CIRCLE TO CIRCLE TRANSITION WITH A SINGLE PH QUINTIC SPIRAL

*Zulfiqar Habib and †Manabu Sakai

Received July 16, 2005; revised March 19, 2007

Abstract. This paper theoretically describes how to compose a single Pythagorean hodograph (PH) quintic Bézier spiral segment, between two circles with one circle inside the other. A spiral is free of local curvature extrema, making spiral design an interesting mathematical problem with importance for both physical and aesthetic applications. The curvature of a spiral varies monotonically with arc-length. A polynomial curve with a PH has the properties that its arc-length is a polynomial of its parameter, and its offset is a rational algebraic expression. A quintic is the lowest degree PH curve that may have an inflection point and that inflection point allows a segment of it to be joined to a straight line segment while preserving continuity of curvature, continuity of tangent direction, and continuity of position. A PH quintic spiral allows the design of fair curves in a NURBS based CAD system. It is also suitable for applications such as highway design in which the clothoid has traditionally been used. We simplify and complete the analysis on earlier results on PH quintic spiral segments which are proposed as transition curve elements, and examine techniques for curve design using the new results.

1 Introduction In curve and surface design it is often desirable to have a transition curve of G^2 contact, composed of single cubic spiral segment, between two circles. The purpose may be practical, e.g., in highway design, railway route, satellite path, or aesthetic applications [6, 3]. In the discussion about geometric design standards in AASHO (American Association of State Highway officials), Hickerson [13] (p. 17) states that "Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of *gradually increasing or decreasing radii* without at the same time introducing an appearance of forced alignment". The importance of this design feature is highlighted in [6] that links vehicle accidents to inconsistency in highway geometric design.

Parametric polynomial cubic splines are usually used in CAD/CAM and CAGD (Computer Aided Geometric Design) applications because of its geometric and numerical properties. Many authors have advocated their use in different applications like data fitting and font designing. The importance of using fair curves in the design process is well documented in the literature [4, 10, 11]. Cubic curves, although smoother, are not always helpful since they might have unwanted inflection points and singularities (See [16]). Walton and Meek [19] mentioned the following unwanted features of a cubic segment.

- It may have unwanted curvature extrema.
- It's offset is neither a rational, nor a polynomial algebraic function of its parameter.
- It's arc-length is the integral part of the square root of the polynomial of its parameter.

2000 *Mathematics Subject Classification.* 65D07, 65D10, 65D17, 65D18.
Key words and phrases. Pythagorean Hodograph; Quintic; Bézier; Curvature Extrema; Spiral.

Pythagorean Hodograph (PH) curves introduced by Farouki and Sakkalis [5] do not suffer from the last two of the aforementioned undesirable features. The lowest degree PH curves that have enough flexibility for curve design and geometric modeling in general are the quintics [8]. The first of the aforementioned undesirable features of a general parametric cubic curve is discussed in [9]; suitable restrictions were determined that allowed the use of a single cubic Bézier curve as a G^2 transition curve between two circles. According to Farin [4], curvature extrema of a fair curve "should only occur where explicitly desired by the designer". B-splines and Bézier curves do not normally allow this. However, it can be accomplished when designing with cubic Bézier spirals. The clothoid or Cornu spiral (non-polynomial) has been used in highway design for many years [12]. Fair curves had previously been formed using two curve segments, in particular, two clothoid spiral segments [14], two cubic spiral segments [9], and two PH quintic spiral segments [8].

This paper theoretically describes how to compose a single Pythagorean hodograph (PH) quintic spiral segment, between two circles with one circle inside the other. The term 'spiral' refers to a curve with monotone curvature of constant sign. Such curves are useful for transition between circles and straight lines. A smooth fair curve is made by adding spiral segments to one end of an existing curve. An added segment is constrained by the position of its beginning point, its beginning unit tangent vector and its beginning curvature. The following transition curves of G^2 contact have been considered for joining (i) straight line to circle, (ii) circle to circle with a broken back C transition, (iii) circle to circle with an S transition, (iv) straight line to straight line and (v) circle to circle where one circle lies inside the other with a C oval transition.

Meek & Walton [17, 18, 20] have considered a possibility of cubic and PH (Pythagorean hodograph) quintic splines to join two circles with one circle inside the other. The *fifth* case (v) used in highway design has not been solved on the existence and the uniqueness at this time. Numerical treatment of the case would imply that it does not always seem to have a solution. Furthermore, the algebra required to prove uniqueness should a solution exist, seems unwieldy and has not been done.

We treat the unsolved *fifth* case and releases the additional G^3 contact condition at the point of smaller circle which gives not so easy to handle simultaneous nonlinear equations [17] and derives a family of PH quintic transition curves depending on an angle between the beginning tangent and the tangent at the point of G^2 contact of the larger circle. It enables us first to obtain the spiral condition on the angle and next to derive the lower and upper bounds for the distance between two circles. We give a sufficient condition for the existence of the transition spiral curve of G^3 contact at the smaller circle. Finally, we give a simple example to show how to use theoretical results and draw the PH quintic spiral transition curve(s) between two circles.

The organization of our paper is as follows. Next section gives a brief discussion of the notations and conventions for the PH quintic spiral with some theoretical background, then description of method followed by numerical example, concluding remarks and future research work.

2 Background

2.1 Notations and conventions The usual Cartesian co-ordinate system is presumed. Boldface is used for points and vectors, e.g.,

$$\boldsymbol{a} = (a_x, a_y)\,.$$

The Euclidean norm or length of a vector $\boldsymbol{a}$ is denoted by the notation

$$\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2},$$

and $\boldsymbol{a}\|\boldsymbol{b}$ means the vector $\boldsymbol{a}$ is parallel to vector $\boldsymbol{b}$. The positive angle of a vector $\boldsymbol{a}$ is the counterclockwise angle from the vector $(1,0)^T$ to $\boldsymbol{a}$. The derivative of a function f is denoted by f'. To aid concise writing of mathematical expressions, the symbol $\times$ is used to denote the two-dimensional scalar cross product, e.g.,

$$\boldsymbol{a} \times \boldsymbol{b} = a_x b_y - a_y b_x = \|\boldsymbol{a}\|\boldsymbol{b}\| \sin\theta,$$

where θ is the counterclockwise angle from $\boldsymbol{a}$ to $\boldsymbol{b}$. The signed curvature of a parametric curve $\boldsymbol{P}(t)$ in the plane is

$$\kappa(t) = \frac{\boldsymbol{P}'(t) \times \boldsymbol{P}''(t)}{\left\|\boldsymbol{P}'(t)\right\|^3}, \tag{1}$$

when $\|\boldsymbol{P}'(t)\|$ is non-zero. Positive curvature has the center of curvature on the left as one traverses the curve in the direction of increasing parameter. For non-zero curvature, the radius of curvature, positive by convention, is $1/|\kappa(t)|$.

The term 'spiral' refers to a curved line segment whose curvature varies monotonically with constant sign. A G^2 point of contact of two curves is a point where the two curves meet and where their unit tangent vectors and signed curvatures match, and a G^3 contact is a point where the derivative of curvatures also match.

Based on Kneser theorem [7], any circle of curvature of spiral encloses all smaller circles of curvature and is enclosed by all larger circles of curvature. So we can not find the transition curve with a single spiral segment between the two tangent circles.

We consider two circles Ω_0, Ω_1 centered at C_0, C_1 with radii r_0, r_1, such that Ω_1 is completely contained inside Ω_0 and $\mu = (r_0/r_1)^{1/3} (> 1)$. It is desirable to join the two circles by a single PH quintic spiral such that both points of contact are G^2 as shown in Figure 1. As in [17], define the PH quintic Bézier curve with six degrees of freedom of the form:

$$\boldsymbol{z}'(t) = \left(u^2(t) - v^2(t), 2u(t)v(t)\right), \tag{2}$$

where

$$u(t) = u_0(1-t)^2 + 2u_1 t(1-t) + u_2 t^2, \quad v(t) = v_0(1-t)^2 + 2v_1 t(1-t) + v_2 t^2. \tag{3}$$

Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) \left(= \frac{\boldsymbol{z}'(t) \times \boldsymbol{z}''(t)}{\|\boldsymbol{z}'(t)\|^3} \right) = \frac{2\left\{u(t)v'(t) - u'(t)v(t)\right\}}{(u^2(t) + v^2(t))^2}, \tag{4}$$

where $\times$ stands for the two-dimensional cross product $(x_0, y_0) \times (x_1, y_1) = x_0 y_1 - x_1 y_0$ and $\|\bullet\|$ means the Euclidean norm. Let $\boldsymbol{p}_i, 0 \le i \le 4$ be the control points of quintic curve. To simplify the analysis, we assume $u_1 = u_0, v_1 = 0$ and require the following conditions:

$$(i)\ \boldsymbol{p}_0 = (0,0),\ (ii)\ \kappa(1) = 1/r_1,\ (iii)\ \boldsymbol{z}'(0) \parallel (1,0),\ (iv)\ \boldsymbol{z}'(1) \parallel (\cos\theta, \sin\theta), \tag{5}$$

where $\theta \in (0, \pi)$ is the fixed anti-clockwise angle from the beginning tangent to the ending (at the point of contact of the smaller circle Ω_1) one. The beginning point $\boldsymbol{p}_0$ of the PH quintic curve (2) with conditions in (5) cannot be used as a point of G^2 contact because the curvature there is 0. Therefore, we use an unknown value $m \in (0,1)$ of the parameter t for the point of contact $\boldsymbol{z}(m)$ of the larger circle Ω_0 with the quintic spiral (2) such that

$$\boldsymbol{z}'(m) \parallel (\cos\phi, \sin\phi), \quad (ii)\ \kappa(m) = 1/r_0 \tag{6}$$

where $\phi \in (0, \theta)$ is an anti-clockwise angle between the tangents at start point and the point of contact of the larger circle Ω_0.

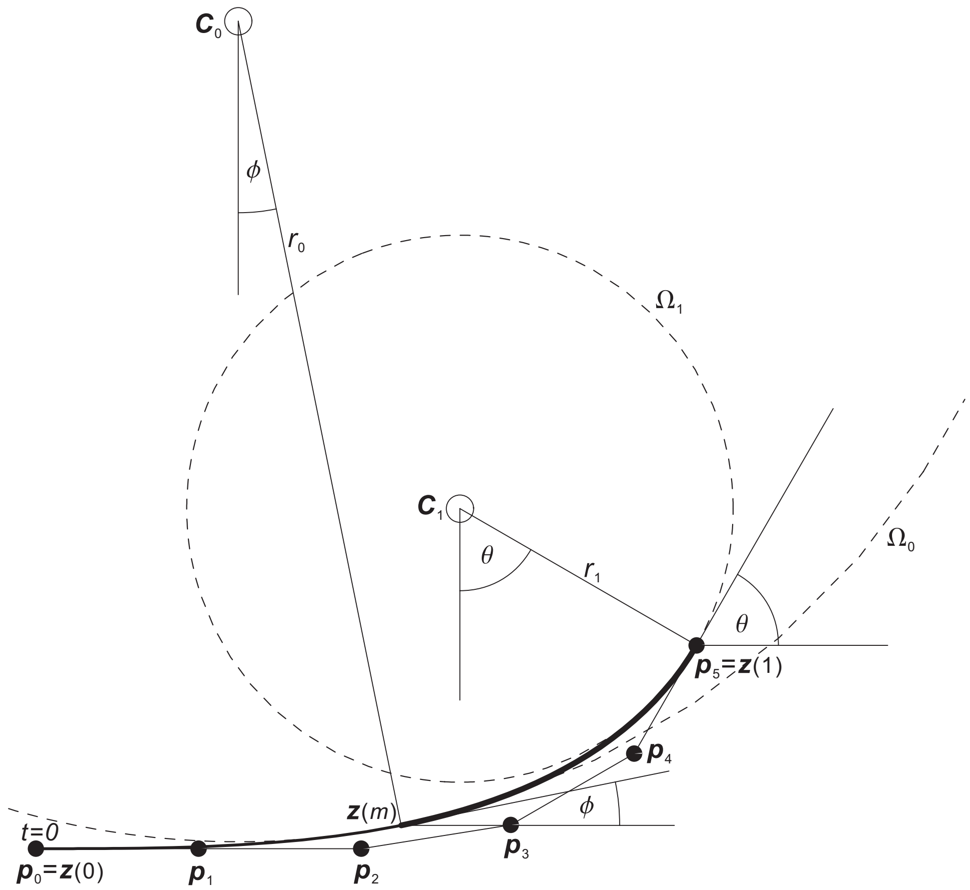


Figure 1: Circle to circle transition with single PH quintic spiral.

3 Description of method The outline of our analysis is first to derive an equation of the unit tangent direction ϕ instead of the value of parameter m [17], and then examine its solvability. First, we impose the conditions in (5) and (6) on PH quintic curve (2) to find

$$v_0 = 0, \quad v_2 = u_2 \tan\frac{\theta}{2}, \quad u_2 = \frac{u_0(1-m^2)}{m^2}\cos\frac{\theta}{2}\csc\frac{\theta-\phi}{2}\sin\frac{\phi}{2}. \tag{7}$$

Next, a simple calculation gives

$$\kappa(1) = \frac{4m^6}{u_0^2(1-m^2)^3}\left(\csc\frac{\phi}{2}\sin\frac{\theta-\phi}{2}\right)^3\sin\frac{\theta}{2}\ \left(=\frac{1}{r_1}\right), \tag{8}$$

and therefore, we have

$$u_0 = \frac{2m^3}{(1-m^2)^{3/2}}\sqrt{r_1\sin\frac{\theta}{2}}\left(\csc\frac{\phi}{2}\sin\frac{\theta-\phi}{2}\right)^{3/2} \tag{9}$$

From 6(ii), we have

$$\kappa(m) = \frac{1}{m^7 r_1}\left(\csc\frac{\theta}{2}\sin\frac{\phi}{2}\right)^4\ \left(=\frac{1}{r_0}\right), \tag{10}$$

to obtain

$$m = \{\mu \csc(\theta/2) \sin(\phi/2)\}^{4/7}. \tag{11}$$

Thus, the transition curve $\boldsymbol{z}(t)$ dependent on a single parameter ϕ satisfies the required conditions in (5) and (6), i.e.,

$$\begin{aligned} u(t) &= \frac{2m\left\{m^2(1-t^2)\cot\frac{\phi}{2}\sin\frac{\theta}{2} - (m^2-t^2)\cos\frac{\theta}{2}\right\}}{(1-m^2)^{3/2}}\sqrt{r_1 \sin\frac{\theta}{2}\csc\frac{\phi}{2}\sin\frac{\theta-\phi}{2}}, \\ v(t) &= \frac{2mt^2}{\sqrt{1-m^2}}\sqrt{r_1\sin\frac{\theta}{2}\csc\frac{\phi}{2}\sin\frac{\theta-\phi}{2}}\sin\frac{\theta}{2}. \end{aligned} \tag{12}$$

To derive the spiral condition on the angle ϕ between the beginning tangent and the tangent at the point of contact of the larger circle Ω_0, we use *Mathematica* to find the derivative of curvature from (4) and obtain

$$\begin{aligned} &\|\boldsymbol{z}'(t)\|^3 \kappa'(t) \left(= 4u_0v_2((1+6t^2-7t^4)u_0^2 - 2t^2(3-7t^2)u_0u_2 - 7t^4(u_2^2+v_2^2))\right) \\ &= \frac{32m^6r_1^2}{(1-m^2)^5(1+s)^4}\csc^5\frac{\phi}{2}\left(\sin\frac{\theta}{2}\sin\frac{\theta-\phi}{2}\right)^3\left\{\sum_{i=0}^{4}c_is^i\right\}, \ t = 1/(1+s) \end{aligned} \tag{13}$$

where

$$\begin{aligned} c_4 &= m^4\{1-\cos(\theta-\phi)\} \ (\geq 0), \quad c_3 = 4c_4 \ (\geq 0), \\ c_2 &= 3m^2\left[m^2\{3-3\cos(\theta-\phi)-\cos\theta+\cos\phi\}+1-\cos(\theta-\phi)+\cos\theta-\cos\phi\right], \\ c_1 &= 2m^2\left[m^2\{5-5\cos(\theta-\phi)-3\cos\theta+3\cos\phi\}+3(1-\cos(\theta-\phi)+\cos\theta \right. \\ &\quad \left. -\cos\phi)\right], \\ c_0 &= (1-m^2)\left[m^2\{3(1-\cos\phi)-4\cos\theta+4\cos(\theta-\phi)\}-7(1-\cos\phi)\right]. \end{aligned}$$

Since $c_1, c_2 \geq 0$ come from $c_0 \geq 0$ or

$$m^2 \geq \frac{7\sin\frac{\phi}{2}}{4\sin(\theta-\frac{\phi}{2})+3\sin\frac{\phi}{2}},$$

to obtain a spiral condition on ϕ. Let ϕ_0 $(\in (0,\theta))$ be the solution of

$$\left\{4\sin(\theta-\frac{\phi}{2})+3\sin\frac{\phi}{2}\right\}m^2 = 7\sin\frac{\phi}{2},$$

or

$$f(\phi)\left(=\mu^8\sin\frac{\phi}{2}\left(4\sin(\theta-\frac{\phi}{2})+3\sin\frac{\phi}{2}\right)^7 - 7^7\sin^8\frac{\theta}{2}\right) = 0, \tag{14}$$

and from (11) with $m=1$, let

$$\phi_1 = 2\arcsin\left(\frac{\sin\frac{\theta}{2}}{\mu}\right) \ (\in (\phi_0,\theta)).$$

Since

$$f(0) = -7^7 \sin^8 \frac{\theta}{2} \ (< 0),$$
$$f(\theta) = 7^7 \left(-1 + \mu^8\right) \sin^8 \frac{\theta}{2} \ (> 0),$$

and

$$f'(\phi) = 2\mu^8 \left\{2 \sin\left(\theta - \frac{\phi}{2}\right) + 3 \sin \frac{\phi}{2}\right\}^6 \times$$
$$\left\{2 \cos\left(\frac{\theta-\phi}{2}\right) + 12 \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\phi}{2}\right)\right\} \sin\left(\frac{\theta-\phi}{2}\right) \ (> 0), \tag{15}$$

i.e., montonically increasing, and therefore only one solution exists.

Note that $c_0 = 0$ or $\kappa'(1) = 0$ in [17].

Theorem 1. *The PH quintic transition curve (2) with (12) is a spiral if*

$$\sqrt{\frac{7 \sin \frac{\phi}{2}}{4 \sin\left(\theta - \frac{\phi}{2}\right) + 3 \sin \frac{\phi}{2}}} \leq m < 1 \tag{16}$$

or $\phi_0 \leq \phi < \phi_1$*, where* ϕ_0 *is the unique solution of (15).*

Proof. This result follows from the above discussion. □

The following Kneser's theorem implies that the smaller circle (Ω_1) of curvature is completely in the interior of the bigger one (Ω_0) for any $\phi \in [\phi_0, \phi_1)$.

Theorem 2. *(Kneser [7], p. 48). Any circle of curvature of a spiral arc contains every smaller circle of curvature of the arc in its interior and in its turn is contained in the interior of every circle of curvature of greater radius.*

Next, we derive the condition on the magnitude of $\|\boldsymbol{C_0} - \boldsymbol{C_1}\|$ for the existence of the circle to circle transition curve (2) of G^2 contact. Since $\boldsymbol{C}_0 = \boldsymbol{z}(m) - r_0(\sin\phi, -\cos\phi)$ and $\boldsymbol{C}_1 = \boldsymbol{z}(1) - r_1(\sin\theta, -\cos\theta)$, *Mathematica* again helps us to find

$$\boldsymbol{C}_1 - \boldsymbol{C}_0 = r_1 \left(g_1(\phi,\theta), g_2(\phi,\theta)\right), \tag{17}$$

where

$$\begin{aligned}
15(1+m)^3 g_1(\phi,\theta) = {}& -2m^2 \csc^3 \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\theta-\phi}{2} \left[-m^2 \left\{-2\left(1+3m+m^2\right)\right.\right.\\
&\left. + \left(2+6m+10m^2+9m^3+3m^4\right)\cos\phi\right\} + \cos\theta \left\{-3-9m-10m^2-6m^3\right.\\
&\left.\left. -2m^4 + \left(3+9m+10m^2+6m^3+10m^4+9m^5+3m^6\right)\cos\phi\right\}\right]\\
&+ \sin\theta \left\{-15(1+m)^3 + 8m^4\left(1+3m+m^2\right)\cot\frac{\phi}{2} \sin\frac{\theta}{2} \sin\frac{\theta-\phi}{2}\right\}\\
&+15(1+m)^3 \mu^4 \sin\phi,
\end{aligned}$$

(18)

$$\begin{aligned}
15(1+m)^2 g_2(\phi,\theta) = {}& -2m^2 \csc^2 \frac{\phi}{2} \sin^2 \frac{\theta}{2} \left\{3+6m+2m^2-2m^3-6m^4-3m^5\right.\\
&+3\left(1+2m+2m^2+2m^3+2m^4+m^5\right)\cos\theta + \left(-3-6m-2m^2+2m^3\right.\\
&\left.\left.+6m^4+3m^5\right)\cos\left(\theta-\phi\right) - 3\left(1+2m+2m^2+2m^3+2m^4+m^5\right)\cos\phi\right\}\\
&+15(1+m)^2 \left\{\cos\theta - \mu^4\cos\phi\right\}.
\end{aligned}$$

With $m = \{\mu \csc(\theta/2)\sin(\phi/2)\}^{4/7}$, we have

$$\|\boldsymbol{C_0} - \boldsymbol{C_1}\| = r_1\sqrt{g_1^2(\phi,\theta) + g_2^2(\phi,\theta)}\,(= r_1 h(\phi,\theta))\,,$$

or

$$\min_{\phi_0\le\phi\le\phi_1}\left[\frac{h(\phi,\theta)}{\mu^4-1}\right](r_0-r_1) < \|\boldsymbol{C_0}-\boldsymbol{C_1}\| < \max_{\phi_0\le\phi\le\phi_1}\left[\frac{h(\phi,\theta)}{\mu^4-1}\right](r_0-r_1). \tag{19}$$

The maximum and minimum values (numerically ≈ 1) would be difficult to get, and so we give a quantity $h(\phi_1,\theta)$ near their values. Since $\phi \to \phi_1$ means $m \to 1$ or $\sin\phi \to 2\arcsin\frac{\sin(\theta/2)}{\mu}$, which follows

$$h^2(\phi_1,\theta) = (\mu^4-1)^2 - \frac{8}{9}\sin^2\frac{\theta}{2}\left(-\cos\frac{\theta}{2} + \sqrt{\mu^2 - \sin^2\frac{\theta}{2}}\right)^3$$
$$\left\{\left(3\mu^2 - 1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2} + \sqrt{\mu^2-\sin^2\frac{\theta}{2}}\left(\mu^2 - 3 + 2\sin^2\frac{\theta}{2}\right)\right\} \tag{20}$$

Theorem 3. *For $\|\boldsymbol{C_0}-\boldsymbol{C_1}\|$ satisfying (19), there exist the desired transition(s) from circle to circle with a single spiral.*

Proof. This result follows from (17) for the value of m in (11). □

In practical application, for given $r_i, i = 0,1$ and θ, first determine $\phi_i, i = 0,1$ and then find the maximum and minimum of $h(\phi,\theta)/(\mu^4-1)$ on $[\phi_0,\phi_1]$ satisfying the inequalities in (19).

Walton & Meek method's [17] being our special case with $\phi = \phi_0$, we obtain the following sufficient condition for the existence of the transition curve of G^3 contact or the bound for $\|\boldsymbol{C_0}-\boldsymbol{C_1}\|$.

Theorem 4. *The spiral segment of G^3 contact exists if*

$$\frac{\sqrt{17+10\mu^2+9\mu^4}}{3(1+\mu^2)}(r_0-r_1) < \|\boldsymbol{C_0}-\boldsymbol{C_1}\| < (r_0-r_1)$$

Proof. Since $(m, \sin(\phi_0/2)) \to (1, 1/\mu)$ as $\theta \to \pi$, we have

$$(g_1(\phi_0,\theta), g_2(\phi_0,\theta)) \to \left(8(\mu^2-1)^{3/2}/3, -(\mu^2-1)(\mu^2-3)\right),$$

from which follows

$$\lim_{\theta\to\pi} h(\phi_0,\theta) = \frac{\sqrt{17+10\mu^2+9\mu^4}}{3\,(1+\mu^2))}\left(\mu^4-1\right). \tag{21}$$

Next, note as $\theta \to 0$

$$(g_1(\phi_0,\theta), g_2(\phi_0,\theta)) \to \left(0, -(\mu^4-1)\right),$$

therefore

$$\lim_{\theta\to 0} h(\phi_0,\theta) = \mu^4 - 1.$$

□

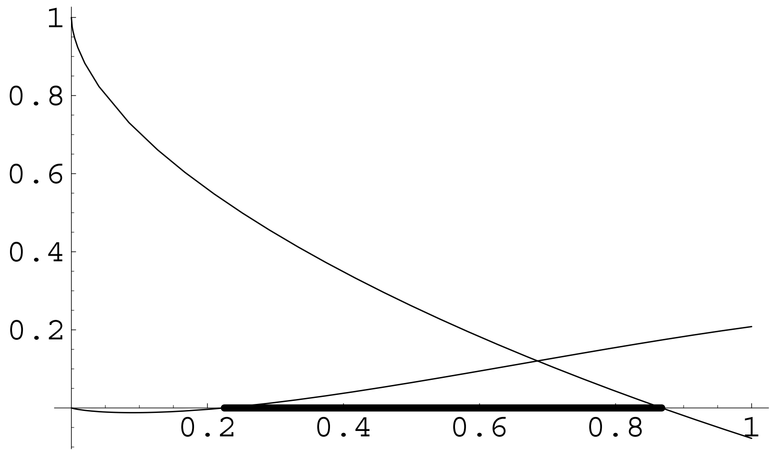
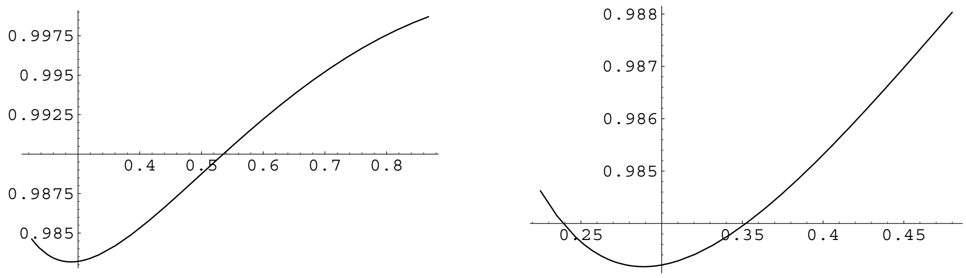


Figure 2: Spiral interval $[\phi_0, \phi_1)(\approx (0.224876, 0.867879))$ for $(\mu, \theta) = (2^{1/4}, \pi/3)$.



(a) On $[\phi_0, \phi_1)$

(b) On $[\phi_0, 0.48)$, (enlarged)

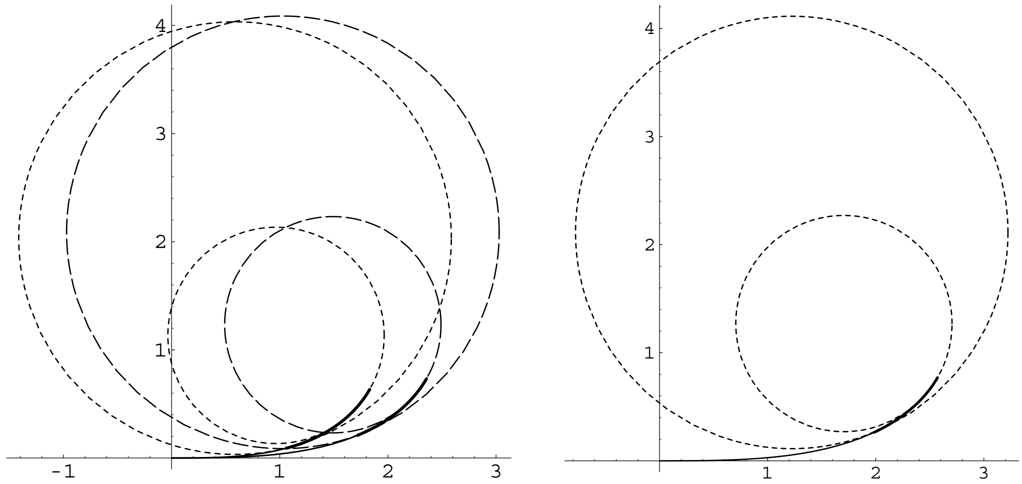
Figure 3: Graph of $h(\phi, \theta)/(\mu^4 - 1)$ for $(\mu, \theta) = (2^{1/4}, \pi/3)$.

4 Numerical example Given $(\mu, \theta) = (2^{1/4}, \pi/3)$, i.e., $(r_0, r_1) = (2, 1)$, find the spiral interval $[\phi_0, \phi_1)(\approx (0.224876, 0.867879))$. The spiral interval on ϕ is shown as a bold line in Figure 2, where $\phi = \phi_0$ means the G^3 contact as in [17]. Figure 3 shows that

1. $0.983175\ldots\big(= \text{Min}\,(h(\phi,\theta))\,/(\mu^4-1)\big) < \phi \le 0.984624\ldots\big(= h(\phi_0,\theta)/(\mu^4-1)\big)$ has two solutions as shown in Figure 3(b).
2. $0.984624\ldots < \phi < 0.998707\ldots\big(= h(\phi_1,\theta)/(\mu^3-1)\big)$ has one solution as shown in Figure 3(a).

For example, for given distance between centers of two circles, e.g., $\|\boldsymbol{C_0} - \boldsymbol{C_1}\| = 0.984(r_0 - r_1)$, there are two solutions: $\phi = (0.238874, 0.352159)$, and therefore two spiral transitions are shown in Figure 4(a). For $\|\boldsymbol{C_0} - \boldsymbol{C_1}\| = 0.985(r_0 - r_1)$, there is one solution: $\phi = 0.389713$, and therefore one spiral transition is shown in Figure 4(b).

Finally, we plot the graph of $\|\boldsymbol{C}_0 - \boldsymbol{C}_1\|,\ \theta \in (0, \pi)$ for the curve of G^3 contact [17] or connect the end points in Figure 3(a). First determine ϕ_0 for $\theta = i\pi/30,\ 1 \le i \le 29$ and interpolate $(\theta, h(\phi_0, \theta))$ to give Figure 5 which would imply the uniqueness of the transition curve.



(a) Two solutions exist

(b) One solution exists

Figure 4: Circle to circle transitions with single PH quintic Bézier spiral.

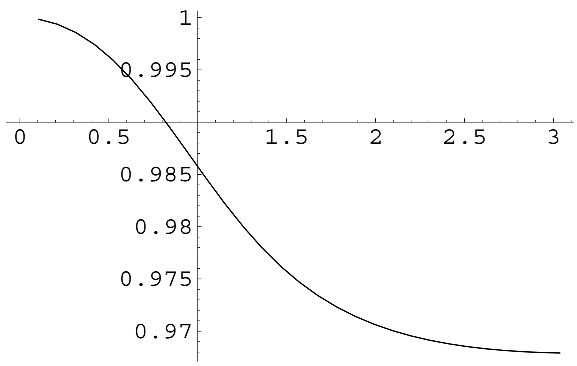


Figure 5: Graph of $h(\phi_0, \theta)/(\mu^4 - 1)$ on $\theta \in (0, \pi)$ for $\mu = 2^{1/4}$.

5 Conclusion The PH quintic spiral is a reasonable alternative to the clothoid [1] for applications such as highway, railway or satellite path design because it is of low degree; being polynomial, unlike the clothoid, it is a special case of a NURBS curve. Since straight line segments and circular arcs also have NURBS representations [15], curves designed using a combination of PH quintic spirals, circular arcs, and straight line segments can be represented entirely by NURBS.

We discussed the fifth unsolved case of Walton & Meek [17, 17] and completed the analysis for most of the practical applications. We proved the existence and uniqueness of PH quintic spiral transition curve(s) of G^2 contact between two circles depending on the given distance between their centers. A simple example is given to prove our analysis and to show how to plot the PH quintic spiral transition curve(s) from circle to circle where one circle lies inside the other, using our theoretical results.

6 Research work in progress Our method uses a parametric value that ranges from a positive value less than one, to 1, because the curvature at the beginning point is 0 and therefore it cannot be used as a point of G^2 contact. However, using available subdivision algorithms this spiral may be reparameterized so that its parameter will range from 0 to 1 [2].

As discussed in [17], there seem to be limitations in extending the idea developed in our paper into 3D non-planar cases. One limitation is the restriction of the first three Bézier control points to a straight line so that the curvature at the beginning of the spiral is zero. This restriction causes the spiral to be planar. A simple way of joining two circles whose planes are not parallel would be to introduce some twist at the beginning points of the pair of transitional spirals (where their curvature are zero). This may however not always be satisfactory, and may also not always work. It can be noted that this limitation is also inherent in the clothoid which by definition is a planar curve. In highway design practice the limitations of extensions into 3D seem to be overcome by designing horizontal and vertical alignments separately but not independently [13].

Work in progress includes an investigation of the behavior of the PH quintic spiral when the point of maximum curvature is moved to values of m greater than 1. Then it may be possible to obtain a curvature vs. arc-length relationship that is closer to linear. Further work will also include research into ways of overcoming the limitations in extending the idea into 3D non-planar cases since such connections are useful in many practical applications.

Acknowledgment The authors owe their gratitude to the referees for their valuable comments and suggestions which have helped in improving the presentation of the paper. This work is supported by Japan Society for the Promotion of Science (JSPS/FF1/315-P04034).

References

[1] K. G. Baass. The use of clothoid templates in highway design. *Transportation Forum*, 1:47–52, 1984.

[2] R. H. Bartels, J. C. Beatty, and B. A. Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling.* Morgan Kaufmann, California, 1987.

[3] H. Burchard, J. Ayers, W. Frey, and N. Sapidis. *Designing fair curves and surfaces*, chapter Approximation with aesthetic constraints. 1993.

[4] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide.* Academic Press, NewYork, 4th edition, 1997.

[5] R. Farouki and T. Sakkalis. Pythagorean hodographs. *IBM Journal of Research and Development*, 34:736–752, 1990.

[6] G. M. Gibreel, S. M. Easa, Y. Hassan, and I. A. El-Dimeery. State of the art of highway geometric design consistency. *ASCE Journal of Transportation Engineering*, 125(4):305–313, 1999.

[7] H. Guggenheimer. *Differential Geometry.* McGraw-Hill, New York, 1963.

[8] Z. Habib and M. Sakai. G^2 PH quintic spiral transition curves and their applications. *Scientiae Mathematicae Japonicae*, 61(2):207–217, 2005. e2004, 263-273, http://www.jams.or.jp/scmjol/2004.html.

[9] Z. Habib and M. Sakai. Spiral transition curves and their applications. *Scientiae Mathematicae Japonicae*, 61(2):195–206, 2005. e2004, 251-262, http://www.jams.or.jp/scmjol/2004.html.

[10] Z. Habib, M. Sakai, and M. Sarfraz. Interactive shape control with rational cubic splines. *Computer-Aided Design & Applications*, 1(1-4):709–718, 2004.

[11] Z. Habib, M. Sarfraz, and M. Sakai. Rational cubic spline interpolation with shape control. *Computers & Graphics*, 29(4):594–605, 2005. http://dx.doi.org/10.1016/j.cag.2005.05.010.

[12] P. Hartman. The highway spiral for combining curves of different radii. *Transactions of the American Society of Civil Engineers*, 122:389–409, 1957.

[13] T. F. Hickerson. *Route Location and Design*. McGraw-Hill, New York, 1964.

[14] D. S. Meek and D. J. Walton. The use of Cornu spirals in drawing planar curves of controlled curvature. *Computational and Applied Mathematics*, 25:69–78, 1989.

[15] L. Piegl and W. Tiller. *The NURBS book*. Springer, 1995.

[16] M. Sakai. Inflection points and singularities on planar rational cubic curve segments. *Computer Aided Geometric Design*, 16:149–156, 1999.

[17] D. J. Walton and D. S. Meek. A Pythagorean hodograph quintic spiral. *Computer Aided Design*, 28(12):943–950, 1996.

[18] D. J. Walton and D. S. Meek. Planar G^2 curve design with spiral segments. *Computer Aided Design*, 30:529–538, 1998.

[19] D. J. Walton and D. S. Meek. Planar G^2 transition with a fair Pythagorean hodograph quintic curve. *Computational and Applied Mathematics*, 138:109–126, 2002.

[20] D. J. Walton, D. S. Meek, and J. M. Ali. Planar G^2 transition curves composed of cubic Bézier spiral segments. *Computational and Applied Mathematics*, 157(2):453–476, 2003.

*Corresponding author: Department of Computer Science, COMSATS Institute of Information Technology, Defence Road, Off Raiwind Road, Lahore, Pakistan.
E-mail addresses: drzhabib@ciitlahore.edu.pk; drzhabib@hotmail.com
†Department of Mathematics and Computer Science, Graduate School of Science and Engineering, Kagoshima University, Kagoshima 890-0065, Japan.
Email address: msakai@sci.kagoshima-u.ac.jp