SOME ANALYTICAL AND STATISTICAL ASPECTS RELATED TO 2D LOGNORMAL DIFFUSION RANDOM FIELDS

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Abstract. This paper introduces 2D lognormal diffusion random fields through their transition densities and studies the main analytical and statistical characteristics of these fields, based on the theoretical formulation for diffusion random fields given in [14]. Lognormal diffusions are characterized here in terms of stochastic partial differential equations, and Kolmogorov's forward equation is obtained.

1 Introduction The formulation of dynamic models using lognormal stochastic processes has been applied in different contexts (see, for example, [18], [20], [19]). Nowadays, increased emphasis on Environmental Sciences and the formulation of dynamic models for transport of chemical or biological agents provides an additional incentive for studying lognormal diffusion random fields.

Lognormal diffusion processes (one-parameter case) have been applied based on establishing statistical inference results which allow model fitting to data obtained from discrete or continuous sampling (see for example, [4], [5], and [11]). The Itô differential equation and both Kolmogorov's forward-backward equations have been used for parameter estimation and hypothesis testing, as well as for first-passage time problems for certain time barriers, which are solved in [9], [6], and [10].

Our concern will be on the study of some analytical and statistical aspects related to 2D (two-parameter) lognormal diffusion random fields. When the parameter space is a subset of $\mathbf{R}_+^2$, [14] introduces a class of 2D random fields which are diffusions on each coordinate and satisfy a particular Markov property related to partial ordering in $\mathbf{R}_+^2$. Using this theory and taking into account that a lognormal random field is commonly introduced as a random field whose logarithm is a Gaussian random field (see, for example, [13], [2], [3]), we first study 2D Gaussian diffusion random fields. The diffusion coefficients and the one-parameter diffusions which appear fixing each coordinate of the parameter space are obtained. Moments and the stochastic partial differential equation (SPDE) for a 2D Gaussian diffusion are given too. Second, the 2D lognormal diffusion random field is introduced in terms of the transition densities. The one-parameter diffusions, the diffusion coefficients and moments are obtained. The lognormal diffusion random field is characterized by its SPDE and the forward Kolmogorov's equation is obtained and the backward Kolmogorov's equation is gave.

2 2D Gaussian Diffusion Random Field Let $\{X(\mathbf{z}) : \mathbf{z} = (s,t) \in I = [0,S] \times [0,T] \subset \mathbf{R}_+^2\}$ be a random field (RF), defined on a probability space $(\Omega, \mathcal{A}, P)$. We will use $X(\mathbf{z})$, $X(s,t)$ or X_{st} to denote the variables of the field. Suppose that the RF is constant in the axes of the parameter space, that is $X(\mathbf{z}) = X(0,0)$, $\forall \mathbf{z} \in \{(s,t) \in I : s = 0 \text{ or } t = 0\}$. We will denote

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$$X\left(\Delta_h^1(\mathbf{z})\right) = X(s+h,t) - X(s,t), \quad X\left(\Delta_k^2(\mathbf{z})\right) = X(s,t+k) - X(s,t),$$
$$X\left(\Delta_{hk}(\mathbf{z})\right) = X(s+h,t+k) - X(s+h,t) - X(s,t+k) + X(s,t),$$
$$\overline{X}\left((s_1,t_1),(s_2,t_2)\right) = \left(X(s_1,t_2), X(s_1,t_1), X(s_2,t_1)\right), \quad (s_1,t_1) \le (s_2,t_2).$$

Let consider the families of σ-fields $\{\mathfrak{F}_{\mathbf{z}} : \mathbf{z} \in \mathbf{R}_+^2\}$, $\{\mathfrak{F}_{\mathbf{z}}^1 : \mathbf{z} \in \mathbf{R}_+^2\}$, and $\{\mathfrak{F}_{\mathbf{z}}^2 : \mathbf{z} \in \mathbf{R}_+^2\}$, where for $\mathbf{z} \in \mathbf{R}_+^2$

$$\mathfrak{F}_{\mathbf{z}} = \sigma\left\{X(\mathbf{z}') : \mathbf{z}' \le \mathbf{z}\right\},$$
$$\mathfrak{F}_{\mathbf{z}}^1 = \sigma\left\{X(s',t') : s' \le s \ , \ t' \in \mathbf{R}_+\right\} = \bigvee_{t' \ge 0} \mathfrak{F}_{st'},$$
$$\mathfrak{F}_{\mathbf{z}}^2 = \sigma\left\{X(s',t') : s' \in \mathbf{R}_+ \ , \ t' \le t\right\} = \bigvee_{s' \ge 0} \mathfrak{F}_{s't}, \text{ and}$$
$$\mathfrak{F}_{\mathbf{z}}^1 \vee \mathfrak{F}_{\mathbf{z}}^2 = \sigma\left\{X(s',t') : s' \le s \text{ or } t' \le t\right\}.$$

We will assume that the following condition holds for every $\mathbf{z} \in \mathbf{R}_+^2$: $\mathfrak{F}_{\mathbf{z}}^1$ y $\mathfrak{F}_{\mathbf{z}}^2$ are conditionally independent given $\mathfrak{F}_{\mathbf{z}}$.

We first study the 2D Gaussian diffusion RF. Let $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ be a 2D Markov RF with a.s. continuous sample paths where $X(0,0)$ is a constant or a Gaussian random variable with $E[X(0,0)] = m_0$ and $var\left(X(0,0)\right) = \sigma_0^2$. We suppose, that there exist a and B, continuous functions in I, such that the transition densities exist and are given by

(1)
$$f\left(y, \mathbf{z} + (h,k) \mid \overline{x}, \mathbf{z}\right) = \frac{1}{\sqrt{2\pi\sigma^2(\mathbf{z};h,k)}} \exp\left\{-\frac{1}{2}\left(\frac{y - x_1 - x_2 + x - m(\mathbf{z};h,k)}{\sigma(\mathbf{z};h,k)}\right)^2\right\},$$

for $\mathbf{z} = (s,t) \in I$, $h,k \ge 0$, $\overline{x} = (x_1, x, x_2) \in \mathbf{R}^3$, $y \in \mathbf{R}$, with

$$m(\mathbf{z};h,k) = \int_s^{s+h}\int_t^{t+k} a(\sigma,\tau)\, d\sigma d\tau, \quad \sigma^2(\mathbf{z};h,k) = \int_s^{s+h}\int_t^{t+k} B(\sigma,\tau)\, d\sigma d\tau.$$

Under these conditions, considering $k = t$ and $t = 0$ in (1), we obtain the transition densities for the stochastic process $\{X(s,t) : s \in [0,S]\}$: For fixed $t \in [0,T]$

$$f_t(y, s+h \mid x, s) = \frac{1}{\sqrt{2\pi\sigma_1^2(\mathbf{z};h)}} \exp\left\{-\frac{1}{2}\left(\frac{y - x - m_1(\mathbf{z};h)}{\sigma_1(\mathbf{z};h)}\right)^2\right\}, \tag{2}$$

for $\mathbf{z} = (s,t) \in I$, $h \ge 0$, $x, y \in \mathbf{R}$, and

$$m_1(\mathbf{z};h) = \int_s^{s+h} a_1(\sigma,t)\, d\sigma, \quad \sigma_1^2(\mathbf{z};h) = \int_s^{s+h} B_1(\sigma,t)\, d\sigma,$$

with

$$a_1(s,t) = \int_0^t a(s,\tau)\, d\tau, \qquad B_1(s,t) = \int_0^t B(s,\tau)\, d\tau.$$

Taking into account the initial distribution (the distribution in the axes of the parameter space), the transition densities and the (one-parameter) Markov property, we can conclude that $\{X(s,t) : s \in [0,S]\}$ is a Gaussian stochastic process where $a_1(\mathbf{z})$ and $B_1(\mathbf{z})$ are the drift and diffusion coefficients, respectively.

Similarly, if we consider $h = s$ and $s = 0$ in (1), we obtain that $\{X(s,t) : t \in [0,T]\}$, for fixed $s \in [0,S]$, is a Gaussian diffusion process, with transition densities given by

$$f_s(y, t+k \mid x, t) = \frac{1}{\sqrt{2\pi\sigma_2^2(\mathbf{z};k)}} \exp\left\{-\frac{1}{2}\left(\frac{y - x - m_2(\mathbf{z};k)}{\sigma_2(\mathbf{z};k)}\right)^2\right\}, \tag{3}$$

for $\mathbf{z} = (s,t) \in I$, $k \geq 0$, x, $y \in \mathbf{R}$, and

$$m_2(\mathbf{z};k) = \int_t^{t+k} a_2(s,\tau)\, d\tau, \quad \sigma_2^2(\mathbf{z};k) = \int_t^{t+k} B_2(s,\tau)\, d\tau,$$

with

$$a_2(s,t) = \int_0^s a(\sigma,t)\, d\sigma, \qquad B_2(s,t) = \int_0^s B(\sigma,t)\, d\sigma$$

being the drift and diffusion coefficients, respectively.

From the above conditions, we can assert the following properties:

1. $X(\Delta_{hk}(\mathbf{z})) \rightsquigarrow N\left(m(\mathbf{z};h,k), \sigma^2(\mathbf{z};h,k)\right)$.
2. $X(\Delta_{hk}(\mathbf{z}))$ is independent of $X(\mathbf{z})$, of $X(s+h,t)$ and of $X(s,t+k)$.
3. $X\left(\Delta_h^1(\mathbf{z})\right) \rightsquigarrow N\left(m_1(\mathbf{z};h), \sigma_1^2(\mathbf{z};h)\right)$.
4. $X\left(\Delta_k^2(\mathbf{z})\right) \rightsquigarrow N\left(m_2(\mathbf{z};k), \sigma_2^2(\mathbf{z};k)\right)$.
5. $X\left(\Delta_h^1(\mathbf{z})\right)$ and $X(\mathbf{z})$ are independent.
6. $X\left(\Delta_k^2(\mathbf{z})\right)$ and $X(\mathbf{z})$ are independent.

Theorem 2.1. $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a Gaussian diffusion RF, with drift coefficient $a(\mathbf{z})$, diffusion coefficient $B(\mathbf{z})$, and with the mixed diffusion coefficients being all null.

Proof See Appendix A.

2.1 SPDE for the 2D Gaussian diffusion RF Under some regularity conditions, it is possible to obtain a SPDE formulation for the 2D diffusion RF. In fact, we need the hypotheses I to V stated in [14] to be satisfied, in order to apply the Theorem 2.8 established in that paper. These hypotheses will be proved for the lognormal diffusion RF (see Appendix B). The proof for the Gaussian case can be carried out in a similar way. Then, applying directly the result, we can conclude that there exists a 2D Wiener process $\{W(\mathbf{z}) : \mathbf{z} \in I\}$ (adjoining, if it is necessary, a new probability space) such that the Gaussian diffusion RF $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is the only diffusion RF satisfying the SPDE

$$\frac{\partial^2 X_{st}}{\partial s \partial t} = a(s,t) + B^{1/2}(s,t)\frac{\partial^2 W_{st}}{\partial s \partial t}.$$

The uniqueness of the solution is the result of applying Theorem 3.9 of Yeh (see [16], p. 282).

3 2D Lognormal Diffusion Random Field We will consider in the following development $\{X(\mathbf{z}) : \mathbf{z} \in I = [0,S] \times [0,T]\}$ to be a positive-valued 2D Markov RF with a.s. continuous sample paths, where $X(0,0)$ is a constant or a lognormal random variable with

$E[\ln X(0,0)] = m_0$ and $var(\ln X(0,0)) = \sigma_0^2$. We suppose that there exist $\tilde{a}$ and $\tilde{B}$, continuous functions in I, such that the transition densities are given by

$$g(y, \mathbf{z} + (h,k) \mid \overline{x}, \mathbf{z}) = \frac{1}{y\sqrt{2\pi\sigma^2(\mathbf{z}; h,k)}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln\left(\frac{yx}{x_1 x_2}\right) - m(\mathbf{z}; h,k)}{\sigma(\mathbf{z}; h,k)} \right)^2 \right\}, \tag{4}$$

for $\mathbf{z} = (s,t) \in I$, $h, k \geq 0$, $\overline{x} = (x_1, x, x_2) \in \mathbf{R}_+^3$, $y \in \mathbf{R}_+$, and

$$m(\mathbf{z}; h,k) = \int_s^{s+h} \int_t^{t+k} \tilde{a}(\sigma,\tau)\, d\sigma d\tau, \quad \sigma^2(\mathbf{z}; h,k) = \int_s^{s+h} \int_t^{t+k} \tilde{B}(\sigma,\tau)\, d\sigma d\tau.$$

The RF $\{Y(\mathbf{z}) : \mathbf{z} \in I\}$ defined as $Y(\mathbf{z}) = \ln X(\mathbf{z})$ has a transition density given by (1). Therefore, $\{Y(\mathbf{z}) : \mathbf{z} \in I\}$ is a Gaussian diffusion RF, with $\tilde{a}(\mathbf{z})$ and $\tilde{B}(\mathbf{z})$ being the drift and diffusion coefficients, respectively. Thus, we can conclude that $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a lognormal RF. We will prove next that it is also a diffusion RF.

We denote $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{B}_1$ and $\tilde{B}_2$ the one-parameter diffusion coefficients of Y and $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{d}$ its mixed diffusions coefficients. Using the results of the previous section we can conclude that

$$\begin{aligned}
\tilde{a}_1(s,t) &= \int_0^t \tilde{a}(s,\tau)\, d\tau, \qquad \tilde{B}_1(s,t) = \int_0^t \tilde{B}(s,\tau)\, d\tau, \qquad (s,t) \in I, \\
\tilde{a}_2(s,t) &= \int_0^s \tilde{a}(\sigma,t)\, d\sigma, \qquad \tilde{B}_2(s,t) = \int_0^s \tilde{B}(\sigma,t)\, d\sigma, \qquad (s,t) \in I, \\
\tilde{c}_1(s,t) &= \tilde{c}_2(s,t) = \tilde{d}(s,t) = 0, \qquad (s,t) \in I.
\end{aligned}$$

On the other hand, denoting a_1, a_2, B_1 and B_2 the one-parameter diffusion coefficient for X, we have

$$\begin{aligned}
a_1(\mathbf{z})\, x &= \left(\tilde{a}_1(\mathbf{z}) + \tfrac{1}{2}\tilde{B}_1(\mathbf{z})\right) x, \qquad B_1(\mathbf{z})\, x^2 = \tilde{B}_1(\mathbf{z})\, x^2, \\
a_2(\mathbf{z})\, x &= \left(\tilde{a}_2(\mathbf{z}) + \tfrac{1}{2}\tilde{B}_2(\mathbf{z})\right) x, \qquad B_2(\mathbf{z})\, x^2 = \tilde{B}_2(\mathbf{z})\, x^2,
\end{aligned} \tag{5}$$

and the infinitesimal moments of order larger than two are all null. In fact,

$$\begin{aligned}
\frac{E\left[X\left(\Delta_h^1(\mathbf{z})\right) \mid X(\mathbf{z}) = x\right]}{h} &= \frac{x}{h}\left(E\left[e^{Y\left(\Delta_h^1(\mathbf{z})\right)} \mid Y(\mathbf{z}) = \ln x\right] - 1\right) \\
&= \frac{x}{h}\left(E\left[e^{Y\left(\Delta_h^1(\mathbf{z})\right)}\right] - 1\right) \\
&= \frac{x}{h}\left(\exp\left\{\int_s^{s+h} \tilde{a}_1(\sigma,t)\, d\sigma + \frac{1}{2}\int_s^{s+h} \tilde{B}_1(\sigma,t)\, d\sigma, \right\} - 1\right),
\end{aligned}$$

using that $Y\left(\Delta_h^1(\mathbf{z})\right)$ is independent of $Y(\mathbf{z})$, and the distribution of $Y\left(\Delta_h^1(\mathbf{z})\right)$. Taking the limit when $h \to 0$,

$$a_1(\mathbf{z}, x) = x\left(\tilde{a}_1(\mathbf{z}) + \frac{1}{2}\tilde{B}_1(\mathbf{z})\right) = x a_1(\mathbf{z}).$$

The proof is similar for the remaining coefficients. We can then conclude that $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a lognormal diffusion RF.

If we consider $k = t$ and $t = 0$ in (4), the transition densities for the lognormal diffusion RF $\{X(s,t) : s \in [0,S]\}$, $t \in [0,T]$ fixed, are given by

$$g_t(y, s+h \mid x, s) = \frac{1}{y\sqrt{2\pi\sigma_1^2(\mathbf{z};h)}} \exp\left\{-\frac{1}{2}\left(\frac{\ln\left(\frac{y}{x}\right) - m_1(\mathbf{z};h)}{\sigma_1(\mathbf{z};h)}\right)^2\right\},$$

for $\mathbf{z} = (s,t) \in I$, $h \geq 0$, x, $y \in \mathbf{R}_+$, and

$$m_1(\mathbf{z};h) = \int_s^{s+h} \tilde{a}_1(\sigma,t)\, d\sigma, \quad \sigma_1^2(\mathbf{z};h) = \int_s^{s+h} \tilde{B}_1(\sigma,t)\, d\sigma.$$

Similarly, if we consider $h = s$ and $s = 0$ in (4), the transition densities for the lognormal diffusion RF $\{X(s,t) : t \in [0,T]\}$, $s \in [0,S]$ fixed, are given by

$$g_s(y, t+k \mid x, t) = \frac{1}{y\sqrt{2\pi\sigma_2^2(\mathbf{z};k)}} \exp\left\{-\frac{1}{2}\left(\frac{\ln\left(\frac{y}{x}\right) - m_2(\mathbf{z};k)}{\sigma_2(\mathbf{z};k)}\right)^2\right\},$$

where $\mathbf{z} = (s,t) \in I$, $k \geq 0$, $x, y \in \mathbf{R}_+$, and

$$m_2(\mathbf{z};h) = \int_t^{t+k} \tilde{a}_2(s,\tau)\, d\tau, \quad \sigma_2^2(\mathbf{z};k) = \int_t^{t+k} \tilde{B}_2(s,\tau)\, d\tau.$$

3.1 Two-parameter diffusion coefficients Hypotheses I to IV stated in [14] hold (see Appendix B) and applying Proposition 2.4 established in that paper, the two-parameter diffusion coefficients for the lognormal and Gaussian diffusion RFs could be obtained in terms of the one-parameter diffusion coefficients. That is, the two-parameter diffusion coefficients of the lognormal diffusion RF $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ are given by

$$\begin{aligned}
a(\mathbf{z},x) &= a(\mathbf{z})\,x, \text{ with } a(\mathbf{z}) = \frac{\partial a_1(\mathbf{z})}{\partial t} + a_1(\mathbf{z})\,a_2(\mathbf{z}) = \frac{\partial a_2(\mathbf{z})}{\partial s} + a_1(\mathbf{z})\,a_2(\mathbf{z}) \\
B(\mathbf{z},x) &= B(\mathbf{z})\,x^2, \text{ with } B(\mathbf{z}) = \frac{\partial B_1(\mathbf{z})}{\partial t} + B_1(\mathbf{z})\,B_2(\mathbf{z}) = \frac{\partial B_2(\mathbf{z})}{\partial s} + B_1(\mathbf{z})\,B_2(\mathbf{z}), \\
c_1(\mathbf{z},x) &= c_1(\mathbf{z})\,x^2, \text{ with } c_1(\mathbf{z}) = a_2(\mathbf{z})\,B_1(\mathbf{z}), \\
c_2(\mathbf{z},x) &= c_2(\mathbf{z})\,x^2, \text{ with } c_2(\mathbf{z}) = a_1(\mathbf{z})\,B_2(\mathbf{z}), \text{ and} \\
d(\mathbf{z},x) &= d(\mathbf{z})\,x^3, \text{ with } d(\mathbf{z}) = B_1(\mathbf{z})\,B_2(\mathbf{z}),
\end{aligned}$$
(6)

and the two-parameter diffusion coefficients of the Gaussian diffusion RF $\{Y(\mathbf{z}) : \mathbf{z} \in I\}$ are given by

$$\begin{aligned}
&\tilde{a}(\mathbf{z},y) = \tilde{a}(\mathbf{z}) = a(\mathbf{z}) - a_1(\mathbf{z})\,a_2(\mathbf{z}) - \frac{1}{2}\left[B(\mathbf{z}) - B_1(\mathbf{z})\,B_2(\mathbf{z})\right], \\
&\tilde{B}(\mathbf{z},y) = \tilde{B}(\mathbf{z}) = B(\mathbf{z}) - B_1(\mathbf{z})\,B_2(\mathbf{z}), \text{ and} \\
&\tilde{c}_1 = \tilde{c}_2 = \tilde{d} = 0.
\end{aligned} \tag{7}$$

3.2 Moments of the 2D lognormal diffusion RF The moments of a 2D lognormal diffusion are useful in the derivation of the forward equation. We can calculate them using the moment generating function of the Gaussian diffusion RF. Denoting $\mathbf{z}_0 = (s_0, t_0)$, $\mathbf{z} = (s,t)$, $\overline{X}(\mathbf{z}_0, \mathbf{z}) = (X(s_0,t),\ X(\mathbf{z}_0),\ X(s,t_0))$ and $\overline{Y}(\mathbf{z}_0, \mathbf{z}) = (\ln X(s_0,t),\ \ln X(\mathbf{z}_0),$

$\ln X(s,t_0))$, and using that the distribution of $Y(\mathbf{z}) = \ln X(\mathbf{z})$ conditional to $\overline{y} = (y_1, y, y_2)$ is $N\left(m_{\mathbf{z}_0\,\mathbf{z}}(\overline{y}), \sigma^2(\mathbf{z}_0, \mathbf{z})\right)$, with

$$m_{\mathbf{z}_0\,\mathbf{z}}(\overline{y}) = y_1 + y_2 - y + \int_{s_0}^{s}\int_{t_0}^{t} \tilde{a}(\sigma,\tau)\,d\sigma d\tau = \ln\left(\frac{x_1 x_2}{x}\right) + \int_{s_0}^{s}\int_{t_0}^{t} \tilde{a}(\sigma,\tau)\,d\sigma d\tau,$$

$$\sigma^2(\mathbf{z}_0,\mathbf{z}) = \int_{s_0}^{s}\int_{t_0}^{t} \tilde{B}(\sigma,\tau)\,d\sigma d\tau,$$

we have

$$E\left[X^k(\mathbf{z}) \mid \overline{X}(\mathbf{z}_0,\mathbf{z})\right] = E\left[e^{k\ln X(\mathbf{z})} \mid \overline{X}(\mathbf{z}_0,\mathbf{z})\right]$$

$$(8)\quad = \left[\frac{X(s_0,t)\,X(s,t_0)}{X(\mathbf{z}_0)}\right]^k \exp\left\{k\int_{s_0}^{s}\int_{t_0}^{t} \tilde{a}(\sigma,\tau)\,d\sigma d\tau + \frac{k^2}{2}\int_{s_0}^{s}\int_{t_0}^{t} \tilde{B}(\sigma,\tau)\,d\sigma d\tau\right\}.$$

3.3 SPDE for the 2D lognormal diffusion RF Hypotheses I to V stated in [14] hold (see Appendix B). Therefore we can apply the Theorem 2.8 established in that paper to obtain the SPDE for X and conclude that there exists a 2D Wiener RF $\{W(\mathbf{z}) : \mathbf{z} \in I\}$ (adjoining, if it is necessary, a new probability space) such that the lognormal diffusion RF $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is the only diffusion RF satisfying the following SPDE:

$$\frac{\partial^2 X_{st}}{\partial s \partial t} - X^{-1}(\mathbf{z})\frac{\partial X_{st}}{\partial s}\frac{\partial X_{st}}{\partial t} - \frac{\partial a_2(s,t)}{\partial s}X_{st} = \left(\frac{\partial B_2(s,t)}{\partial s} + B_1(s,t)\,B_2(s,t)\right)^{1/2} X_{st}\frac{\partial^2 W_{st}}{\partial s \partial t}.$$

The uniqueness of the solution is the result of observing that $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a bijective transformation ($X(\mathbf{z}) = \exp Y(\mathbf{z})$) of a diffusion RF $\{Y(\mathbf{z}) : \mathbf{z} \in I\}$ which satisfies a SPDE with a unique solution (see the proof of Theorem 4.3 of [12]).

3.4 Kolmogorov's forward-backward equations for the lognormal diffusion RF In this section, for a 2D lognormal diffusion RF, the Kolmogorov's forward equation is obtained and the Kolmogorov's backward equation is given (see [17] for diffusion processes).

Proposition 3.1. The transition densities of the lognormal diffusion RF $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ satisfy the following Kolmogorov's forward equation:

$$\frac{\partial^2 g}{\partial s \partial t} = \mu_1(\mathbf{z})\frac{\partial(yg)}{\partial y} + \mu_2(\mathbf{z})\frac{\partial^2\left(y^2 g\right)}{\partial y^2} + \mu_3(\mathbf{z})\frac{\partial^3\left(y^3 g\right)}{\partial y^3} + \mu_4(\mathbf{z})\frac{\partial^4\left(y^4 g\right)}{\partial y^4},$$

where $\mu_i(\mathbf{z})$ are given by

$$\begin{aligned}
\mu_1(\mathbf{z}) &= -a(\mathbf{z}) = -\frac{\partial a_1(\mathbf{z})}{\partial t} - a_1(\mathbf{z})\,a_2(\mathbf{z}),\\
\mu_2(\mathbf{z}) &= a_1(\mathbf{z})\,a_2(\mathbf{z}) + \frac{1}{2}\left(\frac{\partial B_1(\mathbf{z})}{\partial t} + B_1(\mathbf{z})\,B_2(\mathbf{z})\right) + a_1(\mathbf{z})\,B_2(\mathbf{z}) + B_1(\mathbf{z})\,a_2(\mathbf{z}),\\
\mu_3(\mathbf{z}) &= -\frac{1}{2}a_1(\mathbf{z})\,B_2(\mathbf{z}) - \frac{1}{2}B_1(\mathbf{z})\,a_2(\mathbf{z}) - B_1(\mathbf{z})\,B_2(\mathbf{z}),\\
\mu_4(\mathbf{z}) &= \frac{1}{4}B_1(\mathbf{z})\,B_2(\mathbf{z}).
\end{aligned}$$

(9)

Proof See Appendix C.

In a similar way, the Kolmogorov's backward equation can be obtained for a 2D lognormal diffusion RF.

Proposition 3.2. The transition densities of the lognormal diffusion RF $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ satisfy the following Kolmogorov's backward equation:

$$\begin{aligned}\frac{\partial^2 g}{\partial s_0 \partial t_0} &= \lambda_1 \left\{ x\frac{\partial g}{\partial x} + x_1\frac{\partial g}{\partial x_1} + x_2\frac{\partial g}{\partial x_2} \right\} \\ &+\lambda_2 \left\{ x^2\frac{\partial^2 g}{\partial x^2} + x_1^2\frac{\partial^2 g}{\partial x_1^2} + x_2^2\frac{\partial^2 g}{\partial x_2^2} + 2xx_1\frac{\partial^2 g}{\partial x \partial x_1} + 2xx_2\frac{\partial^2 g}{\partial x \partial x_2} \right\} \\ &+\lambda_3 x_1 x_2 \frac{\partial^2 g}{\partial x_1 \partial x_2} + \lambda_4 x_1^2 x_2 \frac{\partial^3 g}{\partial x_1^2 \partial x_2} + \lambda_5 x_1 x_2^2 \frac{\partial^3 g}{\partial x_1 \partial x_2^2} + \lambda_6 x_1^2 x_2^2 \frac{\partial^4 g}{\partial x_1^2 \partial x_2^2},\end{aligned}$$

where

$$\begin{aligned}\lambda_1 &= \tilde{a}(z_0) + \frac{1}{2}\tilde{B}(z_0), \qquad \lambda_2 = \frac{1}{2}\tilde{B}(z_0), \\ \lambda_3 &= \tilde{B}(z_0) + \left(\int_{t_0}^{t} \tilde{a}(s_0,\tau)\,d\tau + \frac{1}{2}\int_{t_0}^{t} \tilde{B}(s_0,\tau)\,d\tau\right) \\ &\quad \times \left(\int_{s_0}^{s} \tilde{a}(\sigma,t_0)\,d\sigma + \frac{1}{2}\int_{s_0}^{s} \tilde{B}(\sigma,t_0)\,d\sigma\right), \\ \lambda_4 &= \frac{1}{2}\left(\int_{t_0}^{t} \tilde{B}(s_0,\tau)\,d\tau\right)\left(\int_{s_0}^{s} \tilde{a}(\sigma,t_0)\,d\sigma + \frac{1}{2}\int_{s_0}^{s} \tilde{B}(\sigma,t_0)\,d\sigma\right), \\ \lambda_5 &= \frac{1}{2}\left(\int_{s_0}^{s} \tilde{B}(\sigma,t_0)\,d\sigma\right)\left(\int_{t_0}^{t} \tilde{a}(s_0,\tau)\,d\tau + \frac{1}{2}\int_{t_0}^{t} \tilde{B}(s_0,\tau)\,d\tau\right), \\ \lambda_6 &= \frac{1}{4}\left(\int_{t_0}^{t} \tilde{B}(s_0,\tau)\,d\tau\right)\left(\int_{s_0}^{s} \tilde{B}(\sigma,t_0)\,d\sigma\right).\end{aligned}$$

4 Conclusions In this paper, we give the theoretical formulations for a 2D lognormal diffusion RF based on the definition of diffusion RF introduced by [14]. Such a model can be useful to describe, predict and simulate real phenomena like transport of pollutant agents in environmental studies.

This investigation is continued with the study of techniques for estimation and prediction of 2D lognormal diffusion RFs, using exogenous factors in the formulation of the drift and the diffusion coefficients (see [7]). An extension for a 2D lognormal diffusion RF with non-constant values at the boundary axes (see [8]) is also under consideration. The investigation is also addressed to introduce spatio-temporal formulation in this context.

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Appendix

A Proof of Theorem 2.1 It is clear that $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a diffusion RF. Let (s_1,t_1), (s_2,t_2), ...,$(s_n,t_n) \in I$, with $s_1 \leq s_2 \leq ... \leq s_n \in [0,S]$, $t_1,t_2,...,t_n \in [0,T]$. We next prove that the distribution of $(X(s_1,t_1), X(s_2,t_2), ..., X(s_n,t_n))^T$ is Gaussian, by induction on $n := \textit{cardinal}\{s_1,s_2,...,s_n\}$. If $n=1$, then $s_1 = s_2 = ... = s_n := s$ and the result is straightforward from the one-parameter case. If $n=2$, then the random vector has the form $(X(s_1,t_1),...,X(s_1,t_j),X(s_2,t_{j+1}),...,X(s_2,t_n))^T$. We will suppose $t_1 \le ... \le t_j$ and $t_{j+1} \le ... \le t_n$. We denote $h_p := s_p - s_{p-1}$, $k_p := t_p - t_{p-1}$, $p = 2,...,n$. For simplicity, when $n = 2$ we denote $h := s_2 - s_1$. The characteristic function of the random vector $(X(s_1,t_1),...,X(s_1,t_j), X(s_2,t_{j+1}), ...,X(s_2,t_n))^T$ is given by

$$\begin{aligned}
&E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+u_{j+1}X(s_2,t_{j+1})+...+u_nX(s_2,t_n))}\right] \\
&\quad= E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+u_{j+1}X(s_2,t_{j+1})+...+u_{n-1}X(s_2,t_{n-1}))}\right. \\
&\qquad\qquad \left.\times E\left[e^{iu_nX(s_2,t_n)} \mid \mathfrak{F}^1_{(s_1,t_{n-1})} \vee \mathfrak{F}^2_{(s_1,t_{n-1})}\right]\right] \\
&\quad= E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+u_{j+1}X(s_2,t_{j+1})+...+(u_{n-1}+u_n)X(s_2,t_{n-1}))}\right. \\
&\qquad\qquad \times e^{i(u_nX(s_1,t_n)-u_nX(s_1,t_{n-1}))} \\
&\qquad\qquad \left.\times E\left[e^{iu_n(X(\Delta_{hk_n}(s_1,t_{n-1})))} \mid \overline{X}((s_1,t_{n-1}),(s_2,t_n))\right]\right] \\
&\quad= E\left[e^{iu_n(X(\Delta_{hk_n}(s_1,t_{n-1})))}\right] \\
&\qquad \times E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+u_{j+1}X(s_2,t_{j+1})+...+(u_{n-1}+u_n)X(s_2,t_{n-1}))}\right. \\
&\qquad\qquad \left.\times\, e^{i(u_nX(s_1,t_n)-u_nX(s_1,t_{n-1}))}\right],
\end{aligned}$$

where we have used that $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a Markov RF and that $X(\Delta_{hk_n}(s_1,t_{n-1}))$ is independent of $\overline{X}((s_1,t_{n-1}),(s_2,t_n))$. Repeating the previous calculations $n-j-2$ times, we can obtain

$$\begin{aligned}
&E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+u_{j+1}X(s_2,t_{j+1})+...+u_nX(s_n,t_n))}\right] \\
&\quad= E\left[e^{iu_n(X(\Delta_{hk_n}(s_1,t_{n-1})))}\right] E\left[e^{i(u_n+u_{n-1})X\left(\Delta_{hk_{n-1}}(s_1,t_{n-2})\right)}\right]\cdots \\
&\qquad \times E\left[e^{i(u_n+...+u_{j+2})X\left(\Delta_{hk_{j+2}}(s_1,t_{j+1})\right)}\right] \\
&\qquad \times E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+(u_{j+1}+...+u_n)X(s_2,t_{j+1})-(u_{j+2}+...+u_n)X(s_1,t_{j+1}))}\right. \\
&\qquad\qquad \left.\times e^{i(u_nX(s_1,t_n)+u_{n-1}X(s_1,t_{n-1})+...+u_{j+2}X(s_1,t_{j+2}))}\right] \\
&\quad= E\left[e^{iu_n(X(\Delta_{hk_n}(s_1,t_{n-1})))}\right] E\left[e^{i(u_n+u_{n-1})X\left(\Delta_{hk_{n-1}}(s_1,t_{n-2})\right)}\right]\cdots \\
&\qquad \times E\left[e^{i(u_n+...+u_{j+2})X\left(\Delta_{hk_{j+2}}(s_1,t_{j+1})\right)}\right] \\
&\qquad \times E\left[E\left[e^{i(u_1X(s_1,t_1)+...+u_jX(s_1,t_j)+(u_{j+1}+...+u_n)X(s_2,t_{j+1})-(u_{j+2}+...+u_n)X(s_1,t_{j+1}))}\right.\right. \\
&\qquad\qquad \left.\left.\times e^{i(u_nX(s_1,t_n)+u_{n-1}X(s_1,t_{n-1})+...+u_{j+2}X(s_1,t_{j+2}))} \mid \mathfrak{F}^1_{(s_1,0)} \vee \mathfrak{F}^2_{(s_1,0)}\right]\right] \\
&\quad= E\left[e^{iu_n(X(\Delta_{hk_n}(s_1,t_{n-1})))}\right] E\left[e^{i(u_n+u_{n-1})X\left(\Delta_{hk_{n-1}}(s_1,t_{n-2})\right)}\right]\cdots
\end{aligned}$$

$$\times E\left[e^{i(u_n+\ldots+u_{j+2})X\left(\Delta_{hk_{j+2}}(s_1,t_{j+1})\right)}\right]$$
$$\times E\left[e^{i(u_1X(s_1,t_1)+\ldots+u_jX(s_1,t_j)+u_{j+1}X(s_1,t_{j+1})+u_{j+2}X(s_1,t_{j+2})+\ldots+u_nX(s_1,t_n))}\right.$$
$$\left.\times E\left[e^{i(u_{j+1}+\ldots+u_n)(X(s_2,t_{j+1})-X(s_1,t_{j+1}))}\mid X\left(s_1,t_{j+1}\right)\right]\right]$$
$$= E\left[e^{iu_n(X(\Delta_{hk_n}(s_1,t_{n-1})))}\right]E\left[e^{i(u_n+u_{n-1})X\left(\Delta_{hk_{n-1}}(s_1,t_{n-2})\right)}\right]\cdots$$
$$\times E\left[e^{iu(u_n+\ldots+u_{j+2})X\left(\Delta_{hk_{j+2}}(s_1,t_{j+1})\right)}\right]E\left[e^{i(u_n+\ldots+u_{j+1})\left(X\left(\Delta_h^1(s_1,t_{j+1})\right)\right)}\right]$$
$$\times E\left[e^{i(u_1X(s_1,t_1)+\ldots+u_jX(s_1,t_j)+u_{j+1}X(s_1,t_{j+1})+u_{j+2}X(s_1,t_{j+2})+\ldots+u_nX(s_1,t_n))}\right],$$

Using the expressions of the characteristics functions for the increments, applying the case $n=1$ and taking into account that $t_1\le\ldots\le t_j$ and $t_{j+1}\le\ldots\le t_n$, it is easy to conclude the result. When *cardinal* $\{s_1,s_2,\ldots,s_n\}=n$, there exists a single point with coordinate s_n on the axis OX, that we have denoted (s_n,t_n). In this case the characteristic function of $(X(s_1,t_1),X(s_2,t_2),\ldots,X(s_n,t_n))^T$ is

$$E\left[e^{i(u_1X(s_1,t_1)+\ldots+u_{n-1}X(s_{n-1},t_{n-1})+u_nX(s_n,t_n))}\right]$$
$$= E\left[E\left[e^{i(u_1X(s_1,t_1)+\ldots+u_{n-1}X(s_{n-1},t_{n-1})+u_nX(s_n,t_n))}\mid \mathfrak{F}^1_{(s_{n-1},0)}\vee\mathfrak{F}^2_{(s_{n-1},0)}\right]\right]$$
$$= E\left[e^{i(u_1X(s_1,t_1)+\ldots+u_{n-1}X(s_{n-1},t_{n-1})+u_nX(s_{n-1},t_n))}\right.$$
$$\left.\times E\left[e^{iu_n(X(s_n,t_n)-X(s_{n-1},t_n))}\mid X\left(s_{n-1},t_n\right)\right]\right]$$
$$= E\left[e^{iu_nX\left(\Delta^1_{h_n}(s_{n-1},t_n)\right)}\right]E\left[e^{i(u_1X(s_1,t_1)+\ldots+u_{n-1}X(s_{n-1},t_{n-1})+u_nX(s_{n-1},t_n))}\right]$$

where we have used the fact that $\{X(s,t):s\in[0,S]\}$ is a Markov process and that $X\left(\Delta_h^1(\mathbf{z})\right)$ is independent of $X(\mathbf{z})$. Finally, using the expression of characteristic function of the increment, applying the induction hypothesis and taking into account that $s_1<s_2<\ldots<s_n$, we can deduce that the distribution of $(X(s_1,t_1),X(s_2,t_2),\ldots,X(s_n,t_n))^T$ is Gaussian.

B Hypothesis I to V stated in [14] hold For obtaining the two-parameter diffusion coefficients in terms of the one-parameter diffusion coefficients, it is necessary to prove the hypotheses I to IV stated in [14]. And for obtaining the SPDE for a 2D diffusion RF an extra hypothesis, hypothesis V stated in [14], is also needed. In this appendix we prove that a 2D lognormal diffusion RF satisfies these hypotheses. The proof for the Gaussian diffusion RF can be carried out in a similar way. Here, the distribution function of the law $N(0,1)$ will be denoted by Φ.

Hypothesis I holds: It is obvious that a_1, a_2, B_1 and B_2 have continuous partial derivatives with respect to s and t and are fourth continuously differentiable in x. Thus, the hypothesis I holds.

Hypothesis II holds: We prove that the condition

$$\int_{|y-x|>\varepsilon}g_t\left(y,s+h\mid x,s\right)dy=o\left(h\right)$$

is satisfied uniformly with respect to $(s,t) \in I$. Indeed,

$$\begin{aligned} f_{\varepsilon;h}(\mathbf{z}) &:= \int_{|y-x|>\varepsilon} g_t(y, s+h \mid x, s)\, dy = P\left[\left|X\left(\Delta_h^1(\mathbf{z})\right)\right| > \varepsilon \;\mid\; X(\mathbf{z}) = x\right] \\ &= P\left[X\left(\Delta_h^1(\mathbf{z})\right) \le -\varepsilon \;\mid\; X(\mathbf{z}) = x\right] + P\left[X\left(\Delta_h^1(\mathbf{z})\right) \ge \varepsilon \;\mid\; X(\mathbf{z}) = x\right] \\ &= P\left[Y\left(\Delta_h^1(\mathbf{z})\right) \le \ln\left(1 - \frac{\varepsilon}{x}\right)\right] + P\left[Y\left(\Delta_h^1(\mathbf{z})\right) \ge \ln\left(1 + \frac{\varepsilon}{x}\right)\right] \\ &= 1 + \Phi\left(\frac{\ln\left(1 - \frac{\varepsilon}{x}\right) - m_1(\mathbf{z};h)}{\sigma_1(\mathbf{z};h)}\right) - \Phi\left(\frac{\ln\left(1 + \frac{\varepsilon}{x}\right) - m_1(\mathbf{z};h)}{\sigma_1(\mathbf{z};h)}\right) \end{aligned}$$

where we have used that $1 - \frac{\varepsilon}{x} \ge 0$, since $|y-x| > \varepsilon$, $Y\left(\Delta_h^1(\mathbf{z})\right)$ is independent of $Y(\mathbf{z})$, and the distribution of $Y\left(\Delta_h^1(\mathbf{z})\right)$. Since the functions m_1, σ_1 and Φ are continuous in a compact I, there exists $\mathbf{z}_1 \in I$ such that

$$\max_{\mathbf{z}\in I} f_{\varepsilon;h}(\mathbf{z}) = f_{\varepsilon;h}(\mathbf{z}_1) = o(h).$$

The condition $\int_{|y-x|>\varepsilon} g_s(y, t+k \mid x, t)\, dy = o(k)$ can be similarly proved to hold uniformly with respect to $(s,t) \in I$.

Let us now consider

$$\begin{aligned} f_{\varepsilon;h,k}(\mathbf{z}) &= \int_{|y-x_1-x_2+x|>\varepsilon} g\left(y, (s+h, t+k) \mid (x_1, x, x_2), (s,t)\right) \\ &\qquad \times g_t(x_2, s+h \mid x, s)\, g_s(x_1, t+k \mid x, t)\, dx_1 dx_2 dy \\ &= P\left[\left|X\left(\Delta_{hk}(\mathbf{z})\right)\right| > \varepsilon \mid X(\mathbf{z}) = x\right]. \end{aligned}$$

Since $\mathfrak{F}_{\mathbf{z}} \subset \mathfrak{F}_{\mathbf{z}}^1 \vee \mathfrak{F}_{\mathbf{z}}^2$,

$$\begin{aligned} P\left[\left|X\left(\Delta_{hk}(\mathbf{z})\right)\right| > \varepsilon \mid X(\mathbf{z})\right] &= P\left[\left|X\left(\Delta_{hk}(\mathbf{z})\right)\right| > \varepsilon \mid \mathfrak{F}_{\mathbf{z}}\right] \\ &= E\left[P\left[\left|X\left(\Delta_{hk}(\mathbf{z})\right)\right| > \varepsilon \mid \mathfrak{F}_{\mathbf{z}}^1 \vee \mathfrak{F}_{\mathbf{z}}^2\right] \;\mid\; \mathfrak{F}_{\mathbf{z}}\right]. \end{aligned}$$

Taking into account that $Y(\Delta_{hk}(\mathbf{z}))$ is independent of $\overline{Y}(\mathbf{z}, (s+h, t+k))$, the distribution of $Y(\Delta_{hk}(\mathbf{z}))$, the distributions of $X\left(\Delta_h^1(\mathbf{z})\right)$ and $X\left(\Delta_k^2(\mathbf{z})\right)$ we have, after some simplifications,

$$\begin{aligned} f_{\varepsilon;h,k} = 1 + \int_{\mathbf{R}^2} &\left[\Phi\left(\frac{\ln\left(\frac{-\varepsilon + e^{\sigma_1(\mathbf{z};h)u + m_1(\mathbf{z};h)}x + e^{\sigma_2(\mathbf{z};k)u + m_2(\mathbf{z};k)}x}{e^{\sigma_1(\mathbf{z};h)u + m_1(\mathbf{z};h)}e^{\sigma_2(\mathbf{z};k)u + m_2(\mathbf{z};k)}x}\right) - m(\mathbf{z};h,k)}{\sigma^2(\mathbf{z};h,k)}\right)\right. \\ &\left. -\Phi\left(\frac{\ln\left(\frac{\varepsilon + e^{\sigma_1(\mathbf{z};h)u + m_1(\mathbf{z};h)}x + e^{\sigma_2(\mathbf{z};k)u + m_2(\mathbf{z};k)}x}{e^{\sigma_1(\mathbf{z};h)u + m_1(\mathbf{z};h)}e^{\sigma_2(\mathbf{z};k)u + m_2(\mathbf{z};k)}x}\right) - m(\mathbf{z};h,k)}{\sigma^2(\mathbf{z};h,k)}\right)\right] e^{-\frac{u^2+v^2}{2}} dudv. \end{aligned}$$

Therefore, $f_{\varepsilon;h,k}(\mathbf{z})$ is continuous in the compact I, and hence its extreme values lie in I, that is, there exists $\mathbf{z}_1 \in I$ such that $\max_{\mathbf{z}\in I} f_{\varepsilon;h,k}(\mathbf{z}) = f_{\varepsilon;h,k}(\mathbf{z}_1) = o(hk)$. This completes the proof of hypothesis II.

Hypothesis III holds: **(a)** To establish the inequality

$$\left|\int_{|y-x|\le\varepsilon} (y-x)\, g_t(y, s+h \mid x, s)\, dy\right| + \int_{|y-x|\le\varepsilon} (y-x)^2 g_t(y, s+h \mid x, s)\, dy \le l'h \tag{10}$$

we will find a bound for each integral separately.

(a.1) Using the transition densities, making the change of variable $u = \ln\left(\frac{y}{x}\right)$, we have

$$\begin{aligned}
&\int_{|y-x|\leq\varepsilon} (y-x)\, g_t\left(y, s+h \mid x, s\right) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} \int_{|y-x|\leq\varepsilon} \frac{(y-x)}{y} \exp\left\{-\frac{1}{2}\left(\frac{\ln\left(\frac{y}{x}\right) - m_1\left(\mathbf{z};h\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)^2\right\} dy \\
&= \frac{x}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} \left(e^u - 1\right) \exp\left\{-\frac{1}{2}\left(\frac{u - m_1\left(\mathbf{z};h\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)^2\right\} du \\
&= \frac{x}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} \exp\left\{u - \frac{1}{2}\left(\frac{u - m_1\left(\mathbf{z};h\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)^2\right\} du \\
&\quad - \frac{x}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} \exp\left\{-\frac{1}{2}\left(\frac{u - m_1\left(\mathbf{z};h\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)^2\right\} du \\
&= x\left[\exp\left\{\frac{\sigma_1^2\left(\mathbf{z};h\right)}{2} + m_1\left(\mathbf{z};h\right)\right\}\right. \\
&\quad \times\left\{\Phi\left(\frac{\ln\left(1+\frac{\varepsilon}{x}\right) - \left(m_1\left(\mathbf{z};h\right) + \sigma_1^2\left(\mathbf{z};h\right)\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)\right. \\
&\quad \left. - \Phi\left(\frac{\ln\left(1-\frac{\varepsilon}{x}\right) - \left(m_1\left(\mathbf{z};h\right) + \sigma_1^2\left(\mathbf{z};h\right)\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)\right\} \\
&\quad \left. - \left\{\Phi\left(\frac{\ln\left(1+\frac{\varepsilon}{x}\right) - m_1\left(\mathbf{z};h\right)}{\sigma_1\left(\mathbf{z};h\right)}\right) - \Phi\left(\frac{\ln\left(1-\frac{\varepsilon}{x}\right) - m_1\left(\mathbf{z};h\right)}{\sigma_1\left(\mathbf{z};h\right)}\right)\right\}\right].
\end{aligned}$$

Since

$$\Phi\left(\frac{\ln\left(1+\frac{\varepsilon}{x}\right) - \left(m_1\left(\mathbf{z};h\right) + \sigma_1^2\left(\mathbf{z};h\right)\right)}{\sigma_1\left(\mathbf{z};h\right)}\right) - \Phi\left(\frac{\ln\left(1-\frac{\varepsilon}{x}\right) - \left(m_1\left(\mathbf{z};h\right) + \sigma_1^2\left(\mathbf{z};h\right)\right)}{\sigma_1\left(\mathbf{z};h\right)}\right) \leq 1,$$

we obtain

$$\begin{aligned}
\left|\int_{|y-x|\leq\varepsilon} (y-x)\, g_t\left(y, s+h \mid x, s\right) dy\right| &\leq x\left|\exp\left\{\frac{\sigma_1^2\left(\mathbf{z};h\right)}{2} + m_1\left(\mathbf{z};h\right)\right\} + f_{\mathbf{z},\varepsilon}\left(h\right) - 1\right| \\
&\leq x\left(\left|\exp\left\{\frac{\sigma_1^2\left(\mathbf{z};h\right)}{2} + m_1\left(\mathbf{z};h\right)\right\} - 1\right| + \left|f_{\mathbf{z},\varepsilon}\left(h\right)\right|\right).
\end{aligned}$$

Applying the Mean Value Theorem to $\exp\left\{\frac{\sigma_1^2(\mathbf{z};h)}{2} + m_1\left(\mathbf{z};h\right)\right\}$ and $f_{\mathbf{z},\varepsilon}\left(h\right)$, and using

$$\lim_{h\to 0} \exp\left\{\frac{\sigma_1^2\left(\mathbf{z};h\right)}{2} + m_1\left(\mathbf{z};h\right)\right\} = 1 \qquad \text{and} \qquad \lim_{h\to 0} f_{\mathbf{z},\varepsilon}\left(h\right) = 0,$$

there exist constants l_1^* and l_1^{**} such that

$$\left|\exp\left\{\frac{\sigma_1^2\left(\mathbf{z};h\right)}{2} + m_1\left(\mathbf{z};h\right)\right\} - 1\right| \leq l_1^* h \quad \text{and} \quad \left|f_{\mathbf{z},\varepsilon}\left(h\right)\right| \leq l_1^{**} h.$$

Therefore,

$$\left| \int_{|y-x|\le\varepsilon} (y-x)\, g_t(y, s+h \mid x, s)\, dy \right| \le x\left[l_1^* h + l_1^{**} h\right] = l_2 h. \tag{11}$$

(a.2) To find a bound of the integral we make the change of variables $u = \ln\left(\frac{y}{x}\right)$:

$$\begin{aligned}
&\int_{|y-x|\le\varepsilon} (y-x)^2 g_t(y, s+h \mid x, s)\, dy \\
&\quad = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{|y-x|\le\varepsilon} \frac{(y-x)^2}{y} \exp\left\{ -\frac{1}{2}\left(\frac{\ln\left(\frac{y}{x}\right) - m_1(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right)^2 \right\} dy, \\
&\quad = \frac{x^2}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} (e^u - 1)^2 \exp\left\{ -\frac{1}{2}\left(\frac{u - m_1(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right)^2 \right\} du \\
&\quad = \frac{x^2}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} \exp\left\{ 2u - \frac{1}{2}\left(\frac{u - m_1(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right)^2 \right\} du \\
&\qquad - \frac{2x^2}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} \exp\left\{ u - \frac{1}{2}\left(\frac{u - m_1(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right)^2 \right\} du \\
&\qquad + \frac{x^2}{\sqrt{2\pi}\sigma_1} \int_{\ln\left(1-\frac{\varepsilon}{x}\right)}^{\ln\left(1+\frac{\varepsilon}{x}\right)} \exp\left\{ -\frac{1}{2}\left(\frac{u - m_1(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right)^2 \right\} du.
\end{aligned}$$

Using

$$\begin{aligned}
2u - \frac{1}{2}\left(\frac{u - m_1(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right)^2 &= -\frac{1}{2}\left(\frac{u - \left(m_1(\mathbf{z}; h) + 2\sigma_1^2(\mathbf{z}; h)\right)}{\sigma_1(\mathbf{z}; h)} \right)^2 \\
&\quad + \sigma_1^2(\mathbf{z}; h) + 2m_1(\mathbf{z}; h),
\end{aligned}$$

the previous results, and applying the *Mean Value Theorem* to the next functions:

$$\exp\left\{ \sigma_1^2(\mathbf{z}; h) + 2m_1(\mathbf{z}; h) \right\} - 1$$

and

$$\begin{aligned}
\exp\left\{ \frac{\sigma_1^2(\mathbf{z}; h)}{2} + m_1(\mathbf{z}; h) \right\} &\left[\Phi\left(\frac{\ln\left(1+\frac{\varepsilon}{x}\right) - m_1(\mathbf{z}; h) + \sigma_1^2(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right) \right. \\
&\left. - \Phi\left(\frac{\ln\left(1-\frac{\varepsilon}{x}\right) - m_1(\mathbf{z}; h) + \sigma_1^2(\mathbf{z}; h)}{\sigma_1(\mathbf{z}; h)} \right) \right] - 1,
\end{aligned}$$

we conclude that there exists a constant l_3 such that

$$\int_{|y-x|\le\varepsilon} (y-x)^2 g_t(y, s+h \mid x, s)\, dy \le l_3 h. \tag{12}$$

Finally, linking (11) and (12), we have that for every compact $K \subset \mathbf{R}_+$, and for every $x \in K$, there exists a constant $l' = l_2 + l_3$ such that (10) holds. Similarly, it can be proved that there exists a constant l'' such that

$$\left| \int_{|y-x|\leq\varepsilon} (y-x)\, g_s(y,t+k \mid x,t)\, dy \right| + \int_{|y-x|\leq\varepsilon} (y-x)^2\, g_s(y,t+k \mid x,t)\, dy \leq l''k.$$

(b) If K is a compact of $\mathbf{R}$, there exists η such that $K \subset]-\eta,\eta[$. Take $c=\varepsilon+\eta$ and let x be such that $|x|>c$. Then

$$\begin{aligned} P_1^t(K,s+h \mid x,s) &= P\left[X(s+h,t)\in K \ \mid\ X(\mathbf{z})=x\right] \\ &= P\left[X\left(\Delta_h^1(\mathbf{z})\right)\in K-x \ \mid\ X(\mathbf{z})=x\right] \\ &\leq P\left[\left|X\left(\Delta_h^1(\mathbf{z})\right)\right|>\varepsilon \ \mid\ X(\mathbf{z})=x\right]. \end{aligned}$$

Therefore, there exists a constant l^* (see the proof of hypothesis II) such that

$$\sup_{|x|>c} P_1^t(K,s+h \mid x,s) \leq l^*h, \quad \forall (s,t)\in I.$$

Similarly, there exist another constant l^{**} such that

$$\sup_{|x|>c} P_2^s(K,t+k \mid x,t) \leq l^{**}k, \quad \forall (s,t)\in I.$$

Finally taking $c=\varepsilon+\eta$ and $l=\max\{l',l'',l^*,l^{**}\}$ we conclude the proof of hypothesis III.

Hypothesis IV holds: Defining

$$X\left(\Delta_h^1(\mathbf{z})\right)_\varepsilon := X\left(\Delta_h^1(\mathbf{z})\right)\mathbf{1}_{\left\{\left|X\left(\Delta_h^1(\mathbf{z})\right)\right|\leq\varepsilon\right\}}$$

and using the Jensen's inequality, we have

$$\begin{aligned} &\psi_\varepsilon(\mathbf{z};h,x,\xi,\tau) = \\ &= \left|E\left[X\left(\Delta_h^1(s,\tau)\right)_\varepsilon X\left(\Delta_h^1(s,t)\right)_\varepsilon \mid X(s,t)=x, X(s,\tau)=\xi\right]\right| \\ &\leq E\left[\left|X\left(\Delta_h^1(s,\tau)\right)_\varepsilon X\left(\Delta_h^1(s,t)\right)_\varepsilon\right| \mid X(s,t)=x, X(s,\tau)=\xi\right]. \end{aligned}$$

Using now Schwarz's inequality,

$$\begin{aligned} \psi_\varepsilon(\mathbf{z};h,x,\xi,\tau) \leq\ & \left(E\left[X^2\left(\Delta_h^1(s,\tau)\right)_\varepsilon \mid X(s,t)=x, X(s,\tau)=\xi\right]\right)^{1/2} \\ & \times\left(E\left[X^2\left(\Delta_h^1(s,t)\right)_\varepsilon \mid X(s,t)=x, X(s,\tau)=\xi\right]\right)^{1/2} \end{aligned}$$

$$\begin{aligned} &= \left(E\left[X^2\left(\Delta_h^1(s,\tau)\right)_\varepsilon \mid X(s,\tau)=\xi\right]\right)^{1/2} \\ &\quad \times\left(E\left[X^2\left(\Delta_h^1(s,t)\right)_\varepsilon \mid X(s,t)=x\right]\right)^{1/2} \\ &= \left(\int_{|\xi_2-\xi|\leq\varepsilon} (\xi_2-\xi)^2\, g_\tau(\xi_2,s+h \mid \xi,s)\, d\xi_2\right)^{1/2} \\ &\quad \times\left(\int_{|y-x|\leq\varepsilon} (y-x)^2\, g_t(y,s+h \mid x,s)\, dy\right)^{1/2} \\ &= \left(\frac{1}{\mu}\sigma_1^2(s,\tau;h)\, f_\varepsilon(s,\tau;h)\right)^{1/2}\left(\frac{1}{\mu}\sigma_1^2(\mathbf{z};h)\, f_\varepsilon(\mathbf{z};h)\right)^{1/2} \\ &\leq l'h, \qquad \text{for} \rho\leq h, \end{aligned}$$

as can be checked by using the proof of hypothesis III-**(a)** with $\mathbf{z} = (s,t)$ and $\mathbf{z} = (s,\tau)$. Thus, there exists a constant l' such that, for all $(s,t) \in I$ and for all $\tau \in [0,t)$,

$$\left| E\left[X\left(\Delta_h^1(s,\tau)\right)_\varepsilon X\left(\Delta_h^1(s,t)\right)_\varepsilon \mid X(s,t) = x, X(s,\tau) = \xi \right] \right| \le l'h.$$

Similarly, there exists a constant l'' such that, for all $(s,t) \in I$ and for all $\sigma \in [0,s)$,

$$\left| E\left[X\left(\Delta_k^2(\sigma,t)\right)_\varepsilon X\left(\Delta_k^2(s,t)\right)_\varepsilon \mid X(s,t) = x, X(\sigma,t) = \eta \right] \right| \le l''k.$$

Taking $l = \max\{l', \ l''\}$ we conclude the proof of hypothesis IV.

Hypothesis V holds: a) Clearly, the coefficients B_1 and B_2 are twice continuous differentiable with respect to x and have continuous derivatives with respect to s and t. b) Let us show $\hat{B}_1(s,t,\tau,x,\xi) = \frac{x}{\xi} B_1(s,\tau,\xi)$. In fact,

$$\begin{aligned}
&\int_{\substack{|x_2-x|\le\varepsilon \\ |\xi_2-\xi|\le\varepsilon}} (x_2-x)(\xi_2-\xi)\, g(x_2,(s+h,t) \mid (x,\xi,\xi_2),(s,\tau))\, g_\tau(\xi_2, s+h \mid \xi, s,)\, dx_2 d\xi_2 \\
&= \int_{|\xi_2-\xi|\le\varepsilon} (\xi_2-\xi)\, g_\tau(\xi_2, s+h \mid \xi, s,)\, d\xi_2 \\
&\quad \times \left(\int_{|x_2-x|\le\varepsilon} (x_2-x) \frac{1}{x_2\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln\left(\frac{x_2\xi}{x\xi_2}\right) - m}{\sigma} \right) \right\} dx_2 \right)
\end{aligned}$$

where we have written $m := m(\mathbf{z};h,k)$ and $\sigma := \sigma(\mathbf{z};h,k)$ for simplicity. Making the change of variable $u = \frac{1}{\sigma}\left(\ln\left(\frac{x_2\xi}{x\xi_2}\right) - m\right)$ and denoting

$$u_1 := \frac{1}{\sigma}\left(\ln\left(\left(1-\frac{\varepsilon}{x}\right)\frac{\xi}{\xi_2}\right) - m\right), \qquad u_2 := \frac{1}{\sigma}\left(\ln\left(\left(1+\frac{\varepsilon}{x}\right)\frac{\xi}{\xi_2}\right) - m\right),$$

we obtain

$$\begin{aligned}
&\int_{|x_2-x|\le\varepsilon} (x_2-x) \frac{1}{x_2\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln\left(\frac{x_2\xi}{x\xi_2}\right) - m}{\sigma} \right) \right\} dx_2 \\
&= \frac{1}{\sqrt{2\pi}} \int_{u_1}^{u_2} x \left(\frac{\xi_2 e^{u\sigma+m}}{\xi} - 1 \right) e^{-\frac{1}{2}u^2} du \\
&= x\left[\left(\frac{\xi_2}{\xi} \frac{e^m}{\sqrt{2\pi}} \int_{u_1}^{u_2} e^{u\sigma - \frac{1}{2}u^2} du \right) - (\Phi(u_2) - \Phi(u_1)) \right] \\
&= x\left[\left(\frac{\xi_2}{\xi} e^{m+\frac{\sigma^2}{2}} \left\{ \Phi(u_2-\sigma) - \Phi(u_1-\sigma) \right\} \right) - (\Phi(u_2) - \Phi(u_1)) \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\hat{B}_1(s,t,\tau,x,\xi) &= \lim_{h\to 0} \frac{1}{h} \int_{|\xi_2-\xi|\le\varepsilon} x\Big[\left(\frac{\xi_2}{\xi} e^{m+\frac{\sigma^2}{2}} \left\{ \Phi(u_2-\sigma) - \Phi(u_1-\sigma) \right\} \right) \\
&\quad - (\Phi(u_2) - \Phi(u_1)) \Big] (\xi_2-\xi)\, g_\tau(\xi_2, s+h \mid \xi, s,)\, d\xi_2.
\end{aligned}$$

Since $\lim_{h\to 0} u_1 = -\infty$ and $\lim_{h\to 0} u_2 = +\infty$,

$$\begin{aligned}
\hat{B}_1(s,t,\tau,x,\xi) &= \lim_{h\to 0} \frac{1}{h} \int_{|\xi_2-\xi|\le\varepsilon} x\left[\frac{\xi_2}{\xi} - 1 \right] (\xi_2-\xi)\, g_\tau(\xi_2, s+h \mid \xi, s,)\, d\xi_2 \\
&= \frac{x}{\xi} \lim_{h\to 0} \frac{1}{h} \int_{|\xi_2-\xi|\le\varepsilon} (\xi_2-\xi)^2 g_\tau(\xi_2, s+h \mid \xi, s,)\, d\xi_2 = \frac{x}{\xi} B_1(s,\tau,\xi).
\end{aligned}$$

Similarly it can be proved that $\hat{B}_2(s,t,\sigma,x,\eta) = \frac{x}{\xi} B_2(\sigma,t,\eta)$.

c) Finally it is clear that

$$\lim_{\substack{t \searrow \tau \\ x \to \xi}} \hat{B}_1(s,t,\tau,x,\xi) = B_1(s,\tau,\xi), \qquad \lim_{\substack{s \searrow \sigma \\ x \to \eta}} \hat{B}_2(s,t,\sigma,x,\eta) = B_2(\sigma,t,\eta),$$

and hence hypothesis V holds.

C Proof of Proposition 3.1 Let $\mathbf{z}_0 = (s,t)$ and $\mathbf{z} = (s+h, t+k)$. The equation will be derived in five steps.

Step 1: $\partial^2 g(y,\mathbf{z} \mid \overline{x}, \mathbf{z}_0) \mid \partial s \partial t$ exists and

$$\frac{\partial^2 g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)}{\partial s \partial t} = \lim_{h,k \to 0} \frac{\Delta_{hk}(g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0))}{hk}, \tag{13}$$

where

$$\begin{aligned} \Delta_{hk}(g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)) &= g(y,\mathbf{z}' \mid \overline{x},\mathbf{z}_0) - g(y,(s+h,t) \mid \overline{x},\mathbf{z}_0) \\ &\quad - g(y,(s,t+k) \mid \overline{x},\mathbf{z}_0) + g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0). \end{aligned} \tag{14}$$

Step 2: Let R be an arbitrary C^∞ (infinitely differentiable) function with compact support. Multiplying both sides of the equality (13) by $R(y)$ and integrating with respect to y, we have

$$\int_{\mathbf{R}_+} R(y) \frac{\partial^2 g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)}{\partial s \partial t} dy = \lim_{h,k \to 0} \int_{\mathbf{R}_+} R(y) \frac{\Delta_{hk}(g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0))}{hk} dy. \tag{15}$$

Step 3: Let us define

$$\varphi(\mathbf{z}) = E\left[R(X(\mathbf{z})) \mid \overline{X}(\mathbf{z}_0,\mathbf{z}) = \overline{x}\right] = \int_{\mathbf{R}_+} R(y)\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy.$$

Equation (15) can be written as

$$\int_{\mathbf{R}_+} R(y) \frac{\partial^2 g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)}{\partial s \partial t} dy = \lim_{h,k \to 0} \frac{1}{hk} \left[\varphi(\mathbf{z}') - \varphi(s+h,t) - \varphi(s,t+k) + \varphi(\mathbf{z})\right], \tag{16}$$

with

$$\begin{aligned} &\varphi(\mathbf{z}') - \varphi(s+h,t) - \varphi(s,t+k) + \varphi(\mathbf{z}) = \\ &= \int_{\mathbf{R}_+} R(y')\, g(y',\mathbf{z}' \mid \overline{x},\mathbf{z}_0)\, dy' - \int_{\mathbf{R}_+} R(y_2)\, g(y_2,(s+h,t) \mid \overline{x},\mathbf{z}_0)\, dy_2 \\ &\quad - \int_{\mathbf{R}_+} R(y_1)\, g(y_1,(s,t+k) \mid \overline{x},\mathbf{z}_0)\, dy_1 + \int_{\mathbf{R}_+} R(y)\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy \\ &= \int_{\mathbf{R}_+} R(y') [\iiint_{\mathbf{R}_+^3} g(y',\mathbf{z}' \mid (y_1,y,y_2),\mathbf{z})\, g(y_2,(s+h,t) \mid (y,x_2,x_2),(s,t_0)) \\ &\qquad \times g(y_1,(s,t+k) \mid (x_1,x_1,y),(s_0,t))\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy_2 dy_1 dy] dy' \\ &\quad - \int_{\mathbf{R}_+} R(y_2) [\int_{\mathbf{R}_+} g(y_2,(s+h,t) \mid (y,x_2,x_2),(s,t_0))\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy] dy_2 \\ &\quad - \int_{\mathbf{R}_+} R(y_1) [\int_{\mathbf{R}_+} g(y_1,(s,t+k) \mid (x_1,x_1,y),(s_0,t))\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy] dy_1 \\ &\quad + \int_{\mathbf{R}_+} R(y)\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy. \end{aligned} \tag{17}$$

where we have used Chapman-Kolmogorov's equations (see [14] and [15])

Step 4: Now, using the above expression of the increment $\varphi(\mathbf{z}')-\varphi(s+h,t)-\varphi(s,t+k)+\varphi(\mathbf{z})$, the Taylor expansion of the function R and the binomial expansion we obtain a general expression of the forward equation (*Kinetic forward equation*). We will calculate the coefficients of this expression in step 5.

The function R is C^{∞}. Applying the Taylor expansion of this function, simplifying the term $\int_{\mathbf{R}_+} R(y)\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy$ and using the binomial expansion, we obtain

$$
\begin{aligned}
&\varphi(\mathbf{z}') - \varphi(s+h,t) - \varphi(s,t+k) + \varphi(\mathbf{z}) = \\
&= \sum_{n=1}^{\infty} \int_{\mathbf{R}_+} \Bigg[\iiint_{\mathbf{R}_+^3} \frac{(y'-y)^n}{n!} g(y',\mathbf{z}' \mid (y_1,y,y_2),\mathbf{z}) \\
&\qquad \times g(y_2,(s+h,t) \mid (y,x_2,x_2),(s,t_0)) \\
&\qquad \times g(y_1,(s,t+k) \mid (x_1,x_1,y),(s_0,t))\, dy_2 dy_1 dy' \\
&\quad - \int_{\mathbf{R}_+} \frac{(y_2-y)^n}{n!} g(y_2,(s+h,t) \mid (y,x_2,x_2),(s,t_0))\, dy_2 \\
&\quad - \int_{\mathbf{R}_+} \frac{(y_1-y)^n}{n!} g(y_1,(s,t+k) \mid (x_1,x_1,y),(s_0,t))\, dy_1 \Bigg] R^{(n)}(y)\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy,
\end{aligned}
$$

$$
= \sum_{n=1}^{\infty} \int_{\mathbf{R}_+} \mu_n^{h,k}(\mathbf{z})\, (-y)^n R^{(n)}(y)\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy, \tag{18}
$$

where

$$
\begin{aligned}
\mu_n^{h,k}(\mathbf{z}) = \sum_{i=0}^{n} \frac{(-1)^i}{i!\,(n-i)!} \Bigg[&\iiint_{\mathbf{R}_+^3} \left(\frac{y'}{y}\right)^i g(y',\mathbf{z}' \mid (y_1,y,y_2),\mathbf{z}) \\
&\times g(y_2,(s+h,t) \mid (y,x_2,x_2),(s,t_0)) \\
&\times g(y_1,(s,t+k) \mid (x_1,x_1,y),(s_0,t))\, dy_2 dy_1 dy' \\
&- \int_{\mathbf{R}_+} \left(\frac{y_2}{y}\right)^i g(y_2,(s+h,t) \mid (y,x_2,x_2),(s,t_0))\, dy_2 \\
&- \int_{\mathbf{R}_+} \left(\frac{y_1}{y}\right)^i g(y_1,(s,t+k) \mid (x_1,x_1,y),(s_0,t))\, dy_1 \Bigg].
\end{aligned}
\tag{19}
$$

Therefore, substituting (18) in (16), we have

$$
\int_{\mathbf{R}_+} R(y) \frac{\partial^2 g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)}{\partial s \partial t} dy = \sum_{n=1}^{\infty} (-1)^n \int_{\mathbf{R}_+} R^{(n)}(y)\, y^n \mu_n(\mathbf{z})\, g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)\, dy, \tag{20}
$$

where

$$
\mu_n(\mathbf{z}) = \lim_{h,k\to 0} \frac{\mu_n^{h,k}(\mathbf{z})}{hk}. \tag{21}
$$

Integrating by parts in the right-hand side of the inequality (20) and using $R^{(k)}(\pm\infty) = 0$, we obtain

$$
\int_{\mathbf{R}_+} R(y) \frac{\partial^2 g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0)}{\partial s \partial t} dy = \int_{\mathbf{R}_+} R(y) \left(\sum_{n=1}^{\infty} \mu_n(\mathbf{z}) \frac{\partial^n (y^n g(y,\mathbf{z} \mid \overline{x},\mathbf{z}_0))}{\partial y^n} \right) dy.
$$

Since R is an arbitrary function, finally we have

$$\frac{\partial^2 g\left(y,\mathbf{z}\mid \overline{x},\mathbf{z}_0\right)}{\partial s\partial t}=\sum_{n=1}^{\infty}\mu_n\left(\mathbf{z}\right)\frac{\partial^n\left(y^n g\left(y,\mathbf{z}\mid \overline{x},\mathbf{z}_0\right)\right)}{\partial y^n}. \tag{22}$$

Step 5: Next, we calculate $\mu_n\left(\mathbf{z}\right)$ using the conditional moments given in (8).
First, substituting (19) in (21)

$$\begin{aligned}\mu_n\left(\mathbf{z}\right) &= \lim_{h,k\to 0}\frac{1}{hk}\sum_{i=0}^{n}\frac{(-1)^i}{i!\,(n-i)!}\left[\iiint_{\mathbf{R}_+^3}\left(\frac{y'}{y}\right)^i g\left(y',\mathbf{z}'\mid \left(y_1,y,y_2\right),\mathbf{z}\right)\right.\\ &\qquad\times g\left(y_2,\left(s+h,t\right)\mid \left(y,x_2,x_2\right),\left(s,t_0\right)\right)\\ &\qquad\times g\left(y_1,\left(s,t+k\right)\mid \left(x_1,x_1,y\right),\left(s_0,t\right)\right)dy_2dy_1dy'\\ &\qquad-\int_{\mathbf{R}_+}\left(\frac{y_2}{y}\right)^i g\left(y_2,\left(s+h,t\right)\mid \left(y,x_2,x_2\right),\left(s,t_0\right)\right)dy_2\\ &\qquad\left.-\int_{\mathbf{R}_+}\left(\frac{y_1}{y}\right)^i g\left(y_1,\left(s,t+k\right)\mid \left(x_1,x_1,y\right),\left(s_0,t\right)\right)dy_1\right]\\ &= \lim_{h,k\to 0}\frac{1}{hk}\sum_{i=0}^{n}\frac{(-1)^i}{i!\,(n-i)!}\left[A-B-C\right]. \qquad (23)\end{aligned}$$

where A, B and C denote the corresponding integrals. To calculate A, we use the definition of conditional expectation and (8):

$$\begin{aligned}A &= \iiint_{\mathbf{R}_+^3}\left(\frac{y'}{y}\right)^i g\left(y',\mathbf{z}'\mid \left(y_1,y,y_2\right),\mathbf{z}\right)g\left(y_2,\left(s+h,t\right)\mid \left(y,x_2,x_2\right),\left(s,t_0\right)\right)\\ &\qquad\times g\left(y_1,\left(s,t+k\right)\mid \left(x_1,x_1,y\right),\left(s_0,t\right)\right)dy_2dy_1dy'\\ &= \iint_{\mathbf{R}_+^2}\frac{1}{y^i}E\left[\left(X\left(\mathbf{z}'\right)\right)^i\mid X\left(s,t+k\right)=y_1,X\left(s,t\right)=y,X\left(s+h,t\right)=y_2\right]\\ &\qquad\times g\left(y_2,\left(s+h,t\right)\mid \left(y,x_2,x_2\right),\left(s,t_0\right)\right)\\ &\qquad\times g\left(y_1,\left(s,t+k\right)\mid \left(x_1,x_1,y\right),\left(s_0,t\right)\right)dy_2dy_1\\ &= \exp\left\{\int_s^{s+h}\int_t^{t+k}\left(i\tilde{a}\left(\sigma,\tau\right)+\frac{i^2}{2}\tilde{B}\left(\sigma,\tau\right)\right)d\sigma d\tau\right\}\\ &\qquad\times\int_{\mathbf{R}_+}\left(\frac{y_2}{y}\right)^i g\left(y_2,\left(s+h,t\right)\mid \left(y,x_2,x_2\right),\left(s,t_0\right)\right)dy_2\\ &\qquad\times\int_{\mathbf{R}_+}\left(\frac{y_1}{y}\right)^i g\left(y_1,\left(s,t+k\right)\mid \left(x_1,x_1,y\right),\left(s_0,t\right)\right)dy_1\\ &= \exp\left\{\int_s^{s+h}\int_t^{t+k}\left(i\tilde{a}\left(\sigma,\tau\right)+\frac{i^2}{2}\tilde{B}\left(\sigma,\tau\right)\right)d\sigma d\tau\right\}\times B\times C. \qquad (24)\end{aligned}$$

Then, we need to calculate B and C first. Using (8),

$$B=\exp\left\{\int_s^{s+h}\int_{t_0}^{t}\left(i\tilde{a}\left(\sigma,\tau\right)+\frac{i^2}{2}\tilde{B}\left(\sigma,\tau\right)\right)d\sigma d\tau\right\}. \tag{25}$$

$$C=\exp\left\{\int_{s_0}^{s}\int_t^{t+k}\left(i\tilde{a}\left(\sigma,\tau\right)+\frac{i^2}{2}\tilde{B}\left(\sigma,\tau\right)\right)d\sigma d\tau\right\}. \tag{26}$$

Substituting (24), (25), and (26) in (23), we obtain

$$\begin{aligned}\mu_n(\mathbf{z}) = \lim_{h,k\to 0}\frac{1}{hk}\sum_{i=0}^{n}\frac{(-1)^i}{i!\,(n-i)!}\Bigg[&\exp\left\{\int_s^{s+h}\int_t^{t+k}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau\right\}\\ &\times\exp\left\{\int_s^{s+h}\int_{t_0}^{t}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau\right\}\\ &\times\exp\left\{\int_{s_0}^{s}\int_t^{t+k}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau\right\}\\ &-\exp\left\{\int_s^{s+h}\int_{t_0}^{t}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau\right\}\\ &-\exp\left\{\int_{s_0}^{s}\int_t^{t+k}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau\right\}\Bigg].\end{aligned}\tag{27}$$

On the other hand, we can write

$$\begin{aligned}\int_{t_0}^{t}\tilde{a}(\sigma,\tau)\,d\tau &= \tilde{a}_1(\sigma,t)-\tilde{a}_1(\sigma,t_0), \quad \int_{t_0}^{t}\tilde{B}(\sigma,\tau)\,d\tau=\tilde{B}_1(\sigma,t)-\tilde{B}_1(\sigma,t_0),\\ \int_{s_0}^{s}\tilde{a}(\sigma,\tau)\,d\sigma &= \tilde{a}_2(s,\tau)-\tilde{a}_2(s_0,\tau), \quad \int_{s_0}^{s}\tilde{B}(\sigma,\tau)\,d\sigma=\tilde{B}_2(s,\tau)-\tilde{B}_2(s_0,\tau).\end{aligned}$$

Since $X(s,\tau)$ is constantly equal to x_1 when $\tau\in[t,t+k]$, and $X(\sigma,t)$ is constantly equal to x_2 when $\sigma\in[s,s+h]$,

$$\left.\begin{array}{l}\tilde{a}_1(\sigma,t_0)=0\\ \tilde{B}_1(\sigma,t_0)=0\end{array}\right\}\text{ when }\sigma\in[s,s+h],$$
$$\left.\begin{array}{l}\tilde{a}_2(s_0,\tau)=0\\ \tilde{B}_2(s_0,\tau)=0\end{array}\right\}\text{ when }\tau\in[t,t+k],$$

and then,

$$\begin{aligned}\int_s^{s+h}\int_{t_0}^{t}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau &= \int_s^{s+h}\left(i\tilde{a}_1(\sigma,t)+\frac{i^2}{2}\tilde{B}_1(\sigma,t)\right)d\sigma,\\ \int_{s_0}^{s}\int_t^{t+k}\left(i\tilde{a}(\sigma,\tau)+\frac{i^2}{2}\tilde{B}(\sigma,\tau)\right)d\sigma d\tau &= \int_t^{t+k}\left(i\tilde{a}_2(s,\tau)+\frac{i^2}{2}\tilde{B}_2(s,\tau)\right)d\tau.\end{aligned}$$

By the Taylor expansion,

$$\begin{aligned}\mu_n(\mathbf{z}) = \sum_{i=0}^{n}\frac{(-1)^i}{i!\,(n-i)!}\lim_{h,k\to 0}\frac{1}{hk}\Bigg\{-1+\Bigg[&\left(i\tilde{a}(\mathbf{z})+\frac{i^2}{2}\tilde{B}(\mathbf{z})\right)\\ &+\left(i\tilde{a}_1(\mathbf{z})+\frac{i^2}{2}\tilde{B}_1(\mathbf{z})\right)\left(i\tilde{a}_2(\mathbf{z})+\frac{i^2}{2}\tilde{B}_2(\mathbf{z})\right)\Bigg]hk+o(hk)\Bigg\}.\end{aligned}\tag{28}$$

Using now

$$\sum_{i=0}^{n}\frac{(-1)^i}{i!\,(n-i)!}=\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\,1^{n-i}=\frac{1}{n!}(-1+1)^n=0,$$

taking limit in (28) and substituting the expressions of $\tilde{a}_1(s,t)$, $\tilde{a}_2(s,t)$, $\tilde{B}_1(s,t)$ and $\tilde{B}_2(s,t)$, given in (5), and the expressions of $\tilde{a}(s,t)$ and $\tilde{B}(s,t)$, given in (7), we have

$$\begin{aligned}\mu_n(\mathbf{z}) &= \sum_{i=0}^{n} \frac{(-1)^i}{i!\,(n-i)!}\left[i\tilde{a}(\mathbf{z}) + \frac{i^2}{2}\tilde{B}(\mathbf{z}) + \left(i\tilde{a}_1(\mathbf{z}) + \frac{i^2}{2}\tilde{B}_1(\mathbf{z})\right)\left(i\tilde{a}_2(\mathbf{z}) + \frac{i^2}{2}\tilde{B}_2(\mathbf{z})\right)\right] \\ &= \sum_{i=1}^{n} \frac{(-1)^i}{(i-1)!\,(n-i)!}\left[a(\mathbf{z}) - a_1(\mathbf{z})a_2(\mathbf{z}) + \frac{i-1}{2}\left(B(\mathbf{z}) - B_1(\mathbf{z})B_2(\mathbf{z})\right)\right. \\ &\qquad \left. + i\left(a_1(\mathbf{z}) + \frac{i-1}{2}B_1(\mathbf{z})\right)\left(a_2(\mathbf{z}) + \frac{i-1}{2}B_2(\mathbf{z})\right)\right],\end{aligned}$$

It is easy to obtain (9) and, for $n \geq 5$, $\mu_n(\mathbf{z}) = 0$, because

$$\begin{aligned}\sum_{i=1}^{n} \frac{(-1)^i}{(i-1)!\,(n-i)!} &= 0, \qquad \sum_{i=1}^{n} \frac{(-1)^i\, i}{(i-1)!\,(n-i)!} = 0, \\ \sum_{i=1}^{n} \frac{(-1)^i\, i\,(i-1)}{(i-1)!\,(n-i)!} &= 0, \qquad \sum_{i=1}^{n} \frac{(-1)^i\, i\,(i-1)\,(i-1)}{(i-1)!\,(n-i)!} = 0.\end{aligned}$$

References

[1] Arnold, L. (1974) *Stochastic Differential Equations: Theory and Applications*. John Wiley, New York.

[2] Christakos, G. (1992) *Random Field Models in Earth Sciences*. Academic Press, San Diego.

[3] Cressie, N. (1993, rev. ed.) *Statistics for Spatial Data*. John Wiley & Sons, New York.

[4] Gutiérrez, R. Angulo, J. M. González, A. and Pérez, R. (1991). Inference in lognormal multidimensional diffusion processes with exogenous factors: Application to modelling in Economics. *Appl. Stoch. Models and Data Analysis* **7**, no. 4, 295–316.

[5] Gutiérrez, R. González, A. and Torres, F. (1997). Estimation in multivariate lognormal diffusion process with exogenous factors. *Appl. Statist.* **46**, no. 1, 140–146.

[6] Gutiérrez, R. Ricciardi, L. Román, P. and Torres, F. (1997). First-passage-time densities for time-non-homogeneous diffusion processes. *J. of Appl. Prob.* **34**, no. 3, 623–631.

[7] Gutierrez, R. Roldán, C. Gutiérrez-Sánchez, R. and Angulo, J. M. (2005). Estimation and prediction of a 2D lognormal diffusion random field. *Stoch. Environ. Res. and Risk Assess.* **19**, no. 4, 258–265.

[8] Gutierrez, R. Roldán, C. Gutiérrez-Sánchez, R. and Angulo, J. M. (2004). Lognormal diffusion random fields with non-constant values at the boundary axes. Workshop on Spatio-Temporal Modelling of Environmental Processes. 1-3 June, Granada, Spain.

[9] Gutiérrez, R. Román, P. and Torres, F. (1995). A note on the Volterra integral equation for the first-passage-time density. *J. of Appl. Prob.* **32**, no. 3, 635–648.

[10] Gutiérrez, R. Román, P. and Torres, F. (1999). Inference and first-passage-times for the lognormal diffusion process with exogenous factors. Applications to modelling in Economics. *Appl. Stoch. Models in Business and Industry* **15**, no. 4, 325–332.

[11] Gutiérrez, R. Román, P. and Torres, F. (2001). Inference on some parametric functions in the univariate lognormal diffusion process with exogenous factors. *Test* **10**, no.2, 357–373.

[12] Hajek, B. (1982) Stochastic equations of hyperbolic type and a two-parameter stratonovich calculus. *The Ann. Prob.* **10**, no. 2, 451–463.

[13] Journel, A. G. and Huijbregts, C. J. (1978) *Mining Geostatistics*. Academic Press, London.

[14] Nualart, D. (1983) Two-parameter diffusion processes and martingales. *Stoch. Proc. Appl.* **15**, 31–57.

[15] Nualart, D. and Sanz, M. (1979). A Markov property for two-parameter Gaussian processes. *Stochastica* **III,** no. 1, 1–16.

[16] Rao, M. M. (1986) *Real and Stochastic Analysis*. John Wiley, New York.

[17] Ricciardi, L. M. (1977) *Diffusion Processes and Related Topics in Biology. Lecture Notes in Biomathematics*. Springer-Verlag, Berlin-Heidelberg-New York.

[18] Smith, J. A. and De Veaux, R. D. (1992) The temporal and spatial variability of rainfall power. *Environmetrics* **3**, no. 1, 29–53.

[19] Tjeng, T. T. and Chai, C. C. (1999) Fade statistics in Nakagami-lognormal channels. *IEEE Trans. Commun.* **47**, no. 12, 1769–1772.

[20] Zielinski, P. A. and Ponnambalam, K. (1994) Computational methods in stochastic river waters quality modeling. In: Hipel KW (ed) *Stoch. and Statist. Methods in Hydrology and Environ. Engineering* **2**, 187–195. Kluwer Academic Publishers, Dordrecht.

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