BASIC BCI-ALGEBRAS AND ABELIAN GROUPS ARE EQUIVALENT

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ABSTRACT. In this paper we prove that Abelian groups and basic BCI-algebras are equivalent and by using it we have obtain more results.

1. Introduction

The notion of BCI- algebras was formulated in 1966 by K. Iséki[3] and since then a lot of work has been done on subalgebras, ideals and homomorphism, but it is not much known about relations of BCI-algebras to other algebra systems. T. Lei and C. Xi[8] displayed the relationship of p-semi simple BCI-algebras to Abelian groups. In any BCI-algebra, there exist two important subsets. One of them is BCK-part and another one is a set consisting of incomparable elements containing 0. It makes the basic BCI-algebra introduced by K. Iséki [4]. Also B-algebras and 0-commutative B-algebras were introduced for the first time by J. Neggers, H. S. Kim and H. G. Park[9, 7]. In this paper, we prove that above two concepts namely basic-BCI algebras and Abelian groups are equivalent and we obtain more results about them.

2. Preliminaries

Definition 2.1. [4] A *basic BCI-algebra* is an algebra $(X, *, 0)$ of type (2,0) which satisfies the following axioms:
For all $x, y, z \in X$.
(B-BCI1) $(x * y) * (x * z) = z * y$,
(B-BCI2) $x * (x * y) = y$,
(B-BCI3) $x * y = 0$ implies $x = y$.

Definition 2.2. [7] A 0-commutative B-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms:
For all $x, y, z \in X$. (B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B3) $(x * y) * z = x * (z * (0 * y))$,
(B4) $x * (0 * y) = y * (0 * x)$.

Theorem 2.3. [7] *Let $(X, *, 0)$ be an algebra of type (2,0). Then the following statements are equivalent.*

(i) $(X, *, 0)$ *is a* 0*-commutative B-algebra,*
(ii) $(X, *, 0)$ *is a p-semi simple BCI-algebras,*
(iii) $(X, *, 0)$ *is an Abelian group.*

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3. Basic BCI-algebras and 0-Commutative B-algebras are equivalent

Proposition 3.1. [4] *Let $(X, *, 0)$ be a basic BCI-algebra. Then, for all $x, y, z \in X$,*

(i) $x * 0 = x$,
(ii) $0 * (0 * x) = x$,
(iii) $0 * x = 0 * y$ *implies* $x = y$,
(iv) $(x * y) * z = (x * z) * y$,
(v) $x * (0 * y) = y * (0 * x)$.

Lemma 3.2. [9, 7] *Let $(X, *, 0)$ be a 0-commutative B-algebra. Then for all $x, y \in X$,*

(i) $x * y = 0$ *implies* $x = y$,
(ii) $x * (x * y) = y$,
(iii) $(x * y) * (x * z) = z * y$.

Corollary 3.3. *Any 0-commutative B-algebra is a basic BCI-algebra.*

Proof. The proof comes from Lemma 3.2. □

Proposition 3.4. *Let $(X, *, 0)$ be a basic BCI-algebra. Then, for all $x, y \in X$,*

(i) $0 * (x * y) = y * x$,
(ii) $0 * (x * y) = (0 * x) * (0 * y)$,
(iii) $(0 * x) * (0 * y) = y * x$.

Proof. (i) Let $x, y \in X$. Then,

$$\begin{aligned} 0 * (x * y)) &= (x * (x * 0)) * (x * y), \text{ (by B-BCI2)} \\ &= (x * (x * y)) * (x * 0), \text{ (by Proposition 3.1(iv))} \\ &= y * x. \text{ (by B-BCI2 and Proposition 3.1(i))} \end{aligned}$$

(ii) Let $x, y \in X$. Then,

$$\begin{aligned} 0 * (x * y) &= y * x, \text{ (by (i))} \\ &= (0 * (0 * y)) * x, \text{ (by B-BCI2)} \\ &= (0 * x) * (0 * y). \text{ (by Proposition 3.1(iv))} \end{aligned}$$

(iii) Let $x, y \in X$. Then, by (i) and (ii) we have

$$(0 * x) * (0 * y) = 0 * (x * y) = y * x$$

□

Theorem 3.5. *$(X, *, 0)$ is a basic BCI-algebra if and only if $(X, *, 0)$ is a 0-commutative B-algebra.*

Proof. By Corollary 3.3, any 0-commutative B-algebra is a basic BCI-algebra.
Conversely, let $(X, *, 0)$ be a basic BCI-algebra. Then for all $x \in X$,

$$\begin{aligned} x * x &= x * (0 * (0 * x)), \text{ (by B-BCI2)} \\ &= (0 * x) * (0 * x), \text{ (by Proposition 3.1(v))} \\ &= (0 * x) * ((0 * x) * 0), \text{ (by Proposition 3.1(i))} \\ &= 0. \text{ (by (B-BCI2))} \end{aligned}$$

Moreover, by Proposition 3.1, $x * 0 = x$. Now, we should prove the axiom B3. For this, we will prove $0 * ((x * y) * z) = 0 * (x * (z * (0 * y)))$. Let $x, y, z \in X$. Then,

$$\begin{aligned} 0*((x*y)*z) &= 0*((x*(0*(0*y)))*z), \text{ (by B-BCI2)} \\ &= 0*(((0*y)*(0*x))*z), \text{ (by Proposition 3.1(v))} \\ &= 0*(((0*y)*z)*(0*x)), \text{ (by Proposition 3.1(iv))} \\ &= (0*((0*y)*z)))*(0*(0*x)), \text{ (by Proposition 3.4(ii))} \\ &= (z*(0*y))*x, \text{ (by Propositions 3.4(i) and 3.1(ii))} \\ &= 0*(x*(z*(0*y))). \text{ (by Proposition 3.4(i))} \end{aligned}$$

Now, by Proposition 3.1(iii), $(x*y)*z = x*(z*(0*y))$ and so we have axiom B3. Finally, by Proposition 3.1(v), we have axiom B4. Therefore, $(X,*,0)$ is a 0-commutative B-algebra. □

Corollary 3.6. *Let X be a non-empty set and "$*$" is a binary operation on X. Then the following statements are equivalent:*

(i) $(X,*,0)$ *is a basic BCI-algebra,*
(ii) $(X,*,0)$ *is a 0-commutative B-algebra,*
(iii) $(X,*,0)$ *is a p-semi simple BCI-algebra,*
(iv) $(X,*,0)$ *is an Abelian group.*

Proof. The proof comes from by Theorems 3.5 and 2.3.

□

References

[1] M. Abe and K. Iseki, *A survey on BCK and BCI-algebras* , Congeresso de Logica Aplcada Techologia, LAPTEC 2000, 431-443.
[2] T. Guo, *P-semi simple BCI-algebras Of N-type*, Mathematicae Japonica, **47**, (1998), 241-245.
[3] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV*, Proc. Japan Academy, **42** (1966) 19-22.
[4] K. Iseki, *On an algebra obtained from BCI*, Scientiae Mathematicae Japonicae **57**, (2003), 69-71.
[5] K. Iseki, *On BCI-algebras*, Seminar Math **8**, (1980), 125-130.
[6] K. Iseki, *Some fundamental theorems on BCK words*, Semigroups and Transductions, Festschrift in Hornor of Gabriel Thierrin, World Sci, (2001), 231-238.
[7] H. S. Kim and H. G. Park, *On 0-commutative B-algebras*, Math. J, (2005), 31-36.
[8] T. Lei and C. Xi, *P-radical in BCI-algebras*, Math. Japonicae, **30**,(1985), 511-517.
[9] J. Neggers and H. S. Kim, *On B-algebras* , Math. Vesnik, **54**, (2002), 21-29.

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