THE BEST CONSTANT OF SOBOLEV INEQUALITY WHICH CORRESPONDS TO A BENDING PROBLEM OF A STRING WITH PERIODIC BOUNDARY CONDITION

Yoshinori Kametaka[*], Kohtaro Watanabe[†], Atsushi Nagai[‡]

Hiroyuki Yamagishi[§], Kazuo Takemura[¶]

Received December 15, 2006

Abstract. The Green function of periodic boundary value problem with supplementary orthogonality conditions for bending of a string is obtained. The best constant of corresponding Sobolev inequality is found.

0 Preparation We first prepare some notations used throughout this paper. The eigenvalue problem

$$\begin{cases} -u'' = \lambda u \qquad (0 < x < 1) & (0.1) \\ u^{(i)}(1) - u^{(i)}(0) = 0 \qquad (i = 0, 1) & (0.2) \end{cases}$$

has countably many eigenvalues

$$\lambda_j = a_j^2 \qquad (j = 0, 1, 2, \cdots) \tag{0.3}$$

where we put $a_j = 2\pi j \quad (j \in \mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\})$. The eigenspace corresponding to eigenvalue $\lambda_0 = 0$ is one-dimensional and

$$\varphi(0, x) = 1 \qquad (0 < x < 1) \tag{0.4}$$

is an eigenfunction. For $j = 1, 2, 3, \cdots$, the eigenspace corresponding to eigenvalue λ_j is two-dimensional. We choose

$$\varphi(\pm j, x) = \exp\big(\sqrt{-1}\, a_{\pm j}\, x\big) \qquad (0 < x < 1) \tag{0.5}$$

as base of this eigenspace. As is well known, the system of eigenfunctions $\big\{\varphi(j, x) \,\big|\, j \in \mathbf{Z}\big\}$ is a C.O.N.S. in $L^2(0, 1)$.

Every function $u(x) \in L^2(0, 1)$ can be expanded as follows.

$$u(x) = \sum_{j \in \mathbf{Z}} \widehat{u}(j)\, \varphi(j, x) \tag{0.6}$$

where

$$\widehat{u}(j) = \int_0^1 u(y)\, \overline{\varphi}(j, y)\, dy \qquad (j \in \mathbf{Z}) \tag{0.7}$$

2000 *Mathematics Subject Classification.* 46E35, 41A44, 34B27.

Key words and phrases. best constant, Sobolev inequality, Green function, Reproducing kernel .

[*]He has retired at March 2004, and now he is an emeritus professor of Osaka University.

For every pair of functions $u(x), v(x) \in L^2(0,1)$, we have Parseval equality

$$\int_0^1 u(x)\,\overline{v}(x)\,dx \;=\; \sum_{j\in\mathbf{Z}} \widehat{u}(j)\,\overline{\widehat{v}}(j) \tag{0.8}$$

Moreover if $u'(x) \in L^2(0,1)$ and $u(1)-u(0)=0$ hold then we have

$$u'(x) \;=\; \sum_{j\in\mathbf{Z}} \sqrt{-1}\,a_j\,\widehat{u}(j)\,\varphi(j,x) \tag{0.9}$$

and

$$\int_0^1 |\,u'(x)\,|^2\;dx \;=\; \sum_{j\in\mathbf{Z}} a_j^2\;|\,\widehat{u}(j)\,|^2 \tag{0.10}$$

1 Conclusion For any bounded continuous function $f(x)$ defined on an interval $0 < x < 1$ which satisfies suitable solvability condition (S), we consider the following boundary value problem

BVP

$$\begin{cases} -\,u'' \,+\, q\,u \;=\; f(x) \qquad (0<x<1) & (1.1)\\ u^{(i)}(1) \,-\, u^{(i)}(0) \;=\; 0 \qquad (i=0,1) & (1.2)\\ \text{orthogonality condition (O) for } u(x) & (1.3)\end{cases}$$

q is a real parameter. Conditions (S) and (O) depend on the value of q and are shown later in Theorem 1.1.

In case of $q > 0$, this boundary value problem has the following meaning. A straight string is supported by the elastic membrane with uniformaly distributed spring with spring constant q. One of the end of this membrane is fixed on the horizontal ceiling. $f(x)$ is a density of a load, $u(x)$ is a bending of this string.

We consider this problem in the following four cases, which cover all the real values of q.

Case I $\quad q \,=\, a^2, \qquad a > 0$

Case II $\quad q \,=\, -\,a^2 \qquad (\,a_N < a < a_{N+1}\,)$

where one of the $N = 0,1,2,\cdots$ is fixed

Case III $\quad q \,=\, 0$

Case IV $\quad q \,=\, -\,a_N^2 \qquad$ where one of the $N = 1,2,3,\cdots$ is fixed

The first conclusion is as follows.

Theorem 1.1 *For any bounded continuous function $f(x)$ defined on an interval $0 < x < 1$ which satisfies condition* (S)*,* BVP *has one and only one classical solution $u(x)$ given by*

$$u(x) \;=\; \int_0^1 G(x,y)\,f(y)\,dy \qquad (0<x<1) \tag{1.4}$$

using Green function $G(x,y) = G(x-y)$.

(I) (S),(O) *nothing.*

$$G(x) = \sum_{j\in\mathbf{Z}} (a_j^2 + a^2)^{-1} \varphi(j,x) = \cosh(a(|x|-1/2)) \Big/ \big(2a \sinh(a/2) \big) \tag{1.5}$$

(II) (S) $$\int_0^1 f(y)\,\overline{\varphi}(j,y)\,dy = 0 \qquad (|j| \le N) \tag{1.6}$$

(O) $$\int_0^1 u(x)\,\overline{\varphi}(j,x)\,dx = 0 \qquad (|j| \le N) \tag{1.7}$$

$$G(x) = \sum_{|j|\ge N+1} (a_j^2 - a^2)^{-1} \varphi(j,x) =$$

$$\sum_{|j|\le N} (a^2 - a_j^2)^{-1} \varphi(j,x) - \cos(a(|x|-1/2)) \Big/ \big(2a \sin(a/2) \big) \tag{1.8}$$

(III) (S) $$\int_0^1 f(y)\,dy = 0 \tag{1.9}$$

(O) $$\int_0^1 u(x)\,dx = 0 \tag{1.10}$$

$$G(x) = \sum_{\substack{j\in\mathbf{Z}\\ j\neq 0}} a_j^{-2} \varphi(j,x) = \frac{1}{2}|x|^2 - \frac{1}{2}|x| + \frac{1}{12} = b_2(|x|) \tag{1.11}$$

where $b_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ *is a Bernoulli polynomial of second order.*

(IV) (S) $$\int_0^1 f(y)\,\overline{\varphi}(j,y)\,dy = 0 \qquad (|j| \le N) \tag{1.12}$$

(O) $$\int_0^1 u(x)\,\overline{\varphi}(j,x)\,dx = 0 \qquad (|j| \le N) \tag{1.13}$$

$$G(x) = \sum_{|j|\ge N+1} (a_j^2 - a_N^2)^{-1} \varphi(j,x) = \sum_{|j|\le N-1} (a_N^2 - a_j^2)^{-1} \varphi(j,x) +$$

$$\frac{(-1)^N}{2a_N^2} \Big[\cos(a_N(|x|-1/2)) + 2a_N (|x|-1/2) \sin(a_N(|x|-1/2)) \Big] \tag{1.14}$$

Now we consider the following function space.

$$H = \Big\{ u(x) \,\Big|\, u(x),\ u'(x) \in L^2(0,1),\ u(1) - u(0) = 0,$$
$$\text{orthogonality condition (O) for } u(x) \Big\} \tag{1.15}$$

The condition (O) is the same as that in Theorem 1.1. For any pair of functions $u(x)$, $v(x)$ in H, we attach the following sesquilinear form.

$$(u,v)_H \;=\; \int_0^1 \left[u'(x)\,\overline{v}'(x) \;+\; q\,u(x)\,\overline{v}(x) \right] dx \tag{1.16}$$

It is necessary to show that the above sesquilinear form is an inner product. We show this fact in the cases II, III, IV ($q \le 0$) because it is obvious in the case I ($q > 0$). In these cases, we have

$$q = -\,a^2 \qquad (\,a_N \le a < a_{N+1}\,) \tag{1.17}$$

for some $N = 0, 1, 2, \cdots$. Using Parseval equality and the orthogonality condition (O) we have

$$(u,u)_H \;=\; \int_0^1 \left[\,|\,u'(x)\,|^2 \;-\; a^2\,|\,u(x)\,|^2 \right] dx \;=\; \sum_{|j|\ge N+1} (a_j^2 \;-\; a^2)\;|\,\widehat{u}(j)\,|^2 \;\ge$$
$$(a_{N+1}^2 - a_N^2) \sum_{|j|\ge N+1} |\,\widehat{u}(j)\,|^2 \;=\; (a_{N+1}^2 - a_N^2) \int_0^1 |\,u(x)\,|^2\,dx$$

This inequality shows that $(\cdot,\cdot)_H$ is an inner product. H is a Hilbert space with inner product $(\cdot,\cdot)_H$.

The second main theorem is as follows.

Theorem 1.2 *If $u(x), u'(x) \in L^2(0,1)$ then we have*

$$\int_0^1 \left[u'(x)\,\partial_x \;+\; q\,u(x) \right] G(x,y)\,dx \;=\; u(y) \;+\; \Big(u(1) - u(0) \Big) \partial_x G(x,y) \Big|_{x=0} \;-$$
$$\begin{cases} 0 & (\,\mathrm{I}\,) \\ \displaystyle\sum_{|j|\le N} \varphi(j,y) \int_0^1 u(x)\,\overline{\varphi}(j,x)\,dx & (\,\mathrm{II},\;\mathrm{III},\;\mathrm{IV}\,) \end{cases}$$
$$(0 < y < 1) \tag{1.18}$$

Especially if $u(x) \in H$ then we have

$$\int_0^1 \left[u'(x)\,\partial_x \;+\; q\,u(x) \right] G(x,y)\,dx \;=\; u(y) \qquad (0 < y < 1) \tag{1.19}$$

This means that Green function $G(x,y)$ is a reproducing kernel for H and $(\cdot,\cdot)_H$.

Finally we define Sobolev functional

$$S(u) \;=\; \left(\sup_{0\le y\le 1} |u(y)| \right)^2 \Big/ \int_0^1 \left[|u'(x)|^2 \;+\; q\;|u(x)|^2 \right] dx \tag{1.20}$$

for $u \in H$ with $u \not\equiv 0$.

The most important conclusion of this paper is as follows.

Theorem 1.3

(1) $$\sup_{u\in H,\; u\not\equiv 0} S(u) \;=\; C_0 \tag{1.21}$$

$$C_0 = G(0) = \begin{cases} \displaystyle\sum_{j\in\mathbf{Z}} (\, a_j^2 + a^2\,)^{-1} \,=\, 1 \big/ \big(\, 2a \,\tanh(a/2)\,\big) & \text{(I)} \\ \displaystyle\sum_{|j|\geq N+1} \big(\, a_j^2 - a^2\,\big)^{-1} = \sum_{|j|\leq N} (\, a^2 - a_j^2\,)^{-1} \,-\, 1 \big/ \big(\, 2a \,\tan(a/2)\,\big) & \text{(II)} \\ \displaystyle\sum_{j\in\mathbf{Z},\ j\neq 0} a_j^{-2} \,=\, 1 \big/ 12 & \text{(III)} \\ \displaystyle\sum_{|j|\geq N+1} \big(\, a_j^2 - a^2\,\big)^{-1} \,=\, 3 \big/ (2a^2) \,+\, 2\sum_{j=1}^{N-1} \big(\, a^2 - a_j^2\,\big)^{-1} & \text{(IV)} \end{cases} \tag{1.22}$$

For any fixed y such that $0 \leq y \leq 1$, if we define $u(x) = const.\, G(x-y)$ then we have $S(u) = C_0$.

(2) $$\inf_{u\in H,\ u\not\equiv 0} S(u) \,=\, 0 \tag{1.23}$$

The engineering meaning of this result is that the square of the maximal bending of a string is estimated from above by the constant multiple of the potential energy of a string. The best constant is the diagonal value of corresponding Green function (impulse response). The best constant depends only on the spring constant q.

The main part (1) of this theorem is proved later in section 6, but the second part (2) is easy to prove. In fact, if $|j|$ is sufficiently large then we have $\varphi(j,x) \in H$ and we have

$$S(\varphi(j,x)) \,=\, \frac{1}{a_j^2 + q} \tag{1.24}$$

It is easy to confirm that

$$\lim_{|j|\to\infty} S(\varphi(j,x)) \,=\, 0$$

2 Solvability and orthogonality conditions In the following sections, we prove our main theorems. We omit the proof in case I because it is easy. Hereafter we fix $N = 0, 1, 2, \cdots$ and assume

$$q \,=\, -a^2 \qquad (a_N \leq a < a_{N+1}) \tag{2.1}$$

From the theory of Fourier series, the solvability condition (S) is equivalent to

$$\widehat{f}(j) \,=\, 0 \qquad (\,|j| \leq N\,) \tag{2.2}$$

BVP is equivalent to

$$\begin{cases} (a_j^2 \,-\, a^2)\,\widehat{u}(j) \,=\, \widehat{f}(j) \qquad (\,|j| \geq N+1\,) & (2.3) \\ \widehat{u}(j) \,=\, 0 \qquad (\,|j| \leq N\,) & (2.4) \end{cases}$$

So we can obtain easily Fourier series expansion of Green function

$$G(x) \,=\, \sum_{|j|\geq N+1} \frac{1}{a_j^2 - a^2}\, \varphi(j,x) \tag{2.5}$$

This formula is valid also in the case of $a = a_N$.

First of all we treat the problem of the uniqueness of a solution to BVP in cases of III and IV without using the theory of Fourier series. We have the following conclusion.

Theorem 2.1 *We assume*

$$a = a_N \qquad (N = 0, 1, 2, \cdots) \tag{2.6}$$

For a bounded continuous function $f(x)$ defined on an interval $0 < x < 1$, we assume that the following boundary value problem

BVP′

$$\begin{cases} -u'' - a^2 u = f(x) \qquad (0 < x < 1) & (2.7) \\ u^{(i)}(1) - u^{(i)}(0) = 0 \qquad (i = 0, 1) & (2.8) \\ (\mathrm{O}') \quad \widehat{u}(j) = \displaystyle\int_0^1 u(x)\,\overline{\varphi}(j, x)\,dx = 0 \qquad (|j| \le N-1) & \\ \textit{in case of } N = 0,\ (\mathrm{O}') \textit{ nothing} & (2.9) \end{cases}$$

has a classical solution $u(x)$ then the following solvability conditions hold

$$\int_0^1 f(y)\,\overline{\psi}(j, y)\,dy = 0 \quad (|j| \le N-1), \qquad \int_0^1 f(y)\,\overline{\varphi}(\pm N, y)\,dy = 0 \tag{2.10}$$

where

$$\psi(j, x) = \int_0^1 g(x-y)\,\varphi(j, y)\,dy \tag{2.11}$$

The solution $u(x)$ is expressed as follows.

$$u(x) = \alpha_+\,\varphi(N, x) + \alpha_-\,\varphi(-N, x) + \int_0^1 g(x-y)\,f(y)\,dy \qquad (0 < x < 1) \tag{2.12}$$

where

$$g(x) = \begin{cases} -\dfrac{1}{2a}\sin(\,a|x|\,) & (N = 1, 2, 3, \cdots) \\ -\dfrac{1}{2}\,|x| & (N = 0) \end{cases} \tag{2.13}$$

and $\alpha_\pm$ are suitable constants. We call $g(x, y) = g(x-y)$ the proto Green function.

Proof of Theorem 2.1 We omit the proof in the case $N = 0$. We assume

$$a = a_N = 2\pi N \qquad (N = 1, 2, 3, \cdots) \tag{2.14}$$

Introducing new functions

$$u_0 = u, \qquad u_1 = u'$$

We have

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x) \qquad (0 < x < 1) \tag{2.15}$$

Employing the fundamental solution

$$\boldsymbol{E}(x) \;=\; \begin{pmatrix} \cos(ax) & a^{-1}\sin(ax) \\ -a\,\sin(ax) & \cos(ax) \end{pmatrix} \tag{2.16}$$

(2.15) is solved as follows.

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(x) \;=\; \boldsymbol{E}(x)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(0) \;-\; \int_0^x \boldsymbol{E}(x-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} f(y)\,dy \;=$$

$$\boldsymbol{E}(x-1)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(1) \;+\; \int_x^1 \boldsymbol{E}(x-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} f(y)\,dy$$

Taking the average of right hand sides, we have

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(x) \;=\; \frac{1}{2}\,\boldsymbol{E}(x)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(0) \;+\; \frac{1}{2}\,\boldsymbol{E}(x-1)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(1) +$$

$$\int_0^1 -\frac{1}{2}\,\mathrm{sgn}(x-y)\,\boldsymbol{E}(x-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} f(y)\,dy$$

For suitable constant α_0, α_1

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(x) \;=\; \boldsymbol{E}(x)\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \;+\; \int_0^1 -\frac{1}{2}\,\mathrm{sgn}(x-y)\,\boldsymbol{E}(x-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} f(y)\,dy$$

$$(0 < x < 1) \tag{2.17}$$

Now we introduce

$$\begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \end{pmatrix} \;=\; \int_0^1 \frac{1}{2}\,\boldsymbol{E}(-y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} f(y)\,dy \;=\; \int_0^1 \frac{1}{2}\begin{pmatrix} -\,a^{-1}\sin(ay) \\ \cos(ay) \end{pmatrix} f(y)\,dy \tag{2.18}$$

Since $\boldsymbol{E}(1) = \boldsymbol{E}(0)$ then we have

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(0) \;=\; \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \;+\; \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \end{pmatrix} \tag{2.19}$$

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(1) \;=\; \boldsymbol{E}(1)\left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} - \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \end{pmatrix} \right\} \;=\; \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} - \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \end{pmatrix} \tag{2.20}$$

From boundary conditions $u_i(0) = u_i(1)$ $(i = 0, 1)$, we have $\varepsilon_0 = \varepsilon_1 = 0$, or equivalently

$$\widehat{f}(\pm N) \;=\; \int_0^1 f(y)\,\overline{\varphi}(\pm N, y)\,dy \;=\; 0 \tag{2.21}$$

From (2.17), we have

$$u_0(x) \;=\; \alpha_0 \,\cos(ax) \,+\, \alpha_1\, a^{-1} \sin(ax) \,-\, \int_0^1 \frac{1}{2a}\, \sin(\, a|x-y|\,)\, f(y)\, dy \tag{2.22}$$

This proves (2.12).

The remaining relations

$$\int_0^1 f(y)\, \overline{\psi}(j,y)\, dy \;=\; 0 \qquad (|j| \le N-1) \tag{2.23}$$

are proved later in section 3. Thus we have proved Theorem 2.1. ■

3 Proto Green function In this section we list up properties of the proto Green function $g(x,y) = g(x-y)$ which was introduced in the previous section .

Lemma 3.1 *We assume*

$$a = a_N \qquad (N = 1, 2, 3, \cdots) \tag{3.1}$$

The proto Green function

$$g(x,y) \;=\; g(x-y) \;=\; -\,\frac{1}{2a}\, \sin(\, a|x-y|\,) \tag{3.2}$$

satisfies the following properties

(1) $$g(x,y) \;=\; g(y,x) \qquad (0<x,y<1) \tag{3.3}$$

(2) $$(\,-\,\partial_x^2 \,-\, a^2\,)\, g(x,y) \;=\; 0 \qquad (0<x,y<1,\;\; x \neq y) \tag{3.4}$$

(3) $$\begin{cases} g(0,y) \;=\; -\,g(1,y) \;=\; -\,\dfrac{1}{2a}\, \sin(ay) \;=\; -\,\dfrac{1}{\sqrt{-1}\,4a} \Big(\varphi(N,y) \,-\, \varphi(-N,y) \Big) \\[2ex] \partial_x g(x,y)\Big|_{x=0} \;=\; -\,\partial_x g(x,y)\Big|_{x=1} \;=\; \dfrac{1}{2}\, \cos(ay) \;=\; \dfrac{1}{4} \Big(\varphi(N,y) \,+\, \varphi(-N,y) \Big) \end{cases}$$
$$(0<y<1) \tag{3.5}$$

(4) $$\begin{cases} g(x,y)\Big|_{y=x-0} \,-\, g(x,y)\Big|_{y=x+0} \;=\; 0 \\[2ex] \partial_x g(x,y)\Big|_{y=x-0} \,-\, \partial_x g(x,y)\Big|_{y=x+0} \;=\; -\,1 \qquad (0<x<1) \end{cases} \tag{3.6}$$

(5) $$\begin{cases} g(x,y)\Big|_{x=y-0} \,-\, g(x,y)\Big|_{x=y+0} \;=\; 0 \\[2ex] \partial_x g(x,y)\Big|_{x=y-0} \,-\, \partial_x g(x,y)\Big|_{x=y+0} \;=\; 1 \qquad (0<y<1) \end{cases} \tag{3.7}$$

Lemma 3.2 *For any bounded continuous function $f(x)$ on an interval $0 < x < 1$*

$$u(x) \,=\, \int_0^1 g(x,y)\, f(y)\, dy \,=\, \int_0^1 g(x-y)\, f(y)\, dy \tag{3.8}$$

satisfies

$$\begin{cases} -\,u'' \,-\, a^2 u \,=\, f(x) \qquad (0 < x < 1) & (3.9) \\[2mm] u(0) \,=\, -\,u(1) \,=\, -\,\dfrac{1}{\sqrt{-1}\,4a} \displaystyle\int_0^1 \Big(\varphi(N,y) \,-\, \varphi(-N,y)\Big) f(y)\, dy & (3.10) \\[2mm] u'(0) \,=\, -\,u'(1) \,=\, \dfrac{1}{4} \displaystyle\int_0^1 \Big(\varphi(N,y) \,+\, \varphi(-N,y)\Big) f(y)\, dy & (3.11) \end{cases}$$

Concerning $\psi(j,x)$ which was introduced in Theorem 2.1, we have

Lemma 3.3

$$\psi(j,x) \,=\, \int_0^1 g(x,y)\, \varphi(j,y)\, dy \qquad (|j| \le N,\ 0 < x < 1) \tag{3.12}$$

satisfies the following properties

(1) $$(-\,\partial_x^2 \,-\, a^2)\, \psi(j,x) \,=\, \varphi(j,x) \qquad (0 < x < 1,\ |j| \le N) \tag{3.13}$$

(2) $$\psi(j,0) \,=\, -\,\psi(j,1) \,=\, \begin{cases} 0 & (|j| \le N-1) \\ \pm\,\dfrac{1}{\sqrt{-1}\,4a} & (j = \pm N) \end{cases} \tag{3.14}$$

(3) $$\psi'(j,0) \,=\, -\,\psi'(j,1) \,=\, \begin{cases} 0 & (|j| \le N-1) \\ \dfrac{1}{4} & (j = \pm N) \end{cases} \tag{3.15}$$

Lemma 3.4 *On an interval $0 < x < 1$, $\psi(j,x)$ $(|j| \le N)$ can be expressed as follows.*

(1) *For* $|j| \le N-1$ *we have*

$$\psi(j,x) \,=\, \alpha_j\, \varphi(N,x) \,+\, \beta_j\, \varphi(-N,x) \,-\, \gamma_j\, \varphi(j,x) \tag{3.16}$$

$$\textit{where}\quad \alpha_j = \frac{1}{2a\,(a-a_j)}, \quad \beta_j = \frac{1}{2a\,(a+a_j)}, \quad \gamma_j = \frac{1}{a^2 - a_j^2} \tag{3.17}$$

(2) *For* $j = \pm N$ *we have*

$$\psi(N,x) \,=\, -\,\frac{1}{4a^2}\Big(\varphi(N,x) \,-\, \varphi(-N,x)\Big) \,+\, \frac{\sqrt{-1}}{4a}\,(2x-1)\,\varphi(N,x) \tag{3.18}$$

$$\psi(-N,x) \,=\, \frac{1}{4a^2}\Big(\varphi(N,x) \,-\, \varphi(-N,x)\Big) \,-\, \frac{\sqrt{-1}}{4a}\,(2x-1)\,\varphi(-N,x) \tag{3.19}$$

Proof of Lemma 3.4 We treat the case (1). Since

$$\left(-\,\partial_x^2\,-\,a^2\right)\,(\,\psi(j,x)\,+\,\gamma_j\,\varphi(j,x)\,)\,=\,0 \tag{3.20}$$

then we have

$$\psi(j,x)\,=\,\alpha_+\,\varphi(N,x)\,+\,\alpha_-\,\varphi(-N,x)\,-\,\gamma_j\,\varphi(j,x) \tag{3.21}$$

for suitable constants $\alpha_\pm$. From (3.14) and (3.15) we have

$$\begin{cases} 0\,=\,\psi(j,0)\,=\,\alpha_+\,\varphi(N,0)\,+\,\alpha_-\,\varphi(-N,0)\,-\,\gamma_j\,\varphi(j,0)\,=\,\alpha_+\,+\,\alpha_-\,-\,\gamma_j \\ 0\,=\,\psi'(j,0)\,=\,\alpha_+\,\varphi'(N,0)\,+\,\alpha_-\,\varphi'(-N,0)\,-\,\gamma_j\,\varphi'(j,0)\,= \\ \sqrt{-1}\,a\,(\alpha_+\,-\,\alpha_-)\,-\,\sqrt{-1}\,a_j\,\gamma_j \end{cases}$$

Solving the above set of equations with respect to $\alpha_\pm$, we have

$$\alpha_+\,=\,\alpha_j, \qquad \alpha_-\,=\,\beta_j \tag{3.22}$$

This shows (1).

Next we treat the case (2). Since

$$\left(-\,\partial_x^2\,-\,a^2\right)\left(x\,\varphi(N,x)\right)\,=\,-\,\sqrt{-1}\,2a\,\varphi(N,x) \tag{3.23}$$

then we have

$$\psi(N,x)\,=\,\alpha_+\,\varphi(N,x)\,+\,\alpha_-\,\varphi(-N,x)\,+\,\frac{\sqrt{-1}}{2a}\,x\,\varphi(N,x) \tag{3.24}$$

for suitable constants $\alpha_\pm$. From (3.14) and (3.15) with $j=N$ we have

$$\begin{cases} -\,\dfrac{\sqrt{-1}}{4a}\,=\,\psi(N,0)\,=\,\alpha_+\,\varphi(N,0)\,+\,\alpha_-\,\varphi(-N,0)\,=\,\alpha_+\,+\,\alpha_- \\ \dfrac{1}{4}\,=\,\psi'(N,0)\,=\,\alpha_+\,\varphi'(N,0)\,+\,\alpha_-\,\varphi'(-N,0)\,+\,\dfrac{\sqrt{-1}}{2a}\,\varphi(N,0)\,= \\ \sqrt{-1}\,a\,(\alpha_+\,-\,\alpha_-)\,+\,\dfrac{\sqrt{-1}}{2a} \end{cases}$$

Hence we have

$$\alpha_+\,=\,-\,\frac{1}{4a^2}\,-\,\frac{\sqrt{-1}}{4a}, \qquad \alpha_-\,=\,\frac{1}{4a^2} \tag{3.25}$$

(3.18) is shown. Taking the complex conjugate of both sides of (3.18), we obtain (3.19). This shows (2). ■

Remark Lemma 3.4 shows that the solvability condition (2.10) is equivalent to

$$\int_0^1 f(y)\,\overline{\varphi}(j,y)\,dy\,=\,0 \qquad (|\,j\,|\le N) \tag{3.26}$$

4 Derivation of Green function $G(x,y)$ We first consider the case II, that is to say

$$q = -\,a^2 \qquad (a_N < a < a_{N+1}) \tag{4.1}$$

where $N = 0, 1, 2, \cdots$ is fixed. Green function $G(q; x, y) = G(q; x-y)$ is given by

$$G(q;x) \;=\; \sum_{|j|\geq N+1} \frac{1}{a_j^2 - a^2}\,\varphi(j,x) \tag{4.2}$$

On the other hand the formula

$$-\,\frac{1}{2a\,\sin(a/2)}\,\cos(\,a(|x|-1/2)\,) \;=\; \sum_{j\in\mathbf{Z}} \frac{1}{a_j^2 - a^2}\,\varphi(j,x) \tag{4.3}$$

is well known. This is a Green function of BVP without solvability condition (S) and orthogonality condition (O).

We next consider the case $a = a_N$. Green function is obtained by putting $q = -a_N^2$ in (4.2) and is given by

$$G(-a_N^2;x) \;=\; \sum_{|j|\geq N+1} \frac{1}{a_j^2 - a_N^2}\,\varphi(j,x) \;=\; 2\sum_{j=N+1}^{\infty} \frac{1}{a_j^2 - a_N^2}\,\cos(a_j x) \tag{4.4}$$

On the other hand this is also obtained by taking the following limit.

$$G(-a_N^2;x) \;=\; \lim_{a\to a_N+0} \sum_{|j|\geq N+1} \frac{1}{a_j^2 - a^2}\,\varphi(j,x) \;=$$

$$\lim_{a\to a_N+0}\left[-\,\frac{\cos(\,a(|x|-1/2)\,)}{2a\,\sin(a/2)} \;-\; \sum_{|j|\leq N} \frac{1}{a_j^2 - a^2}\,\varphi(j,x)\right] \;=$$

$$\lim_{a\to a_N+0}\left[\frac{1}{a^2} \;+\; 2\sum_{j=1}^{N} \frac{1}{a^2 - a_j^2}\,\cos(a_j x) \;-\; \frac{\cos(\,a(|x|-1/2)\,)}{2a\,\sin(a/2)}\right] \tag{4.5}$$

Proof of Theorem 1.1(III.2) From the above relation (4.5) we have

$$G(0;x) \;=\; \lim_{a\to +0}\left[-\,\frac{\cos(\,a(|x|-1/2)\,)}{2a\,\sin(a/2)} \;+\; \frac{1}{a^2}\right] \;=$$

$$\lim_{a\to +0} \frac{2\,\sin(a/2) \;-\; a\,\cos(\,a(|x|-1/2)\,)}{2a^2\,\sin(a/2)} \tag{4.6}$$

The denominator and numerator of (4.6) are expanded around $a = 0$ as

$$2\,\sin(a/2) \;-\; a\,\cos(\,a(|x|-1/2)\,) \;=\; \left[\frac{1}{2}\,|x|^2 \;-\; \frac{1}{2}\,|x| \;+\; \frac{1}{12}\right] a^3 \;+\; O(a^5)$$

$$2a^2\,\sin(a/2) \;=\; a^3 \;+\; O(a^5)$$

Therefore we obtain

$$G(0;x) \;=\; \frac{1}{2}\,|x|^2 \;-\; \frac{1}{2}\,|x| \;+\; \frac{1}{12} \;=\; b_2(\,|x|\,) \tag{4.7}$$

This completes the proof of Theorem 1.1(III.2). ■

It is interesting to note that the above polynomial

$$b_2(x) \;=\; \frac{1}{2}\,x^2 \;-\; \frac{1}{2}\,x \;+\; \frac{1}{12}$$

is nothing but the Bernoulli polynomial of second order.

Proof of Theorem 1.1(IV.2) Now we calculate the last term of (4.5) for $N = 1, 2, 3, \cdots$. It is obvious that

$$\sum_{|j| \le N-1} \frac{1}{a^2 - a_j^2}\,\varphi(j,x) \;=\; \frac{1}{a^2} \;+\; 2\sum_{j=1}^{N-1} \frac{1}{a^2 - a_j^2}\,\cos(a_j x) \;\underset{a \to a_N+0}{\longrightarrow}$$
$$\frac{1}{a_N^2} \;+\; 2\sum_{j=1}^{N-1} \frac{1}{a_N^2 - a_j^2}\,\cos(a_j x) \tag{4.8}$$

It is enough to show

$$\frac{2\,\cos(a_N x)}{a^2 - a_N^2} \;-\; \frac{\cos(\,a(|x| - 1/2)\,)}{2a\,\sin(a/2)} \;\underset{a \to a_N+0}{\longrightarrow}$$
$$\frac{1}{2a_N^2}\,\cos(a_N x) \;+\; \frac{1}{a_N}\,(|x| - 1/2)\,\sin(a_N|x|) \tag{4.9}$$

Since $a_N = 2\pi N$ we have

$$\sin(a/2) \;=\; (-1)^N \sin(\,(a - a_N)/2\,)$$

$$\cos(\,a_N(|x| - 1/2)\,) \;=\; (-1)^N \cos(a_N x)$$

then we have

$$\frac{2}{a^2 - a_N^2}\,\cos(a_N x) \;-\; \frac{\cos(\,a(|x| - 1/2)\,)}{2a\,\sin(a/2)} \;=$$

$$\frac{2}{a^2 - a_N^2}\,\cos(a_N x) \;-$$

$$\frac{1}{2a\,\sin(a/2)}\left(\cos(\,a(|x| - 1/2)\,) - \cos(\,a_N(|x| - 1/2)\,)\right) \;-$$

$$\frac{1}{2a\,\sin(a/2)}\,\cos(\,a_N(|x| - 1/2)\,) \;=$$

$$\left(\frac{2}{a^2 - a_N^2} \;-\; \frac{1}{2a\,\sin(\,(a - a_N)/2\,)}\right)\cos(a_N x) \;-$$

$$\frac{(-1)^N (a - a_N)}{2a\,\sin(\,(a - a_N)/2\,)}\,\frac{\cos(\,a(|x| - 1/2)\,) - \cos(\,a_N(|x| - 1/2)\,)}{a - a_N} \tag{4.10}$$

we have

$$\frac{2}{a^2-a_N^2} - \frac{1}{2a\,\sin(\,(a-a_N)/2\,)} \underset{a\to a_N+0}{\longrightarrow} \frac{1}{2a_N^2} \tag{4.11}$$

$$\frac{a-a_N}{2a\,\sin(\,(a-a_N)/2\,)} \underset{a\to a_N+0}{\longrightarrow} \frac{1}{a_N} \tag{4.12}$$

$$\frac{\cos(\,a(|x|-1/2)\,)-\cos(\,a_N(|x|-1/2)\,)}{a-a_N} \underset{a\to a_N+0}{\longrightarrow}$$

$$-\,(|x|-1/2)\,\sin(\,a_N(|x|-1/2)\,) \tag{4.13}$$

This proves (4.9) and completes the proof of Theorem 1.1(IV.2). ■

5 Properties of Green function In this section, we list properties of Green function $G(x,y)$. Direct calculations show the following lemma.

Lemma 5.1 *The derivatives of Green function with respect to* x *are expressed as follows for* $0<x,y<1\ ,\ \ x\neq y\ .$

(I)

(I.1) $$G(x,y)\,=\,\frac{1}{2a\,\sinh(a/2)}\,\cosh(\,a(|x-y|-1/2)\,) \tag{5.1}$$

(I.2) $$\partial_x G(x,y)\,=\,\frac{\mathrm{sgn}(x-y)}{2\,\sinh(a/2)}\,\sinh(\,a(|x-y|-1/2)\,) \tag{5.2}$$

(I.3) $$\partial_x^2 G(x,y)\,=\,\frac{a}{2\,\sinh(a/2)}\,\cosh(\,a(|x-y|-1/2)\,) \tag{5.3}$$

(II)

(II.1) $$G(x,y)\,=\,-\,\frac{1}{2a\,\sin(a/2)}\,\cos(\,a(|x-y|-1/2)\,)\,+\sum_{|j|\le N}\frac{1}{a^2-a_j^2}\,\varphi(j,x-y) \tag{5.4}$$

(II.2) $$\partial_x G(x,y)\,=\,\frac{\mathrm{sgn}(x-y)}{2\,\sin(a/2)}\,\sin(\,a(|x-y|-1/2)\,)\,+\sum_{|j|\le N}\frac{\sqrt{-1}\,a_j}{a^2-a_j^2}\,\varphi(j,x-y) \tag{5.5}$$

(II.3) $$\partial_x^2 G(x,y)\,=\,\frac{a}{2\,\sin(a/2)}\,\cos(\,a(|x-y|-1/2)\,)\,-\sum_{|j|\le N}\frac{a_j^2}{a^2-a_j^2}\,\varphi(j,x-y) \tag{5.6}$$

(III)

(III.1) $$G(x,y)\,=\,\frac{1}{2}\,|x-y|^2\,-\,\frac{1}{2}\,|x-y|\,+\,\frac{1}{12} \tag{5.7}$$

(III.2) $$\partial_x G(x,y)\,=\,x\,-\,y\,-\,\frac{1}{2}\,\mathrm{sgn}(x-y) \tag{5.8}$$

(III.3) $$\partial_x^2 G(x,y)\,=\,1 \tag{5.9}$$

(IV)

(IV.1) $$G(x,y) = \frac{1}{a_N}\,(\,|x-y|-1/2\,)\,\sin(\,a_N|x-y|\,) + \frac{1}{2a_N^2}\,\cos(a_N(x-y)) + \sum_{|j|\le N-1} \frac{1}{a_N^2-a_j^2}\,\varphi(j,x-y) \tag{5.10}$$

(IV.2) $$\partial_x G(x,y) = \mathrm{sgn}(x-y)\left[(\,|x-y|-1/2\,)\,\cos(\,a_N(x-y)\,) + \frac{1}{a_N}\,\sin(a_N|x-y|)\right] - \frac{1}{2a_N}\,\sin(a_N(x-y)) + \sum_{|j|\le N-1} \frac{\sqrt{-1}\,a_j}{a_N^2-a_j^2}\,\varphi(j,x-y) \tag{5.11}$$

(IV.3) $$\partial_x^2 G(x,y) = -\,a_N\,(\,|x-y|-1/2\,)\,\sin(\,a_N|x-y|\,) + \frac{3}{2}\,\cos(a_N(x-y)) - \sum_{|j|\le N-1} \frac{a_j^2}{a_N^2-a_j^2}\,\varphi(j,x-y) \tag{5.12}$$

Employing Lemma 5.1, we have the following theorem at once.

Theorem 5.1 *Green function $G(x,y)$ satisfies the following properties.*

(1) $$G(x,y) = \overline{G}(x,y) = G(y,x) = G(1-x,1-y) \qquad (0<x,y<1) \tag{5.13}$$

(2) $$(\,-\partial_x^2 + q\,)\,G(x,y) = \begin{cases} 0 & \text{(I)} \\ -\displaystyle\sum_{|j|\le N} \varphi(j,x-y) & \text{(II, IV)} \\ -1 & \text{(III)} \end{cases}$$
$$(0<x,y<1, \quad x\ne y) \tag{5.14}$$

(3) $$G(1,y) = G(0,y) =$$
$$\begin{cases} \dfrac{\cosh(\,a(y-1/2)\,)}{2a\,\sinh(a/2)} & \text{(I)} \\ -\dfrac{\cos(\,a(y-1/2)\,)}{2a\,\sin(a/2)} + \displaystyle\sum_{|j|\le N} \frac{1}{a^2-a_j^2}\,\varphi(j,y) & \text{(II)} \\ \dfrac{1}{2}\,y^2 - \dfrac{1}{2}\,y + \dfrac{1}{12} & \text{(III)} \\ \dfrac{1}{a_N}\,(y-1/2)\,\sin(a_N y) + \dfrac{1}{2a_N^2}\,\cos(a_N y) + \displaystyle\sum_{|j|\le N-1} \frac{1}{a_N^2-a_j^2}\,\varphi(j,y) & \text{(IV)} \end{cases}$$
$$(0<y<1) \tag{5.15}$$

(4) $\partial_x G(x,y)\Big|_{x=1} = \partial_x G(x,y)\Big|_{x=0} =$

$$\begin{cases} -\dfrac{\sinh(\,a(y-1/2)\,)}{2\,\sinh(a/2)} & \text{(I)} \\ -\dfrac{\sin(\,a(y-1/2)\,)}{2\,\sin(a/2)} - \displaystyle\sum_{|j|\le N} \frac{\sqrt{-1}\,a_j}{a^2-a_j^2}\,\varphi(j,y) & \text{(II)} \\ -(y\,-\,1/2) & \text{(III)} \\ -(y\,-\,1/2)\,\cos(a_N y) - \dfrac{1}{2a_N}\,\sin(a_N y) - \displaystyle\sum_{|j|\le N-1} \frac{\sqrt{-1}\,a_j}{a_N^2-a_j^2}\,\varphi(j,y) & \text{(IV)} \end{cases}$$

$$(0<y<1) \tag{5.16}$$

(5)
$$\begin{cases} G(x,y)\Big|_{y=x-0} - G(x,y)\Big|_{y=x+0} = 0 \\ \partial_x G(x,y)\Big|_{y=x-0} - \partial_x G(x,y)\Big|_{y=x+0} = -1 \qquad (0<x<1) \end{cases} \tag{5.17}$$

(6)
$$\int_0^1 \overline{\varphi}(j,x)\,G(x,y)\,dx = 0 \qquad (0<y<1,\ |j|\le N) \qquad (\mathrm{II,IV}) \tag{5.18}$$

$$\int_0^1 G(x,y)\,dx = 0 \qquad (0<y<1) \qquad (\mathrm{III}) \tag{5.19}$$

(7)
$$\begin{cases} G(x,y)\Big|_{x=y-0} - G(x,y)\Big|_{x=y+0} = 0 \\ \partial_x G(x,y)\Big|_{x=y-0} - \partial_x G(x,y)\Big|_{x=y+0} = 1 \qquad (0<y<1) \end{cases} \tag{5.20}$$

From the above Theorem, we can easily show that $u(x)$ defined by (1.4) is a classical solution to BVP.

Proof of Theorem 1.2 For any function $u(x)$ and $v(x) = G(x,y)$ where y is an arbitrary fixed value satisfying $0<y<1$ we have

$$u'\,v' + q\,u\,v = (\,u\,v'\,)' + u\,(\,-v'' + q\,v\,) \tag{5.21}$$

Integrating this on intervals $0<x<y$ and $y<x<1$ with respect to x we have (1.18). ■

6 Sobolev inequality and the best constant This section is devoted to the proof of the most important Theorem 1.3.

Proof of Theorem 1.3(1) From Theorem 1.2 the following reproducing equality

$$u(y) = \int_0^1 \Big[u'(x)\,\partial_x G(x,y) + q\,u(x)\,G(x,y)\Big]dx \qquad (0<y<1) \tag{6.1}$$

holds for any function $u(x) \in H$. Applying Schwarz inequality we have

$$|\,u(y)\,|^2 \,\le\, \int_0^1 \Big(|\,u'(x)\,|^2 \,+\, q\,|\,u(x)\,|^2 \Big) dx \int_0^1 \Big(|\,\partial_x G(x,y)\,|^2 \,+\, q\,|\,G(x,y)\,|^2 \Big) dx$$

$$(0 < y < 1) \tag{6.2}$$

If we put $u(x) = G(x,y) \in H$ in (6.1) then we have

$$G(y,y) \,=\, \int_0^1 \Big[|\,\partial_x G(x,y)\,|^2 \,+\, q\,|\,G(x,y)\,|^2 \Big] dx \qquad (0 < y < 1) \tag{6.3}$$

Thus we obtained

$$|\,u(y)\,|^2 \,\le\, G(y,y) \int_0^1 \Big[|\,u'(x)\,|^2 \,+\, q\,|\,u(x)\,|^2 \Big] dx \qquad (0 < y < 1) \tag{6.4}$$

Since $G(y,y)$ is a positive constant which is independent of y then we have

$$\sup_{0 \le y \le 1} G(y,y) \,=\, G(y_0, y_0) \tag{6.5}$$

where y_0 is an arbitrarily fixed number satisfying $0 \le y_0 \le 1$. Taking the supremum with respect to y of the above inequality (6.4), we have

$$\Big(\sup_{0 \le y \le 1} |\,u(y)\,| \Big)^2 \,\le\, G(y_0, y_0) \int_0^1 \Big[|\,u'(x)\,|^2 \,+\, q\,|\,u(x)\,|^2 \Big] dx \tag{6.6}$$

Thus we have the following conclusion. For any function $u(x) \in H$ we can take a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality

$$\Big(\sup_{0 \le y \le 1} |\,u(y)\,| \Big)^2 \,\le\, C \int_0^1 \Big[|\,u'(x)\,|^2 \,+\, q\,|\,u(x)\,|^2 \Big] dx \tag{6.7}$$

holds. The best constant C_0 among such C satisfies

$$C_0 \,\le\, G(y_0, y_0) \tag{6.8}$$

Applying Sobolev inequality to $u(x) = G(x, y_0)$, we have

$$\Big(\sup_{0 \le y \le 1} |\,G(y, y_0)\,| \Big)^2 \,\le\, C \int_0^1 \Big[|\,\partial_x G(x, y_0)\,|^2 \,+\, q\;|\,G(x, y_0)\,|^2 \Big]\, dx \,=\, C\,G(y_0, y_0) \tag{6.9}$$

We also have

$$G(y_0, y_0)^2 \,\le\, \Big(\sup_{0 \le y \le 1} |\,G(y, y_0)\,| \Big)^2 \tag{6.10}$$

Combining these two inequalities, we have

$$G(y_0, y_0)^2 \,\le\, \Big(\sup_{0 \le y \le 1} |\,G(y, y_0)\,| \Big)^2 \,\le$$

$$C \int_0^1 \Big[|\,\partial_x G(x, y_0)\,|^2 \,+\, q\;|\,G(x, y_0)\,|^2 \Big]\, dx \,=\, C\,G(y_0, y_0) \tag{6.11}$$

Then we have

$$G(y_0, y_0) \le C_0 \tag{6.12}$$

(6.8) and (6.12) shows that

$$C_0 = G(y_0, y_0) \tag{6.13}$$

Putting $C = C_0$ in (6.11) we have

$$G(y_0, y_0)^2 = \left(\sup_{0 \le y \le 1} |\, G(y, y_0) \,| \right)^2 =$$

$$C_0 \int_0^1 \left[\,|\, \partial_x G(x, y_0) \,|^2 + q \,|\, G(x, y_0) \,|^2 \right] dx = G(y_0, y_0)^2 \tag{6.14}$$

This means that

$$\left(\sup_{0 \le y \le 1} |\, G(y, y_0) \,| \right)^2 = C_0 \int_0^1 \left[\,|\, \partial_x G(x, y_0) \,|^2 + q \,|\, G(x, y_0) \,|^2 \right] dx \tag{6.15}$$

This completes the proof of Theorem 1.3. ■

Acknoledgement One of the authors A. N. is supported by J. S. P. S. Grant-in-Aid for Scientific Research for Young Scientists No. 16740092, K. T. is supported by J. S. P. S. Grant-in-Aid for Scientific Research (C) No. 17540175 and H. Y. is supported by The 21st Century COE Program named "Towards a new basic science : depth and synthesis".

References

[1] Y. Kametaka, K. Watanabe, A. Nagai and S. Pyatkov *The best constant of Sobolev inequality in an n dimensional Euclidean space* , Scientiae Mathematicae Japonicae Online, **e-2004** (2004) pp. 295-303.

[2] K. Watanabe, T. Yamada and W. Takahashi, *Reproducing Kernels of* $H^m(a, b)$ $(m = 1, 2, 3)$ *and Least Constants in Sobolev's Inequalities*, Applicable Analysis **82** (2003) pp. 809–820.

[3] Y. Kametaka, K. Takemura, Y. Suzuki and A. Nagai, *Positivity and hierarchical structure of Green's functions of 2-point boundary value problems for bending of a beam* , Japan J. Ind. Appl. Math. **18** (2001) pp. 543-566.

[4] Y.Kametaka, H.Yamagishi, K.Watanabe, A.Nagai and K.Takemura : *Riemann zeta function, Bernoulli polynomials and the best constant of Sobolev inequality*, Scientiae Mathematicae Japonicae Online **e-2007** (2007) pp. 63-89.

*Faculty of Engineering Science, Osaka University
1-3 Matikaneyamatyo, Toyonaka 560-8531, Japan
E-mail address: kametaka@sigmath.es.osaka-u.ac.jp

†Department of Computer Science, National Defense Academy
1-10-20 Yokosuka 239-8686, Japan
E-mail address: wata@nda.ac.jp

[‡]Liberal Arts and Basic Sciences, College of Industrial Technology
Nihon University, 2-11-1 Shinei, Narashino 275-8576, Japan
E-mail address: a8nagai@cit.nihon-u.ac.jp

[§]Faculty of Engineering Science, Osaka University
1-3 Matikaneyamatyo, Toyonaka 560-8531, Japan
E-mail address: yamagisi@sigmath.es.osaka-u.ac.jp

[¶]School of Media Science, Tokyo University of Technology
1404-1 Katakura, Hachioji 192-0982, Japan
E-mail address: takemura@media.teu.ac.jp