SPECTRAL ORDER OF OPERATORS AND FURUTA INEQUALITY

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Abstract. By virtue of Furuta inequality, we show some characterizations of the spectral order of positive operators on a Hilbert space from the viewpoint of the Kantorovich type inequality and the Riccati equation. Let A and B be positive operators on a Hilbert space. Then the spectral order $A \succeq B$ holds if and only if there exist a unique positive contraction $T_{p,u}$ such that $T_{p,u}A^{\frac{p+u}{2}}T_{p,u} = A^{\frac{u-p}{4}}B^pA^{\frac{u-p}{4}}$ for all $p \geq 0$ and $u \geq 0$. This form interpolates the chaotic order, δ-order and the usual order continuously.

1 Introduction. An operator means a bounded linear operator on a Hilbert space H. An operator A is said to be positive (denoted by $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. In particular, we denote by $A > 0$ if A is positive and invertible. The usual order $A \geq B$ for selfadjoint operators A and B is defined by $A - B \geq 0$.

Let A and B be positive invertible operators. We introduce a function order $A \geq_f B$ for a real valued continuous function f on $[0, \infty)$ defined by

$$A \geq_f B \qquad \Longleftrightarrow \qquad f(A) \geq f(B). \tag{1}$$

If f is operator monotone, then the order $A \geq_f B$ is weaker than the usual order $A \geq B$. The logarithmic function $f(t) = \log t$ yields the chaotic order $A \gg B$ for $A, B > 0$ by $\log A \geq \log B$.

Olson [24] introduced a new order among selfadjoint operators: Let E_t (resp. F_t) be the resolution of the identity of A (resp. B), i.e.,

$$A = \int t dE_t \qquad \text{and} \qquad B = \int t dF_t.$$

Then the spectral order $A \succeq B$ holds if $E_t \leq F_t$ for all t. He showed the following characterization of the spectral order:

Theorem A. *Let A and B be positive operators. Then the spectral order $A \succeq B$ if and only if $A^n \geq B^n$ for all natural numbers $n \in \mathbb{N}$.*

Moreover, Fujii and Kasahara [11] pointed out that the spectral order $A \succeq B$ if and only if

$$f(A) \geq f(B)$$

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for all monotone increasing continuous functions f, also see [12, 27]. If we put $f_r(t) = t^r$ for $0 < r < +\infty$, then the f_r-order for $r \in (0, 1]$ interpolates the usual order and the chaotic one, and for $r \in [1, +\infty)$ the usual order and the spectral one continuously.

The Löwner-Heinz theorem asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ for all $1 \geq p \geq 0$. However $A \geq B$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general. Related to this, Furuta [13] established the following ingenious operator inequality twenty years ago:

Theorem F (Furuta inequality)

If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$

and

(ii) $\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with

$(1 + r)q \geq p + r$.

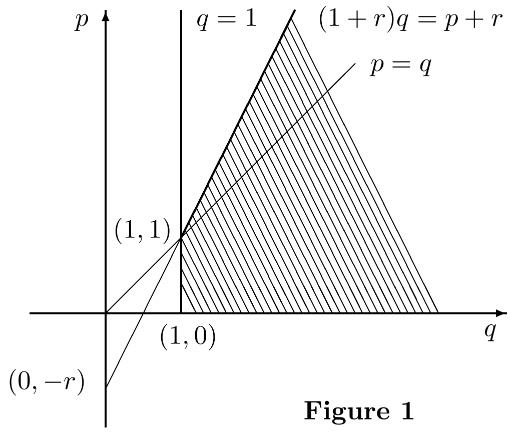

Figure 1

Alternative proofs of Theorem F have been given in [6], [22], and one-page proof in [14]. It is shown in [26] that the domain of the parameters p, q and r drawn in Figure 1 is the best possible for Theorem F.

Consider the following two form

$$f(\Box) = (B^{\frac{r}{2}} \Box B^{\frac{r}{2}})^{\frac{1}{q}} \qquad \text{and} \qquad g(\Box) = (A^{\frac{r}{2}} \Box A^{\frac{r}{2}})^{\frac{1}{q}}.$$

Then Furuta inequality asserts that if $A \geq B \geq 0$, then

$$f(A^p) \geq f(B^p) \qquad \text{and} \qquad g(A^p) \geq g(B^p)$$

hold under the condition p, q and r in Figure 1. We have been turned out that Furuta inequality has immense applications in the operator theory for these 20 years. As an extension of Furuta inequality, the following grand Furuta inequality was established in [17], which interpolates Theorem F itself and Ando-Hiai inequality [2].

Theorem G (The grand Furuta inequality)([17]) . *If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0, 1]$*

$$A^{\frac{(p-t)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for all $s \geq 0$, $p \geq 0$, $q \geq 1$ and $r \geq t$ with $(s-1)(p-1) \geq 0$ and $(1-t+r)q \geq (p-t)s+r$.

In this note, we show some characterizations of the spectral order of positive operators on a Hilbert space from the viewpoint of the Kantorovich type inequality and the Riccati equation: Let A and B be positive operators. Then the spectral order $A \succeq B$ holds if and only if there exist a unique positive contraction $T_{p,u}$ such that

$$T_{p,u} A^{\frac{p+u}{2}} T_{p,u} = A^{\frac{u-p}{4}} B^p A^{\frac{u-p}{4}} \qquad \text{for all } p \geq 0 \text{ and } u \geq 0.$$

This form interpolates the chaotic order, δ-order and the usual order continuously.

2 Kantorovich type inequality. The celebrated Kantorovich inequality asserts that if A is a positive operator on a Hilbert space H satisfying $M \geq A \geq m$ for some scalars $M > m > 0$, then $(A^{-1}x, x)(Ax, x) \leq \frac{(M+m)^2}{4Mm}$ holds for every unit vector x in H. The constant $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant. As an application of the Kantorovich inequality, Fujii, Izumino, Nakamoto and one of the authors [8] showed that t^2 is order preserving in the following sense:

$$A \geq B \geq 0 \quad \text{and} \quad M \geq A \geq m > 0 \quad \text{imply} \quad \frac{(M+m)^2}{4Mm} A^2 \geq B^2. \tag{2}$$

Related to this, Furuta [18] showed the following Kantorovich type operator inequality:

Theorem B. *Let A and B be positive operators satisfying $M \geq A \geq m$ for some scalars $M > m > 0$. If $A \geq B > 0$, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K(m, M, p) A^p \geq B^p \qquad \text{for all } p \geq 1,$$

where a generalized Kantorovich constant $K(m, M, p)$ ([18, 19]) is defined as

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p}\right)^p \tag{3}$$

for any real number $p \in \mathbb{R}$.

We show a characterization of the spectral order via the Kantorovich type inequality, which parametrizes some results on the chaotic order and the usual order by Hashimoto-Yamazaki [20, Theorem 4] and Fujii-Kamei-Seo [10, Theorem 3] respectively.

Theorem 1. *Let A and B be positive invertible operators satisfying $M \geq A \geq m > 0$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i) $A \succeq B$.

(ii) *For each $n \in \mathbb{N}$, $\alpha \in [0, 1]$, $p \geq v \geq 0$ and $u \geq 0$*

$$K(m^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}, M^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}, n+1)\, A^{(p-v+(u+v)\alpha)s} \geq \left(A^{\frac{(u+v)\alpha-v}{2}} B^p A^{\frac{(u+v)\alpha-v}{2}}\right)^s$$

holds for $s \geq 1$ and $(p-v+(u+v)\alpha)s \geq (u+v)(n+\alpha)$, where $K(m, M, p)$ is defined by (3).

To prove this theorem, we need the following lemma:

Lemma 2. *Let A and B be positive operators. Then the spectral order $A \succeq B$ if and only if for each $v \geq 0$*

$$A^{u+v} \geq \left(A^{\frac{u}{2}} B^p A^{\frac{u}{2}}\right)^{\frac{u+v}{p+u}} \tag{4}$$

holds for $u \geq 0$ and $p \geq v$.

Proof. Suppose that $A \succeq B$. For each $v \geq 0$, it follows from Theorem A that $A^{p+u} \geq A^{\frac{u}{2}} B^p A^{\frac{u}{2}}$ for $p \geq v$ and $u \geq 0$. By raising each sides to the power $\frac{u+v}{p+u} \in [0, 1]$, we have

$$A^{u+v} \geq \left(A^{\frac{u}{2}} B^p A^{\frac{u}{2}}\right)^{\frac{u+v}{p+u}}$$

holds for $u \geq 0$ and $p \geq v$.

Conversely, for each $p \geq 0$, put $v = p$ and $u = 0$. By Theorem A, we have $A \succeq B$. □

Proof of Theorem 1. (i) $\Longrightarrow$ (ii). Let $n \in \mathbb{N}$ and $\alpha \in [0,1]$. For each $u \geq 0$ and $p \geq v \geq 0$, put $A_1 = A^{u+v}$ and $B_1 = \left(A^{\frac{u}{2}} B^p A^{\frac{u}{2}}\right)^{\frac{u+v}{p+u}}$. Then we have $A_1 \geq B_1 \geq 0$ by Lemma 2. It follows from Theorem G that for each $t \in [0,1]$

$$A_1^{\frac{(p_1-t)s+r}{q}} \geq \left(A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}} B_1^{p_1} A_1^{-\frac{t}{2}})^s A_1^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{5}$$

holds for $s \geq 1$, $p_1 \geq 1$, $q \geq 1$ and $r \geq 0$ satisfying

$$r \geq t, \tag{6}$$

$$(1-t+r)q \geq (p_1-t)s + r. \tag{7}$$

Put $p_1 = \frac{p+u}{u+v} \geq 1, q = n+1, t = 1-\alpha$ and $r = \frac{(p-v+(u+v)\alpha)s}{n(u+v)} - \frac{n+1}{n}\alpha$. Then (7) is satisfied and (6) is equivalent to the following

$$(p-v+(u+v)\alpha)s \geq (u+v)(n+\alpha). \tag{8}$$

Therefore (5) implies that for each $\alpha \in [0,1]$, $u > 0$ and $p \geq v \geq 0$

$$\begin{aligned} &A^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}} \\ &\geq \left(A^{\frac{(p-v+(u+v)\alpha)s-(n+1)(u+v)\alpha}{2n}} (A^{\frac{(u+v)\alpha-v}{2}} B^p A^{\frac{(u+v)\alpha-v}{2}})^s A^{\frac{(p-v+(u+v)\alpha)s-(n+1)(u+v)\alpha}{2n}}\right)^{\frac{1}{n+1}} \end{aligned}$$

holds for $s \geq 1$, $p \geq v$ with (8). By raising each side to the power $n+1$, it follows from Theorem B that

$$\begin{aligned} &K(m^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}, M^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}, n+1) \left(A^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}\right)^{n+1} \\ &\geq A^{\frac{(p-v+(u+v)\alpha)s-(n+1)(u+v)\alpha}{2n}} (A^{\frac{(u+v)\alpha-v}{2}} B^p A^{\frac{(u+v)\alpha-v}{2}})^s A^{\frac{(p-v+(u+v)\alpha)s-(n+1)(u+v)\alpha}{2n}}. \end{aligned}$$

By rearraging it, we have

$$\begin{aligned} K(m^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}, M^{\frac{(p-v+(u+v)\alpha)s-(u+v)\alpha}{n}}, n+1)\, &A^{(p-v+(u+v)\alpha)s} \\ &\geq \left(A^{\frac{(u+v)\alpha-v}{2}} B^p A^{\frac{(u+v)\alpha-v}{2}}\right)^s, \end{aligned}$$

as desired.

(ii) $\Longrightarrow$ (i). Let $v > 0$. Put $\alpha = 0$ in (ii) and by raising each sides to the power $\frac{1}{s}$, it follows that

$$K(m^{\frac{(p-v)s}{n}}, M^{\frac{(p-v)s}{n}}, n+1)^{\frac{1}{s}} A^{p-v} \geq A^{-\frac{v}{2}} B^p A^{-\frac{v}{2}} \quad \text{for } s \geq 1 \text{ and } (p-v)s \geq (u+v)n. \tag{9}$$

Moreover, since

$$K(m^{\frac{(p-v)s}{n}}, M^{\frac{(p-v)s}{n}}, n+1)^{\frac{1}{s}} \to \left(\frac{M}{m}\right)^{p-v} \qquad \text{as } s \to \infty,$$

it follows from (9) that

$$\left(\frac{M}{m}\right)^{p-v} A^p \geq B^p \qquad \text{for } p \geq v. \tag{10}$$

Put $p = v(> 0)$ in (10) and we have $A^v \geq B^v$ for all $v > 0$, i.e., $A \succeq B$ by Theorem A. $\square$

Remark 3. Theorem 1 unifies the results of both Hashimoto-Yamazaki [20, Theorem 4] and Fujii-Kamei-Seo [10, Theorem 3] in the following sense. Indeed, if we put $v = 0$ (resp. $v = 1$) in (ii) of Theorem 1, then we have the result of Hashimoto-Yamazaki [20, Theorem 4 (ii)](resp. Fujii-Kamei-Seo [10, Theorem 3 (ii)]). In correspondence with them, if we put $v = 0$ (resp. $v = 1$) in (4) of Lemma 2, then (4) of Lemma 2 is equivalent to the chaotic order $A \gg B$ (resp. the usual order $A \geq B$).

Let A and B be positive invertible operators. For each $\alpha \geq 0$ and $\beta \geq 0$, we consider a family $\{A \geq_{(\alpha,\beta)} B\}$ defined by

$$A^{\alpha r+\beta} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\alpha r+\beta}{p+r}} \tag{11}$$

for $p, r \geq 0$ with

$$\begin{cases} p \geq \beta + (\alpha - 1)r & \text{if} \quad \alpha \geq 1 \\ p \geq \beta & \text{if} \quad 1 \geq \alpha \geq 0. \end{cases}$$

The family $\{A \geq_{(\alpha,\beta)} B\}$ has order like property. If $r = 0$, then it is a function order for $f(t) = t^\beta$. In particular, the relation $A \geq_{(1,0)} B$ is just the chaotic order $A \gg B$, the relation $A \geq_{(1,1)} B$ is the usual order and the relation $A \geq_{(1,\beta)} B$ for all $\beta > 0$ is the spectral order $A \succeq B$.

Theorem 4. *For positive operators A and B, the following statements hold:*

(i) *If $A \gg B$, then $A \geq_{(\alpha,0)} B$ for all $\alpha \in (0, 1]$. Conversely, if $A \geq_{(\alpha,0)} B$ for some $\alpha \in (0, 1]$, then $A \gg B$.*

(ii) *If $A^\beta \geq B^\beta$ for a fixed $\beta > 0$, then $A \geq_{(\alpha,\beta)} B$ for all $\alpha \in [0, 1]$. Conversely, if $A \geq_{(\alpha,\beta)} B$ for some $\alpha \in [0, 1]$ and $\beta > 0$, then $A^\beta \geq B^\beta$.*

(iii) *If $A \succeq B$, then $A \geq_{(\alpha,\beta)} B$ for all $\alpha > 1$ and $\beta \geq 0$. Conversely, if $A \geq_{(\alpha,\beta)} B$ for some $\alpha > 1$ and $\beta > 0$, then $A \succeq B$.*

Proof. (i) If $A \gg B$ for $A, B > 0$, it follows from [7] and [15] that

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \qquad \text{for all } p \geq 0 \text{ and } r \geq 0.$$

Since $\alpha \in (0, 1]$, it follows from the Löwner-Heinz theorem that

$$A^{\alpha r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\alpha r}{p+r}}.$$

Therefore we have $A \geq_{(\alpha,0)} B$.

Conversely, suppose that $A \geq_{(\alpha,0)} B$ for some $\alpha \in (0, 1]$. Taking the logarithm in both sides, we have

$$\alpha r \log A \geq \alpha r \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{p+r}} \qquad \text{for all } p, r > 0$$

and hence

$$\log A \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{p+r}} \to \log(B^p)^{\frac{1}{p}} = \log B \quad \text{as } r \to 0,$$

which implies that $A \gg B$.

(ii) If $A^\beta \geq B^\beta$ for a fixed $\beta > 0$, then Theorem F says that for each $r_1 \geq 0$

$$(A^{\beta\frac{r_1}{2}} A^{\beta p_1} A^{\beta\frac{r_1}{2}})^{\frac{1}{q}} \geq (A^{\beta\frac{r_1}{2}} B^{\beta p_1} A^{\beta\frac{r_1}{2}})^{\frac{1}{q}}$$

holds for $p_1 \geq 0$ and $q \geq 1$ with $(1+r_1)q \geq p_1 + r_1$. For each $\alpha \in [0,1]$, $p \geq \beta$ and $r \geq 0$, put $p_1 = \frac{p}{\beta}$, $r_1 = \frac{r}{\beta}$ and $q = \frac{p+r}{\alpha r+\beta}$. Since they satisfy $r_1 \geq 0$, $p_1 \geq 0$, $q \geq 1$ with $(1+r_1)q \geq p_1 + r_1$, we have

$$A^{\alpha r+\beta} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\alpha r+\beta}{p+r}}.$$

The converse is shown by putting $r = 0$ in (11).

(iii) Suppose that $A \succeq B$. Then Theorem A ensures that

$$A^{p+r} \geq A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \qquad \text{for all } p, r \geq 0.$$

For each $\alpha > 1$, $\beta \geq 0$ and $r > 0$, if $p \geq \beta + (\alpha - 1)r$, then $\frac{\alpha r+\beta}{p+r} \in [0,1]$ and so we have (11), i.e., $A \geq_{(\alpha,\beta)} B$.

Conversely, suppose that $A \geq_{(\alpha,\beta)} B$ for some $\alpha > 1$ and $\beta > 0$. For a fixed $p > 0$, we put $r = \frac{p-\beta}{\alpha-1}$. Then $\frac{\alpha r+\beta}{p+r} = 1$ and so

$$A^{\alpha r+\beta} \geq A^{\frac{r}{2}} B^p A^{\frac{r}{2}}.$$

Since $\alpha r + \beta = p + r$, it means that $A^p \geq B^p$, i.e., $A \succeq B$.

□

3 Riccati equation. In 1972, Pedersen-Takesaki [25] discussed the existence of a positive solution of the Riccati equation: For two positive operators B and C on a Hilbert space,

$$XBX = C \tag{12}$$

is called the Riccati equation by several authors, see [3, 21]. If B is invertible, it is known that the geometric mean $B^{-1} \sharp C$ is the unique positive solution of (12):

$$B^{-1} \sharp C = B^{-\frac{1}{2}} (B^{\frac{1}{2}} C B^{\frac{1}{2}})^{\frac{1}{2}} B^{-\frac{1}{2}}, \tag{13}$$

see [23]. Recently J.I.Fujii, M.Fujii, R.Nakamoto [4] and J.I.Fujii, A.Matsumoto, M.Nakamura [5] have been discussed around the Riccati equation.

We show some characterizations of the spectral order via the Riccati equation.

Theorem 5. *Let A and B be positive invertible operators. Then the following statements are mutually equivalent:*

(i) $A \succeq B$.

(ii) *There exists a unique positive contraction $T_{p,u}$ such that*

$$T_{p,u}\, A^{\frac{p+u}{2}}\, T_{p,u} = A^{\frac{u-p}{4}} B^p A^{\frac{u-p}{4}} \qquad \text{for all } p \geq 0 \text{ and } u \geq 0.$$

(iii) $I \geq A^{-\frac{p+u}{2}} \sharp \left(A^{\frac{u-p}{4}} B^p A^{\frac{u-p}{4}} \right)$ *for all $p \geq 0$ and $u \geq 0$.*

Proof. Suppose that $A \succeq B$. It follows from the Löwner-Heinz inequality and Lemma 2 that

$$A^{\frac{p+u}{2}} \geq (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{1}{2}}$$

for all $p \geq 0$ and $u \geq 0$. Put

$$T_{p,u} = A^{-\frac{p+u}{4}} (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{1}{2}} A^{-\frac{p+u}{4}}.$$

Then $T_{p,u}$ is positive contraction and

$$A^{\frac{p+u}{4}}\ T_{p,u}\ A^{\frac{p+u}{4}} = (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{1}{2}}.$$

By taking 2-power of both sides, we have

$$T_{p,u}\ A^{\frac{p+u}{2}}\ T_{p,u} = A^{\frac{u-p}{4}} B^p A^{\frac{u-p}{4}}. \tag{14}$$

Assume that there exists another positive contraction Z such that $Z\ A^{\frac{p+u}{2}}\ Z = A^{\frac{u-p}{4}} B^p A^{\frac{u-p}{4}}$. Then we have

$$\left(A^{\frac{p+u}{4}}\ Z\ A^{\frac{p+u}{4}}\right)^2 = \left(A^{\frac{p+u}{4}}\ T_{p,u}\ A^{\frac{p+u}{4}}\right)^2,$$

so that we have $Z = T_{p,u}$ since A is invertible. Consequently $T_{p,u}$ is the unique invertible positive contraction such that (14) holds.

By the unique positive solution of the Riccati equation, we have (ii) $\Longrightarrow$ (iii).

If we put $u = 0$ in (iii), then $A^{\frac{p}{2}} \geq B^{\frac{p}{2}}$ for all $p \geq 0$ and we have (i) by Theorem A. $\square$

We show some characterizations of a function order $A \geq_\delta B$, i.e., $A^\delta \geq B^\delta$ for a fixed $\delta > 0$ via the Riccati equation.

Theorem 6. *Let A and B be positive invertible operators. Then the following statements are mutually equivalent for a fixed $\delta > 0$:*

(i) $A \geq_\delta B$.

(ii) *There exists a unique positive contraction $T_{p,\delta}$ such that*

$$T_{p,\delta} A^{p-\delta} T_{p,\delta} = A^{-\frac{\delta}{2}} B^p A^{-\frac{\delta}{2}} \text{ for all } p \geq 2\delta.$$

(iii) $I \geq A^{-p+\delta}\ \sharp\ \left(A^{-\frac{\delta}{2}} B^p A^{-\frac{\delta}{2}}\right) \qquad$ *for all $p \geq 2\delta$.*

Proof. If $A^\delta \geq B^\delta$ for a fixed $\delta > 0$, then Theorem F says that for each $r_1 \geq 0$

$$A^{\delta(1+r_1)} \geq \left(A^{\delta \frac{r_1}{2}} B^{\delta p_1} A^{\delta \frac{r_1}{2}}\right)^{\frac{1+r_1}{p_1+r_1}}$$

for all $p_1 \geq 1$. Put $p_1 = \frac{p}{\delta}$ and $r_1 = \frac{r}{\delta}$. Hence we have

$$A^{\delta+r} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{\delta+r}{p+r}}$$

for all $p \geq \delta$ and $r \geq 0$. If we put $\frac{\delta+r}{p+r} = \frac{1}{2}$, then we have $r = p - 2\delta \geq 0$ and

$$A^{p-\delta} \geq \left(A^{\frac{p-2\delta}{2}} B^p A^{\frac{p-2\delta}{2}}\right)^{\frac{1}{2}}$$

for all $p \geq 2\delta$.

The equivalence of the proof is similar to the proof of Theorem 5. $\square$

Remark 7. The following equivalence of the usual order and the chaotic order are due to Furuta [16]: Let A and B be positive invertible operators. Then

(i) The usual order $A \geq B$ if and only if there exists a unique positive contraction T_p such that $T_pA^pT_p = A^{-\frac{1}{2}}B^{p+1}A^{-\frac{1}{2}}$ for all $p \geq 1$.

(ii) The chaotic order $A \gg B$ if and only if there exists a unique positive contraction T_p such that $T_pA^pT_p = B^p$ for all $p \geq 0$.

(iii) $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ if and only if there exists a unique positive contraction T_p such that $T_pA^pT_p = B^p$ for all $p \geq 1$.

Remark 8. If we replace p (resp. u) by $p+1$ (resp. $p-1$) in Theorem 5, then we have the Riccati equation (1) of Remark 7.
If we put $u = p - 2\delta$ in Theorem 5, then we have the Riccati equation (ii) of Theorem 6.
If we put $u = p$ in Theorem 5, then we have the Riccati equation (2) of Remark 7.
If we replace both p and u by $p+1$ in Theorem 5, then we have the Riccati equation (3) of Remark 7.

The following theorem continuously interpolates the chatoic order $A \gg B$ and the relation $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, also see [1].

Theorem 9. *Let A and B be positive invertible operators. Then the following statements are mutually equivalent for a fixed $\delta \in (0,1]$:*

(i) $A^\delta \geq (A^{\frac{\delta}{2}}B^\delta A^{\frac{\delta}{2}})^{\frac{1}{2}}$.

(ii) *There exists a unique positive contraction $T_{p,\delta}$ such that*

$$T_{p,\delta}A^pT_{p,\delta} = B^p \text{ for all } p \geq \delta.$$

(iii) $I \geq A^{-p} \sharp B^p \qquad$ *for all $p \geq \delta$.*

Proof. (i) $\Longleftrightarrow$ (ii). For each $\delta \in (0,1]$, put $A_1 = A^\delta, B_1 = B^\delta$ and $p_1 = \frac{p}{\delta}$. It follows from [16, Theorem 3.2] that the order $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ is equivalent to the following fact that there exists a unique positive contraction T_{p_1} such that $T_{p_1}A_1^{p_1}T_{p_1} = B_1^{p_1}$ for all $p_1 \geq 1$. In other words, there exists a unique positive contraction $T_{p,\delta}$ such that $T_{p,\delta}A^pT_{p,\delta} = B^p$ for all $p \geq \delta$.

(ii) $\Longleftrightarrow$ (iii). The equivalence is just the unique positive solution of the Riccati equation. $\square$

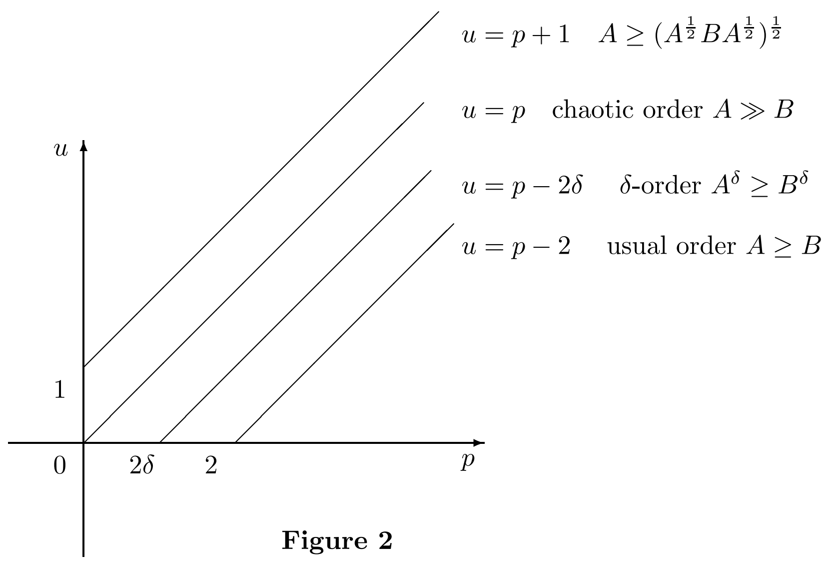

Figure 2

In the frame of the spectral order, the orders among positive operators are lined up with great regularity in quadrant I. By virture of Theorem 5, the domain (p, u) that the spectral order holds is just quadrant I. Under the framework of Figure 2, the chaotic order is represented as a line $u = p$, the δ-order as a line $u = p - 2\delta$ and the usual order as a line $u = p - 2$.

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