\frac{5/2}{1-3p}}\right) > \frac{1}{3}\left(1 + \sqrt{5/2}\right) \approx 0.8604$ if $0 < p < \frac{1}{8}$. Since $f(1/2) > \frac{\overline{p}}{8(1-3p)} > 0 > f(1) = \frac{-p}{1-3p}$, the equation $f(u) = 0$ has a unique root in $(1/2, 1)$. See Figure 1.

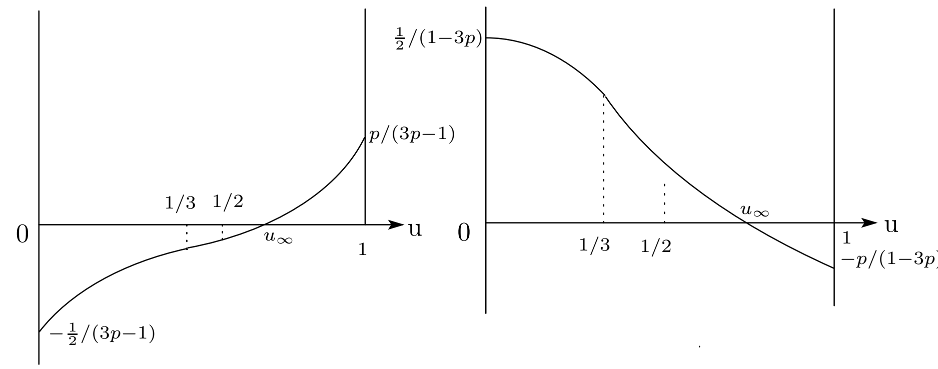

Figure 1.(a) $f(u)$, where $1/3 < p \leq 1/2$ (b) $f(u)$, where $0 \leq p < 1/3$

Therefore, we find from (11) that if, $1/2 < u_{n-1} < u_\infty$ then

$$u_n - u_{n-1} = (3p-1)(u_{n-1}-1)f(u_{n-1}) > 0, \quad \text{when both of } \frac{1}{3} < p \leq \frac{1}{2} \text{ and } 0 \leq p < \frac{1}{3}. \quad \square$$

Considering lemmas 2 ∼ 4 altogether, we obtain

Theorem. *The solution of the* 3*-player game given by* (1) ∼ (4)*, where* $p \in [0, 1/2]$ *is as follows. The common* EQS *for each player is to*
"Choose A (R)*, if his r.v. is* $> (<) u_{n-1}(p)$.
where $\{u_n(p)\}$ *is determined by the recursion* (5)*. The expected payoff to the committee is* $u_n(p)$*. We have* $u_n(p) \uparrow u_\infty(p), \forall p \in [0, 1/2]$
where $u_\infty(p)$ *is a unique root in* $(1/2, 1)$ *of the cubic equation* (10).

Let us check the three special cases of Eq.(10). $u_\infty(0) = 1/\sqrt{2} \approx 0.7071$ (*i.e.*, simple-majority case); $u_\infty(1/3) = 3/5$ (*i.e.*, equal-priority case) and $u_\infty(1/2) \approx 0.5698$(= unique root in $(1/2, 1)$ of the equation $u^{3/2} - u + 1 = 0$ (*i.e.*, majority and minority have equal priority). Computation gives the values of $u_\infty(p)$ for various p.

$p = 0$	0.1	0.2	0.3	1/3	0.35	0.4	1/2
$u_\infty(p) = \frac{1}{\sqrt{2}} \approx 0.7071$	0.6605	0.6304	0.6069	3/5	0.5967	0.5872	0.5698

If the odd-man appears, and has some power of arbitration the committee stands at disadvantage, in the sense that its gain $u_\infty(p) - \frac{1}{2}$ decreases as $p \in \left[0, \frac{1}{2}\right]$ increases. The committee gets less, as odd-man's power of arbitration becomes stronger.

3 Remarks.

Remark 1. The 2-member committee related to our problem stated in Section 1 is discussed in Ref.[1, 4, 5, 6]. Player I and II observe $(X_j, Y_j), j = 1, \cdots, n$, *i.i.d.*, with $U_{[0,1]\times[0,1]}$ distribution. I (II) has priority $p(\overline{p})$, where $p \in \left[\frac{1}{2}, 1\right]$. The case $p = \frac{1}{2}(1)$ means equal-priority (I's dictatorship). The Optimality Equation is

$$(u_n, v_n) = E_{x,y}\,[\text{eq.val. } \mathbf{M}_n(x,y)]$$

$$\mathbf{M}_n(x,y) = \begin{array}{c} \\ R \\ A \end{array} \begin{array}{c} \begin{array}{cc} R & A \end{array} \\ \left[\begin{array}{cc} u_{n-1}, \quad v_{n-1} & \overline{p}(x,y) + p(u_{n-1}, v_{n-1}) \\ p(x,y) + \overline{p}(u_{n-1}, v_{n-1}) & x, \quad y \end{array}\right] \end{array},$$

$$\left(n \geq 1, u_1 = v_1 = \frac{1}{2}\right).$$

It is proven that the eq.strategies in state (n, x, y) are :
"Choose A (R), if $x > (<) u_{n-1}$ " for I.
"Choose A (R), if $y > (<) v_{n-1}$ " for II.
where

$$u_n = \frac{1}{2}\left\{p u_{n-1}^2 + \overline{p}(2u_{n-1} - 1)v_{n-1} + 1\right\}, \quad v_n = \frac{1}{2}\left\{\overline{p} v_{n-1}^2 + p(2v_{n-1} - 1)u_{n-1} + 1\right\},$$

and that $u_n \uparrow u_\infty (= u, \text{ say}), v_n \uparrow v_\infty (= v, \text{ say})$, and (u, v) is a unique root in $(1/2, 1)^2$ of

$$u = \frac{\sqrt{1 - \overline{p}v}}{\sqrt{1 - \overline{p}v} + \sqrt{\overline{p}\ \overline{v}}}, v = \frac{\sqrt{1 - pu}}{\sqrt{1 - pu} + \sqrt{p\overline{u}}}.$$

Computation gives

$p = 0.5$	0.6	0.8	1.0
$u = 2/3$	0.6946	0.7663	1
$v = 2/3$	0.6408	0.5899	1/2

It is interesting and reasonable to find that, in the equal-priority case, the optimal payoff in three-player game 3/5 is less than 2/3 in two-player game.

Some other approaches to the 3-member committee are found in Ref.[2, 3, 7, 8, 9].

Remark 2. We give some interesting open problems around the field of 3-member committee. ① If $X_j(Y_j, Z_j)$ is the ability of management (foreign language, computer technic) of the j-th applicant, then these three r.v.s are not independent. ② The case where each committee member wants $X_j I(X_j \geq a), Y_j I(Y_j \geq b), Z_j I(Z_j \geq c) \to \max$, where $1 > a \geq b \geq c > 0$. The three r.v.s are independent with $U_{[a,1]\times[b,1]\times[c,1]}$ distribution. ③ A fair division of a r.v. $X_j \sim U_{[0,1]}$. If the committee members make different choices, the member(s) who chooses A drops out from the game getting his fair share and the remaining one-or-two members continue the corresponding one-or-two player game thereafter, by facing a new r.v. $Y_{j+1} \sim U_{[0,1]}$. The odd-man, if it appears, has his priority $p \in [0, 1/2]$. The Opt.Eq. will be, instead of (1)∼(4),

$$(1') \qquad (u_n, u_n, u_n) = E_x[\text{EQV of } \mathbf{M}_n(x)], \quad \left(n \geq 1, u_1 = \frac{1}{3}EX = \frac{1}{6}\right)$$

(2′)
$$\mathbf{M}_n(x) \begin{cases} \xrightarrow{\text{R by I}} \mathbf{M}_{nR}(x) \\ \xrightarrow{\text{A by I}} \mathbf{M}_{nA}(x) \end{cases}$$

(3′) $\mathbf{M}_{n,R}(x) =$

	III's R			III's A		
II's R	$u,$	$u,$	u	$\overline{p}U,$	$\overline{p}U,$	px
II's A	$\overline{p}U,$	$px,$	$\overline{p}U$	$pG,$	$(\overline{p}/2)x,$	$(\overline{p}/2)x$

(4′) $\mathbf{M}_{n,A}(x) =$

$px,$	$\overline{p}U,$	$\overline{p}U$	$(\overline{p}/2)x,$	$pG,$	$(\overline{p}/2)x$
$(\overline{p}/2)x,$	$(\overline{p}/2)x,$	pG	$x/3,$	$x/3,$	$x/3$

where, in each cell, the subscripts $n-1$ is omitted from $u_{n-1}, U_{n-1}, G_{n-1}$. U_n is the common EQV of the two-player n-stage game, and G_n is the optimal value of the 1-player n-stage game. $\{G_n\}$ satisfies $G_n = \frac{1}{2}\left(1 + G_{n-1}^2\right)$ $(n \geq 1, G_0 = 0)$, *i.e.*, Moser's sequence. $\{U_n\}$ satisfies the Opt.Eq.

$$(U_n, U_n) = E_y \left[\text{eq.val.} \left\{ \begin{array}{c|cc|cc|} & \text{R} & & \text{A} & \\ \hline \text{R} & U_{n-1}, & U_{n-1} & G_{n-1}, & y \\ \hline \text{A} & y, & G_{n-1} & y/2, & y/2 \\ \hline \end{array} \right\} \right], \quad (n \geq 1, U_0 = G_0 = 0)$$

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