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SEMISIMPLE, ARCHIMEDEAN, AND SEMILOCAL PSEUDO MV-ALGEBRAS

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Abstract. The concepts of semisimple, Archimedean, and semilocal pseudo MV-algebras are investigated and many interesting facts concerning them are given.

1. Introduction

Pseudo MV-algebras were introduced by G. Georgescu and A. Iorgulescu in [6] and independently by J. Rachůnek in [8] (there they are called generalized MV-algebras or, for short, GMV-algebras) as a non-commutative generalization of MV-algebras. This work was intended as an attempt to order some notions appearing in the theory of these algebras. Semisimple pseudo MV-algebras and Archimedean pseudo MV-algebras are examples of such notions. In Section 3 we give some characterizations of semisimple pseudo MV-algebras. Archimedean pseudo MV-algebras are investigated and characterized in Section 4. It is shown that in the case of pseudo MV-algebras the notion of Archimedean is equivalent with the notion of Archimedean in the Belluce sense, that occurs in the theory of MV-algebras, and both are equivalent with the notion of semisimple. Section 5 is devoted to introduce and characterize semilocal pseudo MV-algebras, the concept generalizing a similar one from the theory of MV-algebras. For the convenience of the reader, in Section 2 we give the relevant material needed in the sequel, thus making our exposition self-contained.

2. Preliminaries

Let $A = (A, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. Set $x \cdot y = (y^- \oplus x^-)^\sim$ for any $x, y \in A$. We assume that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(A2) $x \oplus 0 = 0 \oplus x = x$,
(A3) $x \oplus 1 = 1 \oplus x = 1$,
(A4) $1^\sim = 0; 1^- = 0$,
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
(A6) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$,
(A7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$,
(A8) $(x^-)^\sim = x$.

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If the addition $\oplus$ is commutative, then both unary operations $^-$ and $^\sim$ coincide and A can be considered as an MV-algebra.

Throughout this paper A will denote a pseudo MV-algebra. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$\begin{aligned} 0x &= 0 \text{ and } (n+1)\,x = nx \oplus x; \\ x^0 &= 1 \text{ and } x^{n+1} = x^n \cdot x. \end{aligned}$$

Proposition 2.1 (Georgescu and Iorgulescu [6])**.** *The following properties hold for any $x, y \in A$:*
(a) $x \cdot 1 = 1 \cdot x = x$,
(b) $x^- \oplus x = 1, x \oplus x^\sim = 1$,
(c) $x \cdot x^- = 0, x^\sim \cdot x = 0$.

Proposition 2.2 (Georgescu and Iorgulescu [6])**.** *The following properties are equivalent for any $x, y \in A$:*
(a) $x^- \oplus y = 1$,
(b) $y \oplus x^\sim = 1$.

We define

$$x \leqslant y \Longleftrightarrow x^- \oplus y = 1.$$

As it is shown in [6], $(A, \leqslant)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^\sim \cdot y = x \cdot y^- \oplus y, \\ x \wedge y &= x \cdot \left(x^- \oplus y\right) = (x \oplus y^\sim) \cdot y. \end{aligned}$$

For every pseudo MV-algebra A we set $\mathcal{L}(A) = (A, \vee, \wedge, 0, 1)$.

Proposition 2.3 (Georgescu and Iorgulescu [6])**.** *Let A be a pseudo MV-algebra. The following properties hold for any $x, y, z \in A$:*
(a) $x \leqslant y \Longleftrightarrow y^- \leqslant x^- \Longleftrightarrow y^\sim \leqslant x^\sim$,
(b) $x \leqslant y \Longrightarrow z \oplus x \leqslant z \oplus y, x \oplus z \leqslant y \oplus z$,
(c) $(x \oplus z) \cdot y \leqslant x \oplus z \cdot y, y \cdot (x \oplus z) \leqslant y \cdot x \oplus z$.

Definition 2.4. An *ideal* of A is a subset J of A satisfying the following conditions:
(I1) $0 \in J$,
(I2) if $x, y \in J$, then $x \oplus y \in J$,
(I3) if $x \in J$, $y \in A$ and $y \leqslant x$, then $y \in J$.

Under this definition, $\{0\}$ and A are the simplest examples of ideals.

Denote by $\mathrm{Id}(A)$ the set of all ideals of A and note that $\mathrm{Id}(A)$ ordered by set inclusion is a complete lattice.

Remark 2.5. Let $J \in \mathrm{Id}(A)$.
(a) If $x, y \in J$, then $x \cdot y, x \wedge y, x \vee y \in J$,
(b) J is an ideal of the lattice $\mathcal{L}(A)$.

For every subset $W \subseteq A$, the smallest ideal of A which contains W, i.e., the intersection of all ideals $J \supseteq W$, is said to be the ideal *generated* by W, and will be denoted $(W]$.

Proposition 2.6 (Georgescu and Iorgulescu [6])**.** *Let W be a subset of A. If $W = \varnothing$, then $(W] = \{0\}$. If $W \neq \varnothing$, then*

$$(W] = \{x \in A : x \leqslant w_1 \oplus \cdots \oplus w_n \textit{ for some } w_1, \ldots, w_n \in W\}.$$

In particular, for every $z \in A$, the ideal $(z] = (\{z\}]$ is called the *principal ideal generated by* z (see [6]), and we have

$$(z] = \{x \in A : x \leqslant nz \text{ for some } n \in \mathbb{N}\}\,.$$

Definition 2.7. Let J be a proper ideal of A (i.e., $J \neq A$).
(a) J is called *prime* if, for all $J_1, J_2 \in \mathrm{Id}(A)$, $J = J_1 \cap J_2$ implies $J = J_1$ or $J = J_2$.
(b) J is called *regular* iff $J = \bigcap X$ implies that $J \in X$ for every subset X of $\mathrm{Id}(A)$.
(c) J is called *maximal* iff whenever M is an ideal such that $J \subseteq M \subseteq A$, then either $M = J$ or $M = A$.

By definition, each maximal ideal is regular and each regular ideal is prime.

Definition 2.8. An ideal H of A is called *normal* if it satisfies the condition:
(N) for all $x, y \in A$, $x \cdot y^{-} \in H \Longleftrightarrow y^{\sim} \cdot x \in H$.

Denote by $\mathrm{Id}_n\,(A)$ the set of normal ideals of A.

Proposition 2.9 (Georgescu and Iorgulescu [6])**.** *Let A be a pseudo MV-algebra and let H be an ideal of A. Then the following are equivalent:*
(a) *H is normal,*
(b) *for each $x \in A$, $x \oplus H = H \oplus x$ (i.e., for each $h \in H$ there exists $h' \in H$ such that $x \oplus h = h' \oplus x$; and for each $h \in H$ there exists $h'' \in H$ such that $h \oplus x = x \oplus h''$).*

From Propositions 2.6 and 2.9 we obtain the following lemma.

Lemma 2.10. *Let H_1, H_2 be normal ideals of A. Then*

$$(H_1 \cup H_2] = \{x \in A : x \leqslant h_1 \oplus h_2 \textit{ for some } h_1 \in H_1, h_2 \in H_2\}.$$

Lemma 2.11. *Let A be a pseudo MV-algebra and let H be an ideal of A. Then the following are equivalent:*
(a) *H is normal,*
(b) *$(x \oplus h) \cdot x^{-} \in H$ and $x^{\sim} \cdot (h \oplus x) \in H$ for all $x \in A$ and $h \in H$.*

Proof. (a) $\Rightarrow$ (b): Let $x \in A$. By Proposition 2.9, for each $h \in H$ there exists $h' \in H$ such that $x \oplus h = h' \oplus x$. From Propositions 2.3(c) and 2.1(c) we obtain

$$(x \oplus h) \cdot x^{-} = (h' \oplus x) \cdot x^{-} \leqslant h' \oplus x \cdot x^{-} = h' \oplus 0 = h' \in H.$$

Hence $(x \oplus h) \cdot x^{-} \in H$. Similarly, $x^{\sim} \cdot (h \oplus x) \in H$.

(b) $\Rightarrow$ (a): Let $x \in A$ and $h \in H$. Let us set $h' = (x \oplus h) \cdot x^{-}$ and $h'' = x^{\sim} \cdot (h \oplus x)$. By assumption, $h', h'' \in H$. Applying (A6) and Propositions 2.3(b,c) and 2.1(c) we have

$$h' \oplus x = (x \oplus h) \cdot x^{-} \oplus x = x \oplus x^{\sim} \cdot (x \oplus h) \leqslant x \oplus x^{\sim} \cdot x \oplus h = x \oplus h.$$

On the other hand, by Propositions 2.3(c) and 2.1, we get

$$h' \oplus x = x \oplus x^{\sim} \cdot (x \oplus h) \geqslant (x \oplus x^{\sim}) \cdot (x \oplus h) = x \oplus h.$$

Thus $x \oplus h = h' \oplus x$. Similarly, $h \oplus x = x \oplus h''$. Therefore, from Proposition 2.9 we conclude that (a) is true. ☐

Proposition 2.12 (Dvurečenskij and Pulmannova [4])**.** *For any proper normal ideal H of a pseudo MV-algebra A, the following conditions are equivalent:*
(a) *H is maximal,*
(b) *for each $z \in A$, $z \notin H$ iff $(nz)^{-} \in H$ for some $n \in \mathbb{N}$,*
(c) *for each $z \in A$, $z \notin H$ iff $(nz)^{\sim} \in H$ for some $n \in \mathbb{N}$.*

Following [6], for any normal ideal H of A, we define a congruence on A by:

$$x \sim_H y \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in H.$$

We also have

$$x \sim_H y \Longleftrightarrow x^\sim \cdot y \vee y^\sim \cdot x \in H.$$

We denote by x/H the congruence class of an element $x \in A$ and on the set $A/H = \{x/H : x \in A\}$ we define the operations:

$$x/H \oplus y/H = (x \oplus y)/H, \ (x/H)^- = \left(x^-\right)/H, \ (x/H)^\sim = (x^\sim)/H.$$

The resulting quotient algebra $A/H = (A/H, \oplus, ^-, ^\sim, 0/H, 1/H)$ becomes a pseudo MV-algebra, called *the quotient algebra of A by the normal ideal H*.

Lemma 2.13. *Let $H_1, \dots, H_m$ be normal ideals of A such that $(H_i \cup H_j] = A$ for $i, j = 1, \dots, m$ and $i \neq j$. Let $x_1, \dots, x_m \in A$. Then there is $x \in A$ such that $x \sim_{H_i} x_i$ for $i = 1, \dots, m$.*

Proof. First, let $m = 2$. Since $(H_1 \cup H_2] = A$, by Lemma 2.10 there exist $h_{12} \in H_1$ and $h_{21} \in H_2$ such that $h_{12} \oplus h_{21} = 1$. Applying (A8) we get $h_{12} \oplus (h_{21}^-)^\sim = 1$. From Proposition 2.2 we deduce that $h_{21}^- \leqslant h_{12}$. Since $h_{12} \in H_1$, we see that $h_{21}^- \in H_1$. Hence $h_{21} \sim_{H_1} 1$. Take $x = x_1 \cdot h_{21} \oplus x_2 \cdot h_{12}$, where $x_1, x_2 \in A$. We obtain

$$\begin{aligned} x/H_1 &= x_1/H_1 \cdot h_{21}/H_1 \oplus x_2/H_1 \cdot h_{12}/H_1 \\ &= x_1/H_1 \cdot 1/H_1 \oplus x_2/H_1 \cdot 0/H_1 = x_1/H_1. \end{aligned}$$

Thus $x \sim_{H_1} x_1$. Similarly, $x \sim_{H_2} x_2$.

Now let m be arbitrary. For $i, j = 1, \dots, m$ and $i \neq j$, there exist $h_{ij} \in H_i$ and $h_{ji} \in H_j$ such that $h_{ij} \oplus h_{ji} = 1$. Considering $x = \sum_{i=1}^m x_i \cdot h_{1i} \cdots h_{i-1,i} \cdot h_{i+1,i} \cdots h_{mi}$ and reasoning as above we see that $x \sim_{H_i} x_i$ for $i = 1, \dots, m$. □

A pseudo MV-algebra is *simple* iff there is no non-trivial proper ideal of A (i.e., $\mathrm{Id}(A) = \{\{0\}, A\}$).

Proposition 2.14 (Dvurečenskij [3]). *A normal ideal H of a pseudo MV-algebra A is maximal if and only if A/H is a simple pseudo MV-algebra.*

Proposition 2.15 (Georgescu and Iorgulescu [6]). *Let H be a normal ideal of a pseudo MV-algebra A. Then the quotient algebra A/H is a pseudo MV-chain if and only if H is prime.*

The *radical* of a pseudo MV-algebra A is the set

$$\mathrm{Rad}\,(A) = \bigcap \{M : M \text{ is a maximal ideal of } A\}$$

and the *normal radical* of A is the set

$$\mathrm{Rad}_n\,(A) = \bigcap \{M : M \text{ is a maximal and normal ideal of } A\}.$$

If there are no maximal and normal ideals of A, then we set $\mathrm{Rad}_n\,(A) = A$.

Remark 2.16. If A is an MV-algebra, then $\mathrm{Rad}_n\,(A) = \mathrm{Rad}(A)$.

Let I be a nonempty set. The direct product of the pseudo MV-algebras A_i, $i \in I$, denoted by $\prod_{i \in I} A_i$, is the pseudo MV-algebra obtained by endowing the set-theoretical cartesian product of A_i $(i \in I)$ with the pseudo MV-operations defined pointwise. For each $i \in I$, the map $\pi_i : \prod_{i \in I} A_i \to A_i$, defined by

$$\pi_i(x) = x(i) \text{ for all } x \in \prod_{i \in I} A_i,$$

is a homomorphism onto A_i, called the *i-th projection function.*

Proposition 2.17. *Let* $A_1, \dots, A_k$ *be pseudo MV-algebras and let* $A = A_1 \times \cdots \times A_k$*. If* $J_i \in \mathrm{Id}(A_i)$ *for* $i = 1, \dots, k$*, then* $J_1 \times \cdots \times J_k$ *is an ideal of* A*.*
Conversely, if J *is an ideal of* A*, then for* $i = 1, \dots, k$*,* $J_i = \pi_i(J)$ *is an ideal of* A_i*, and* $J = J_1 \times \cdots \times J_k$*.*

Proof. It is straightforward. □

Proposition 2.18. *Let* $A = A_1 \times \cdots \times A_k$*, where* $A_1, \dots, A_k$ *are pseudo MV-algebras. Then:*
(a) $\mathrm{Id}(A) = \mathrm{Id}(A_1) \times \cdots \times \mathrm{Id}(A_k)$,
(b) $\mathrm{Id}_n(A) = \mathrm{Id}_n(A_1) \times \cdots \times \mathrm{Id}_n(A_k)$,
(c) $\mathrm{Rad}(A) = \mathrm{Rad}(A_1) \times \cdots \times \mathrm{Rad}(A_k)$,
(d) $\mathrm{Rad}_n(A) = \mathrm{Rad}_n(A_1) \times \cdots \times \mathrm{Rad}_n(A_k)$.

Proof. (a) Follows from Proposition 2.17.

(b) It is sufficient to prove that $J_1 \times \cdots \times J_k$ is a normal ideal of A if and only if J_i is a normal ideal of A_i for $i = 1, \dots, k$. It is easy to see that if J_i is a normal ideal of A_i for $i = 1, \dots, k$, then $J_1 \times \cdots \times J_k$ is a normal ideal of A. Now, assume that $J = J_1 \times \cdots \times J_k$ is normal. Let $a \in A_i$ and $b \in J_i$. Take $x \in A$ with $x(i) = a$. Define $y \in A$ by $y(i) = b$ and $y(j) = 0$ for $j \neq i$. Then $y \in J$, and we conclude from Lemma 2.11 that $(x \oplus y) \cdot x^- \in J$. We have

$$(a \oplus b) \cdot a^- = [\pi_i(x) \oplus \pi_i(y)] \cdot [\pi_i(x)]^- = \pi_i\left((x \oplus y) \cdot x^-\right) \in \pi_i(J) = J_i.$$

Similarly, $a^\sim \cdot (b \oplus a) \in J_i$. Therefore, by Lemma 2.11, J_i is a normal ideal of A_i for $i = 1, \dots, k$.

(c) It is easy to see that J is a maximal ideal of A if and only if $J = A_1 \times \cdots \times A_{i-1} \times J_i \times A_{i+1} \times \cdots \times A_k$, where J_i is a maximal ideal of A_i for $i = 1, \dots, k$. Hence (c) is true.

(d) Follows from (b) and (c). □

Definition 2.19. A pseudo MV-algebra A is called *normal-valued* if for any regular ideal J of A and any $x \in J^*$, $x \oplus J = J \oplus x$, where J^* is the unique least ideal which properly contains J.

Proposition 2.20. *Let* A *be a normal-valued pseudo MV-algebra and let* M *be a maximal ideal of* A*. Then* M *is normal.*

Proof. Since A is normal-valued and M is a maximal ideal of A, M is regular and $x \oplus M = M \oplus x$ for every $x \in M^* = A$. Hence, by Proposition 2.9, M is normal. □

An element x of a pseudo MV-algebra A is called *infinitesimal* (see [9]) if x satisfies condition

$$nx \leqslant x^- \text{ for each } n \in \mathbb{N}.$$

Let us denote by $\mathrm{Infinit}(A)$ the set of all infinitesimal elements in A.

Proposition 2.21 (Rachůnek [9])**.** *Let* A *be a pseudo MV-algebra. Then:*
(a) $\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A)$*,*
(b) *if* A *is normal-valued, then* $\mathrm{Rad}(A) = \mathrm{Infinit}(A)$.

Proposition 2.22 (Di Nola, Dvurečenskij and Jakubík [1])**.** *Let* A *be a pseudo MV-algebra. Then* $\mathrm{Infinit}(A) \subseteq \mathrm{Rad}_n(A)$.

By Propositions 2.21 and 2.22 we have a ladder of inclusions:

$$\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A) \subseteq \mathrm{Rad}_n(A).$$

Proposition 2.23. *Let A be a normal-valued pseudo MV-algebra. Then*

$$\mathrm{Rad}\,(A) = \mathrm{Infinit}\,(A) = \mathrm{Rad}_n\,(A)\,.$$

Proof. Since A is normal-valued, from Proposition 2.20 we have that every maximal ideal of A is normal. Thus $\mathrm{Rad}(A) = \mathrm{Rad}_n\,(A)$. □

Now we give the definition of an Artinian pseudo MV-algebra.

Definition 2.24. A pseudo MV-algebra A is called *Artinian* if for every descending sequence $J_1 \supseteq J_2 \supseteq \cdots$ of ideals of A there exists $k \in \mathbb{N}$ such that $J_m = J_k$ for all $m \geqslant k$.

Proposition 2.25 (Dymek [5])**.** *If A is Artinian, then A/H is Artinian for every normal ideal H of A.*

At the end of this section we recall some definitions and facts from [7].

Definition 2.26. The *order* of an element $x \in A$ is the least n such that $nx = 1$ if such n exists, and $\mathrm{ord}(x) = \infty$ otherwise.

Remark 2.27. It is easy to see that for any $x \in A$, $\mathrm{ord}(x^-) = \mathrm{ord}(x^\sim)$.

Definition 2.28. A pseudo MV-algebra A is called *local* if

$$\mathrm{ord}\,(x \oplus y) < \infty \text{ implies that } \mathrm{ord}\,(x) < \infty \text{ or } \mathrm{ord}\,(y) < \infty$$

for all $x, y \in A$.

Remark 2.29. If A is local, then $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(x^-) < \infty$ for every $x \in A$.

Let A be a pseudo MV-algebra. We denote by $\mathrm{D}(A) = \{x \in A : \mathrm{ord}(x) = \infty\}$ the set of all elements of infinite order.

Proposition 2.30 (Leuştean [7])**.** *Let A be a pseudo MV-algebra. The following are equivalent:*
(a) *A is local,*
(b) *$\mathrm{D}(A)$ is an ideal of A,*
(c) *$\mathrm{D}(A)$ is the only maximal ideal of A.*

3. Semisimple pseudo MV-algebras

Definition 3.1. A pseudo MV-algebra A is *semisimple* iff $\mathrm{Rad}_n\,(A) = \{0\}$.

Remark 3.2. Every simple pseudo MV-algebra is semisimple.

Example 3.3. Let $A = \{(1, y) \in \mathbb{R}^2 : y \geqslant 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$\begin{aligned}
(a,b) \oplus (c,d) &= \begin{cases} (1, b+d) & \textit{if } a = c = 1, \\ (2, ad+b) & \textit{if } ac = 2 \textit{ and } ad + b \leqslant 0, \\ (2,0) & \textit{in other cases.} \end{cases} \\
(a,b)^- &= \left(\frac{2}{a}, -\frac{2b}{a}\right), \\
(a,b)^\sim &= \left(\frac{2}{a}, -\frac{b}{a}\right).
\end{aligned}$$

Then $A = (A, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra. Let $H = \{(1, y) : y \geqslant 0\}$. Then H is the unique normal maximal ideal of A and hence $\mathrm{Rad}_n\,(A) = H \neq \{\mathbf{0}\}$. Thus A is not semisimple. Moreover, note that by Proposition 2.14, A/H is a simple pseudo MV-algebra. Therefore A/H is semisimple.

Recall that a pseudo MV-algebra A is a *subdirect product of pseudo MV-algebras* A_i, $i \in I$, if there exists an injective homomorphism $h : A \to \prod_{i\in I} A_i$ such that $\pi_i \circ h$ maps A onto A_i for all $i \in I$.

Proposition 3.4. *Let A be a pseudo MV-algebra. The following are equivalent:*
(a) *A is semisimple,*
(b) *there is a family $\{H_i : i \in I\}$ of normal maximal ideals of A with $\bigcap_{i\in I} H_i = \{0\}$,*
(c) *A is a subdirect product of simple pseudo MV-chains.*

Proof. (a) $\Rightarrow$ (b): Follows from definition.

(b) $\Rightarrow$ (c): Suppose that $\{H_i : i \in I\}$ is a family of maximal and normal ideals of A such that $\bigcap_{i\in I} H_i = \{0\}$. Write $A_i := A/H_i$ for $i \in I$. First note that, by Propositons 2.14 and 2.15, A_i are simple pseudo MV-chains. Define $h : A \to \prod_{i\in I} A_i$ by

$$h(x) = (x/H_i : i \in I) \text{ for all } x \in A.$$

Since $\bigcap_{i\in I} H_i = \{0\}$, we have that $\mathrm{Ker}(h) = \{0\}$. Thus h is injective. It is easy to see that $\pi_i \circ h$ maps A onto A_i, where π_i is the i-th projection function. Therefore, A is a subdirect product of the (simple) pseudo MV-chains A_i, $i \in I$.

(c) $\Rightarrow$ (a): Let $h : A \to \prod_{i\in I} A_i$ be an injective homomorphism, where A_i are simple pseudo MV-chains, and let $\pi_i \circ h : A \to A_i$ be surjective. Write $\mathrm{Ker}(\pi_i \circ h) = H_i$ for $i \in I$. Then H_i is a normal ideal of A and $A/H_i \cong A_i$. Consequently, A/H_i is simple. By Proposition 2.14, H_i is maximal. If $x \in \bigcap_{i\in \Gamma} H_i$, then $\pi_i(h(x)) = 0$ for all $i \in I$. This implies that $h(x) = 0$, and since h is injective, we obtain $x = 0$. Therefore $\mathrm{Rad}_n(A) \subseteq \bigcap_{i\in I} H_i = \{0\}$. Hence $\mathrm{Rad}_n(A) = \{0\}$. Thus A is semisimple. □

Now recall that a pseudo MV-algebra A is *representable* (see [6]) if it is a subdirect product of pseudo MV-chains. Thus, by Proposition 3.4, we have the following proposition.

Proposition 3.5. *If a pseudo MV-algebra A is semisimple, then it is representable.*

Proposition 2.19(d) yields

Proposition 3.6. *Let $A = A_1 \times \cdots \times A_k$, where $A_1, \ldots, A_k$ are pseudo MV-algebras. Then A is semisimple if and only if A_i is semisimple for $i = 1, \ldots, k$.*

4. Archimedean pseudo MV-algebras

Definition 4.1. Let A be a pseudo MV-algebra.
(a) A is Archimedean iff $\mathrm{Infinit}(A) = \{0\}$.
(b) A is Archimedean in the Belluce sense iff for each $x, y \in A$, if $nx \leqslant y$ for all $n \geqslant 0$, then $x \cdot y = x$.

Proposition 4.2 (Dvurečenskij [2])**.** *Any Archimedean pseudo MV-algebra is an MV-algebra.*

Proposition 4.3 (Dvurečenskij [2])**.** *A pseudo MV-algebra A has the MacNeille completion as a pseudo MV-algebra if and only if A is Archimedean.*

Recall that a pseudo MV-algebra is *locally finite* if $\mathrm{ord}(x) < \infty$ for every $x > 0$.

Lemma 4.4. *A pseudo MV-algebra A is locally finite if and only if* $\mathrm{Id}(A) = \{\{0\}, A\}$ *(i.e., A is simple).*

Proof. If A is trivial, then the lemma is obvious. Assume that $A \neq \{0\}$. Suppose that A is locally finite. Let $I \neq \{0\}$ be an ideal of A and let $x \in I, x \neq 0$. Then there is $n \in \mathbb{N}$ such that $nx = 1$. Thus $1 \in I$, i.e., $I = A$.

Now suppose that $\mathrm{Id}(A) = \{\{0\}, A\}$ and A is not locally finite. Then there exists $x \in A$ and $x \neq 0$ such that $nx < 1$ for all $n \in \mathbb{N}$. Let us take an ideal

$$(x] = \{y \in A : y \leqslant mx \text{ for some } m \in \mathbb{N}\}$$

generated by x. Then $(x] \neq \{0\}$. Hence $(x] = A$, i.e., $1 \in (x]$. Thus $1 \leqslant mx$ for some $m \in \mathbb{N}$, i.e., $mx = 1$ for some $m \in \mathbb{N}$. This is a contradiction. Therefore A is locally finite. □

Theorem 4.5. *Let A be a pseudo MV-algebra. The following are equivalent:*
(a) *A is semisimple,*
(b) *A is a subdirect product of simple pseudo MV-chains,*
(c) *A is Archimedean in the Belluce sense,*
(d) *A is Archimedean,*
(e) *A has the MacNeille completion.*

Proof. (a) ⇒ (b): Follows by Proposition 3.4.

(b) ⇒ (c): Let $A \subseteq \prod_{i \in I} A_i$ be a subdirect product of simple pseudo MV-chains A_i, $i \in I$. Let $x, y \in A$ and suppose that $nx \leqslant y$ for all $n \geqslant 0$. Then

$$nx(i) = (nx)(i) = \pi_i(nx) \leq \pi_i(y) = y(i)$$

for all $i \in I$ and $n \geqslant 0$. By Lemma 4.4, each A_i is locally finite. Therefore $x(i) = 0$ or $y(i) = 1$ for all $i \in I$. Hence in each A_i we have

$$(x \cdot y)(i) = x(i) \cdot y(i) = \begin{cases} 0 & \text{if} \quad x(i) = 0 \\ x(i) & \text{if} \quad x(i) \neq 0 \end{cases}.$$

Thus $(x \cdot y)(i) = x(i)$ for $i \in I$. It follows that $x \cdot y = x$.

(c) ⇒ (d): Let $x \in \mathrm{Infinit}(A)$. Then $nx \leqslant x^-$ for all $n \in \mathbb{N}$. Since A is Archimedean in the Belluce sense, we obtain $x = x \cdot x^- = 0$. Consequently, $\mathrm{Infinit}(A) = \{0\}$, i.e., A is Archimedean.

(d) ⇔ (e): Follows from Proposition 4.3.

(d) ⇒ (a): Let A be an Archimedean pseudo MV-algebra. By Proposition 4.2, A is an MV-algebra. Hence $\mathrm{Rad}_n(A) = \mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A) = \{0\}$, i.e., $\mathrm{Rad}_n(A) = \{0\}$. Thus A is semisimple. □

Proposition 4.6. *Any subalgebra of a semisimple pseudo MV-algebra is semisimple.*

Proof. Let A be a semisimple pseudo MV-algebra and let B be a subalgebra of A. We have $\mathrm{Infinit}(A) = \{0\}$,because A is Archimedean by Theorem 4.5. Since $\mathrm{Infinit}(B) \subseteq \mathrm{Infinit}(A)$, we see that $\mathrm{Infinit}(B) = \{0\}$. Theorem 4.5 now shows that B is semisimple. □

5. Semilocal pseudo MV-algebras

Definition 5.1. A pseudo MV-algebra is called *semilocal* if it has only finitely many normal maximal ideals.

By Proposition 2.30, we have the following proposition.

Proposition 5.2. *Any local pseudo MV-algebra is semilocal.*

Theorem 5.3. *Let A be a pseudo MV-algebra. The following are equivalent:*
(a) *A is semilocal,*
(b) *$A/\mathrm{Rad}_n(A)$ is trivial or isomorphic to a direct product of finitely many simple pseudo MV-chains,*
(c) *$A/\mathrm{Rad}_n(A)$ has finitely many ideals,*
(d) *$A/\mathrm{Rad}_n(A)$ is Artinian.*

Proof. (a) $\Rightarrow$ (b): Assume that A is semilocal. If A does not have any maximal and normal ideals, then $\mathrm{Rad}_n(A) = A$ and hence $A/\mathrm{Rad}_n(A)$ is trivial. Let $\{H_1, \dots, H_m\}$ be the set of all maximal and normal ideals of A, where m is a natural number. Then $\mathrm{Rad}_n(A) = \bigcap_{i=1}^{m} H_i$. By Propositions 2.14 and 2.15, each A/H_i is a simple pseudo MV-chain. Take the map $\varphi : A/\mathrm{Rad}_n(A) \to \prod_{i=1}^{m} A/H_i$ given by

$$\varphi(x/\mathrm{Rad}_n(A)) = (x/H_1, \dots, x/H_m).$$

Clearly φ is a homomorphism. We prove that φ is an isomorphism. Indeed, since $(H_i \cup H_j] = A$ for $i, j = 1, \dots, m$ and $i \neq j$, we have, by Lemma 2.13, that φ is surjective. Now, suppose that $\varphi(x/\mathrm{Rad}_n(A)) = \varphi(y/\mathrm{Rad}_n(A))$ for $x, y \in A$. Hence $x/H_i = y/H_i$ for each i $(1 \leq i \leq m)$. Then $x \cdot y^- \vee y \cdot x^- \in H_i$ for $i = 1, \dots, m$, i.e., $x \cdot y^- \vee y \cdot x^- \in \mathrm{Rad}_n(A)$. Thus $x/\mathrm{Rad}_n(A) = y/\mathrm{Rad}_n(A)$. Therefore φ is an isomorphism.

(b) $\Rightarrow$ (c): If $A/\mathrm{Rad}_n(A)$ is trivial, then it has only one ideal. Let $A/\mathrm{Rad}_n(A) \cong A_1 \times \cdots \times A_m$, where A_i is a simple (non-trivial) pseudo MV-chain for $i = 1, \dots, m$. From Proposition 2.18 we have $|\mathrm{Id}(A/\mathrm{Rad}_n(A))| = |\mathrm{Id}(A_1) \times \cdots \times \mathrm{Id}(A_m)|$. Since $\mathrm{Id}(A_i)$ has 2 elements for every $i = 1, \dots, m$, we have that $\mathrm{Id}(A/\mathrm{Rad}_n(A))$ has 2^m elements. Thus $A/\mathrm{Rad}_n(A)$ has finitely many ideals.

(c) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (a): Suppose that A has infinitely many maximal and normal ideals $H_1, H_2, \dots$. Then we have a strictly descending sequence $H_1 \supset H_1 \cap H_2 \supset H_1 \cap H_2 \cap H_3 \supset \cdots$ of ideals of A. Hence we obtain a sequence

$$H_1/\mathrm{Rad}_n(A) \supseteq (H_1 \cap H_2)/\mathrm{Rad}_n(A) \supseteq (H_1 \cap H_2 \cap H_3)/\mathrm{Rad}_n(A) \supseteq \cdots$$

of ideals of $A/\mathrm{Rad}_n(A)$. Note that this sequence is strictly descending. Indeed, if J_1, J_2 are maximal and normal ideals of A, then $(J_1 \cap J_2)/\mathrm{Rad}_n(A) \subset J_1/\mathrm{Rad}_n(A)$. Suppose that $(J_1 \cap J_2)/\mathrm{Rad}_n(A) = J_1/\mathrm{Rad}_n(A)$. Let $a \in J_1 - (J_1 \cap J_2)$. Note that there is $b \in J_1 \cap J_2$ such that $a/\mathrm{Rad}_n(A) = b/\mathrm{Rad}_n(A)$. Thus $a \cdot b^- \vee b \cdot a^- \in \mathrm{Rad}_n(A)$ and hence $a \cdot b^- \in \mathrm{Rad}_n(A) \subseteq J_1 \cap J_2$. Since $a \leqslant a \vee b = a \cdot b^- \oplus b \in J_1 \cap J_2$, we have $a \in J_1 \cap J_2$, which is a contradiction. Therefore we get a strictly descending sequence of ideals of Artinian pseudo MV-algebra $A/\mathrm{Rad}_n(A)$, which is impossible. Thus A is semilocal. ☐

Corollary 5.4. *If A is Artinian, then it is semilocal.*

Proof. If A is Artinian, then $A/\mathrm{Rad}_n(A)$ is Artinian by Proposition 2.25. From Theorem 5.3 we see that A is semilocal. ☐

Corollary 5.5. *Let A be semisimple pseudo MV-algebra. Then A is semilocal if and only if A is Artinian.*

Corollary 5.6. *If A is semilocal, then $A/\mathrm{Rad}_n(A)$ is semisimple.*

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