Scientiae Mathematicae Japonicae Online, **e-2007,** 443–453

REGULARITY AND EXHAUSTIVITY FOR FINITELY ADDITIVE FUNCTIONS. THE DIEUDONNÉ' S CONVERGENCE THEOREM

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Received May 11, 2007

Abstract. In this note we state an extension of a Cafiero's criterion and some Dieudonné convergence type theorems for semigroup-valued finitely additive functions by means of the relationships between regularity and exhaustivity.

1. Introduction

A classical theorem of Dieudonné [20] states that a sequence of regular measures on a compact metrizable space, which converges on open sets, converges on all Borel sets; moreover the measures are uniformly regular.

As observed in [1], the development within the so-called area "Topology and Measure" led, in the eighties, to investigate possible extensions of Dieudonné's result in more general setting, respect to the algebraic-topological structure of the domain of the measures as well as respect to the range and the properties of them.

Of the huge range of literature concerning the generalizations of the previous theorem, here we confine our attention to some contributions related to finitely additive regular functions, e.g. [23], [30], [1], [6], [3], [5], [12], [29].

In the seminal paper [12] C. Constantinescu introduced an algebraic notion of regularity apart from topological assumptions (see also [5]) which suggested a new approach to the problem, characterized by a close examination of the relationship between exhaustivity and regularity, e.g. [33], [13], [9], [32], [26], [27], [25].

Recently, some authors have been concerned again with Dieudonné convergence type theorems as well as with the concept of regularity in different context (see, for example, [18], [7], [28], [4], [11], [2], [10], [19]), even connecting these convergence results with a uniform exhaustivity criterion due to F. Cafiero [8].

In this note, starting from the contributions in [9], [27] and [19], we deal with the study of the relationship between regularity and exhaustivity. Subsequently, once established an extended version of Cafiero's criterion, we obtain a Dieudonné 's theorem for a sequence of uniform semigroup-valued finitely additive regular functions defined on an algebra of sets.

2000 Mathematics Subject Classification. 28B10.
Key words and phrases. Regular set function, Additive set function, Cafiero theorem, Dieudonné theorem.
This research was partially supported by G.N.A.M.P.A. of Istituto Nazionale di Alta Matematica (Italy).

2. Basic Definitions and Auxiliary Results

Let X be a non-empty set and let $\mathcal{A}$ be an algebra of subsets of X. Let $\mathcal{F}$ and $\mathcal{G}$ be, respectively, a join semilattice and a meet semilattice of subsets of X, both contained in $\mathcal{A}$ (viz, non-empty families of elements of $\mathcal{A}$ closed, respectively, under finite unions and finite intersections), such that $\mathcal{F}^* := \{X \setminus F |\ F \in \mathcal{F}\} \subseteq \mathcal{G}$.

Moreover, let $\mathcal{S} = (S, +, 0, \mathcal{U})$ be a complete Hausdorff uniform abelian semigroup, *i.e.* $(S, +, 0)$ is a commutative semigroup with neutral element 0, $[S, \mathcal{U}]$ is a Hausdorff uniform space and the function $(x, y) \to x + y$ is an associative, uniformly continuous mapping from $[S \times S, \mathcal{U} \times \mathcal{U}]$ into $[S, \mathcal{U}]$.

We just recall that in this case the uniformity $\mathcal{U}$ can be generated by a family P of semi-invariant continuous $[0, 1]$-valued pseudo-metrics on $\mathcal{S}$ (where a pseudo-metric p on $\mathcal{S}$ is semi-invariant if $p(x + z, y + z) \le p(x, y)$ for every $x, y, z \in \mathcal{S}$). We refer the reader to [31], [21], [34], [35] and [36] for more details.

In the following we denote as $\mathcal{U}[0]$ the set of the uniform neighbourhoods of the neutral element $0 \in \mathcal{S}$ of the form $U = \{s \in S \,|\, (s, 0) \in U^*\}$, where $U^* \in \mathcal{U}$.

A set function $\mu : \mathcal{A} \to \mathcal{S}$ is called *inner $\mathcal{F}$-regular* (briefly *$\mathcal{F}$-regular*) in $\mathcal{H}$, $\mathcal{H} \subseteq \mathcal{A}$, if for every $A \in \mathcal{H}$ and for every $U \in \mathcal{U}[0]$ there exists $F \in \mathcal{F}$ such that $F \subseteq A$ and $\mu(Y) \in U$, for any $Y \in \mathcal{A}_{A\setminus F}$, where $\mathcal{A}_{A\setminus F}$ is the trace of $\mathcal{A}$ on $A \setminus F$ i.e. $\mathcal{A}_{A\setminus F} := \{Y \in \mathcal{A} : Y \subseteq A \setminus F\}$.

First we show

Proposition 2.1. *If $\mu : \mathcal{A} \to \mathcal{S}$ is $\mathcal{F}$-regular in $\mathcal{A}$, then for every $A \in \mathcal{A}$ and for every $U \in \mathcal{U}[0]$ there exists $G \in \mathcal{F}^*$ such that*

$$A \subseteq G, \qquad \mu(Y) \in U \qquad \forall\, Y \in \mathcal{A}_{G\setminus A}. \tag{2.1}$$

Proof. Let $A \in \mathcal{A}$ and $U \in \mathcal{U}[0]$. Since μ is $\mathcal{F}$-regular, there exists $F \in \mathcal{F}$ such that

$$F \subseteq X \setminus A, \qquad \mu(Y) \in U \qquad \forall\, Y \in \mathcal{A}_{(X\setminus A)\setminus F}.$$

It follows that $G := X \setminus F$ belongs to $\mathcal{F}^*$ and satisfies (2.1). □

Because of the lack of duality between $\mathcal{F}$ and $\mathcal{G}$, the converse of the previous proposition does not hold in general. But we do have the following result which gives a useful characterization of finitely additive $\mathcal{F}$-regular set functions.

Proposition 2.2. *Let $\mu : \mathcal{A} \to \mathcal{S}$ be a finitely additive function. Then μ is $\mathcal{F}$-regular in $\mathcal{A}$ if and only if for every $A \in \mathcal{A}$ and for every $U \in \mathcal{U}[0]$ there exist $F \in \mathcal{F}$ and $G \in \mathcal{F}^*$ such that*

$$F \subseteq A \subseteq G, \qquad \mu(Y) \in U \qquad \forall\, Y \in \mathcal{A}_{G\setminus F}. \tag{2.2}$$

Proof. Let $A \in \mathcal{A}$ and $U \in \mathcal{U}[0]$ be given and consider $V \in \mathcal{U}[0]$ such that $V + V \subseteq U$. If μ is $\mathcal{F}$-regular, also for Proposition 2.1, there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that

$$F \subseteq A \subseteq G, \qquad \mu(Y) \in V \qquad \forall\, Y \in \mathcal{A}_{A\setminus F}, \qquad \mu(Y) \in V \qquad \forall\, Y \in \mathcal{A}_{G\setminus A}. \tag{2.3}$$

Since any $Y \in \mathcal{A}_{G\setminus F}$ can be express as disjoint union $Y = Y' \cup Y''$, with $Y' \in \mathcal{A}_{G\setminus A}$ and $Y'' \in \mathcal{A}_{A\setminus F}$, (2.2) follows from the finitely additivity of μ. The other implication is trivial. □

In the following we require $\mathcal{F}$ and $\mathcal{G}$ to satisfy the following “separation property”:

($\mathcal{P}$) for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \subseteq G$, there exist $E \in \mathcal{G}$ and $H \in \mathcal{F}$ such that $F \subseteq E \subseteq H \subseteq G$.

In this context, every finitely additive $\mathcal{F}$-regular set function has the following further property.

Proposition 2.3. *Let $\mu : \mathcal{A} \to \mathcal{S}$ be a finitely additive $\mathcal{F}$-regular function. If ($\mathcal{P}$) holds, then for every $U,\ V \in \mathcal{U}[0]$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $V + V \subseteq U$ and $F \subseteq G$, there exist $E_1, E_2 \in \mathcal{G}$ and $H_2 \in \mathcal{F}$ such that*

$$\begin{cases} F \subseteq E_2 \subseteq H_2 \subseteq E_1 \subseteq G, \\ \mu(Y) \in\ U \qquad \forall\, Y \in \mathcal{A}_{E_1 \setminus F}, \\ \mu(Y) \in\ V \qquad \forall\, Y \in \mathcal{A}_{E_2 \setminus F}. \end{cases} \tag{2.4}$$

Proof. Let U, V, F and G be as in the statement. Since μ is $\mathcal{F}$-regular, according to Proposition 2.1, there exists $E_1 \in \mathcal{G}$ such that

$$F \subseteq E_1, \qquad \mu(Y) \in\ U \qquad \forall\, Y \in \mathcal{A}_{E_1 \setminus F}.$$

Being $\mathcal{G}$ $(\cap f)$-closed, we may assume without less of generality that $E_1 \subseteq G$.

Moreover, by the separation property ($\mathcal{P}$), there exist $E' \in \mathcal{G}$ and $H' \in \mathcal{F}$ such that

$$F \subseteq E' \subseteq H' \subseteq E_1 \subseteq G.$$

So, arguing as above, but with V and E' in place of U and G, respectively, one can establish the existence of $E_2 \in \mathcal{G}$, with $F \subseteq E_2 \subseteq E'$, such that $\mu(Y) \in V$ for every set Y belonging to $\mathcal{A}_{E_2 \setminus F}$. This completes the proof. $\square$

3. Sequences in $\mathbf{ra}_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$

From now on, we employ the notation $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$ to denote the set of all finitely additive $\mathcal{F}$-regular functions from $\mathcal{A}$ into $\mathcal{S}$.

Moreover, let us recall that, if $\mathcal{H} \subseteq \mathcal{A}$, a function μ from $\mathcal{A}$ to $\mathcal{S}$ is said to be *$\mathcal{H}$-exhaustive* (briefly *exhaustive* if $\mathcal{H} = \mathcal{A}$) if $\lim_k \mu(D_k) = 0$ whenever $(D_k)_{k \in \mathbb{N}}$ is a sequence of pairwise disjoint sets from $\mathcal{H}$. Then, a sequence $(\mu_n)_{n \in \mathbb{N}}$ of function from $\mathcal{A}$ to $\mathcal{S}$ is *uniformly $\mathcal{H}$-exhaustive* if $\lim_k \mu_n(D_k) = 0$, uniformly with respect to $n \in \mathbb{N}$, for any disjoint sequence $(D_k)_{k \in \mathbb{N}}$ in $\mathcal{H}$.

In this section we deal with the relationships between the uniform $\mathcal{H}$-exhaustivity -when $\mathcal{H}$ is $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{A}$- and the uniform $\mathcal{F}$-regularity of sequences in $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$.

Lemma 3.1. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy ($\mathcal{P}$). If $(\mu_n)_{n \in \mathbb{N}}$ is uniformly $\mathcal{G}$-exhaustive, then $(\mu_n)_{n \in \mathbb{N}}$ is uniformly $\mathcal{F}$-regular in $\mathcal{F}$, i.e. for every $U \in \mathcal{U}[0]$ and every $F \in \mathcal{F}$ there exists $G \in \mathcal{G}$ such that*

$$F \subseteq G, \qquad \mu_n(Y) \in\ U \qquad \forall\, Y \in \mathcal{A}_{G \setminus F},\ \forall\, n \in \mathbb{N}.$$

Proof. Let $U \in \mathcal{U}[0]$ and $F \in \mathcal{F}$ be given. Let W and V_n, $n \in \mathbb{N}$, be closed symmetric elements of $\mathcal{U}[0]$ such that

$$W + W \subseteq U, \qquad V_1 + V_1 \subseteq W, \qquad \sum_{i=n+1}^{n+k} V_i\ \subseteq\ V_n \qquad \forall\, n \in \mathbb{N}, \forall\, k \in \mathbb{N}.$$

Since μ_1 is $\mathcal{F}$-regular in F, according to Proposition 2.1, there exists $E_1 \in \mathcal{G}$ such that

$$F \subseteq E_1, \qquad \mu_1(Y) \in V_1 \qquad \forall Y \in \mathcal{A}_{E_1 \setminus F}.$$

Suppose to have already selected $E_2, \cdots, E_n \in \mathcal{G}$ and $H_2, \cdots, H_n \in \mathcal{F}$ such that

$$\text{(3.1)} \quad \begin{cases} F \subseteq E_{i+1} \subseteq H_{i+1} \subseteq E_i & \forall i \in \{1, \cdots, n-1\}, \\ \text{for every } i \in \{1, \cdots, n\}: & \mu_j(Y) \in V_i \quad \forall Y \in \mathcal{A}_{E_i \setminus F}, \quad \forall j \in \{1, \cdots, i\}. \end{cases}$$

By the $\mathcal{F}$-regularity of $\mu_1, \cdots, \mu_{n+1}$, there exist $E'_1, \cdots, E'_{n+1} \in \mathcal{G}$ such that

$$\text{(3.2)} \qquad \forall j \in \{1, \cdots, n+1\}: \qquad F \subseteq E'_j, \qquad \mu_j(Y) \in V_{n+1} \qquad \forall Y \in \mathcal{A}_{E'_j \setminus F}.$$

On the other hand, using $(\mathcal{P})$, one can determine $E'' \in \mathcal{G}$ and $H'' \in \mathcal{F}$ such that

$$F \subseteq E'' \subseteq H'' \subseteq E_n.$$

Then, putting

$$E_{n+1} := \left(\cap_{j=1}^{n+1} E'_j \right) \cap E'', \qquad H_{n+1} := H''$$

(note that the first one pertains to $\mathcal{G}$ since it is a meet semilattice), one obtains that

$$\text{(3.3)} \qquad \begin{cases} F \subseteq E_{n+1} \subseteq H_{n+1} \subseteq E_n, \\ \mu_j(Y) \in V_{n+1} \qquad \forall Y \in \mathcal{A}_{E_{n+1} \setminus F}, \quad \forall j \in \{1, \cdots, n+1\}. \end{cases}$$

Thus it is possible to construct, by induction, two decreasing sequences of sets $(E_i)_{i \in \mathbb{N}}$ and $(H_i)_{i \in \mathbb{N}}$ in $\mathcal{G}$ and in $\mathcal{F}$, respectively, such that

$$\text{(3.4)} \qquad \begin{cases} F \subseteq E_{i+1} \subseteq H_{i+1} \subseteq E_i \qquad \forall i \in \mathbb{N}, \\ \mu_j(Y) \in V_i \qquad \forall Y \in \mathcal{A}_{E_i \setminus F}, \quad \forall i \in \mathbb{N}, \quad \forall j \in \{1, \cdots, i\}. \end{cases}$$

Now let us show that

$$(\alpha) \qquad \text{there exists } i_o \in \mathbb{N} \text{ such that:} \qquad \mu_n(Y) \in W \qquad \forall Y \in \mathcal{G}_{E_{i_o} \setminus F}, \quad \forall n \in \mathbb{N}.$$

Suppose (α) to be wrong; then for every $k \in \mathbb{N}$ there exist $Y_k \in \mathcal{G}_{E_k \setminus F}$ and $n_k \in \mathbb{N}$ such that $\mu_{n_k}(Y_k) \notin U$.

Since

$$\mu_{n_k}(Y_k) = \mu_{n_k}(Y_k \setminus H_{i+1}) + \mu_{n_k}(Y_k \cap H_{i+1}) \qquad \forall i \in \mathbb{N},$$

and $Y_k \cap H_{i+1}$ belongs to $\mathcal{A}_{E_i \setminus F}$, from (3.4) it follows that

$$\mu_{n_k}(Y_k \setminus H_{i+1}) \ \notin \ V_1 \qquad \forall k \in \mathbb{N}, \ \forall i \geq n_k.$$

Hence, by induction, it is possible to construct two sequences of natural numbers $(n_{k_l})_{l \in \mathbb{N}}$ and $(i_{k_l})_{l \in \mathbb{N}}$, the second of them increasing, and a sequence $(Y_{k_l})_{l \in \mathbb{N}}$ of elements of $\mathcal{G}$ such that for every $l \in \mathbb{N}$ it results that

$$Y_{k_l} \subseteq E_{i_{k_l}} \setminus F, \qquad \mu_{n_{k_l}}(Y_{k_l} \setminus H_{i_{k_l}+1}) \ \notin \ V_1,$$

but this contradicts the uniform $\mathcal{G}$-exhaustivity of $(\mu_n)_{n \in \mathbb{N}}$. Therefore (α) holds.

Now let $Y \in \mathcal{A}_{E_{i_o} \setminus F}$. For every fixed $n \in \mathbb{N}$, by Proposition 2.1, there exists $G_n \in \mathcal{G}$ such that

$$Y \subseteq G_n, \qquad \mu_n(T) \in W \qquad \forall\, T \in \mathcal{A}_{G_n \setminus Y},$$

and, without less of generality, we may assume that $G_n \subseteq E_{i_o} \setminus F$.

Thus, denoted by p a continuous semi-invariant pseudometric on $\mathcal{S}$ generating $\mathcal{U}$, since

$$p\big(\mu_n(G_n), \mu_n(Y)\big) = p\big(\mu_n(G_n \setminus Y) + \mu_n(Y), \mu_n(Y)\big) \leq p\big(\mu_n(G_n \setminus Y), 0\big),$$

it results that $(\mu_n(G_n), \mu_n(Y)) \in W^*$ as well as, from (α), that $(\mu_n(G_n), 0) \in W^*$.

Hence $(\mu_n(Y), 0) \in W^* \circ W^* \subseteq U^*$, i.e.

$$\mu_n(Y) \in U \qquad \forall\, Y \in \mathcal{A}_{E_{i_o} \setminus F}, \quad \forall\, n \in \mathbb{N},$$

which completes the proof. $\square$

Lemma 3.2. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy $(\mathcal{P})$. If $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{G}$-exhaustive, then for every $U \in \mathcal{U}[0]$, $F \in \mathcal{F}$, and $G \in \mathcal{G}$ containing F, there exist $E \in \mathcal{G}$ and $H \in \mathcal{F}$ such that*

$$F \subseteq E \subseteq H \subseteq G, \qquad \mu_n(Y) \in U \qquad \forall\, Y \in \mathcal{A}_{H \setminus F},\ \forall\, n \in \mathbb{N}.$$

Proof. By Lemma 3.1, there exists $G_o \in \mathcal{G}$ such that

$$F \subseteq G_o, \qquad \mu_n(Y) \in U \qquad \forall\, Y \in \mathcal{A}_{G_o \setminus F},\ \forall\, n \in \mathbb{N},$$

and we can assume that $G_o \subseteq G$. Then the assertion easily follows from the separation property $(\mathcal{P})$. $\square$

Lemma 3.3. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy $(\mathcal{P})$. If $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{G}$-exhaustive, then $(\mu_n)_{n\in\mathbb{N}}$ is also uniformly $\mathcal{F}$-exhaustive.*

Proof. Suppose the contrary. Then we may assume, by passing to a subsequence if necessary, that there exist $U \in \mathcal{U}[0]$ and a disjoint sequence $(C_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{F}$ such that

$$\mu_k(C_k) \notin U \qquad \forall\, k \in \mathbb{N}. \tag{3.5}$$

Let $V \in \mathcal{U}[0]$ and $(V_n)_{n\in\mathbb{N}}$ be a sequence of symmetric members of $\mathcal{U}[0]$ such that

$$V + V + V \subseteq U, \qquad \sum_{i=n+1}^{n+k} V_i \subseteq V_n \subseteq V \qquad \forall\, n \in \mathbb{N}, \forall\, k \in \mathbb{N}.$$

Let $F_n := \cup_{k=1}^{n} C_k$, $n \in \mathbb{N}$; then $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{F}$. From Lemmas 3.1 and 3.2, there exist $E_1 \in \mathcal{G}$ and $H_1 \in \mathcal{F}$ such that

$$F_1 \subseteq E_1 \subseteq H_1, \qquad \mu_k(Y) \in V_1 \qquad \forall\, Y \in \mathcal{A}_{H_1 \setminus F_1},\ \forall\, k \in \mathbb{N}.$$

In a same way it can be claimed that there exist $E_2 \in \mathcal{G}$ and $H_2 \in \mathcal{F}$ such that

$$F_2 \cup H_1 \subseteq E_2 \subseteq H_2, \qquad \mu_k(Y) \in V_2 \qquad \forall\, Y \in \mathcal{A}_{H_2 \setminus (F_2 \cup H_1)},\ \forall\, k \in \mathbb{N}.$$

Since

$$H_2 \setminus F_2 \subseteq \big(H_2 \setminus (F_2 \cup H_1)\big) \cup (H_1 \setminus F_2) \subseteq \big(H_2 \setminus (F_2 \cup H_1)\big) \cup (H_1 \setminus F_1),$$

any $Y \in \mathcal{A}_{H_2 \setminus F_2}$ can be written as disjoint union $Y = Y_1 \cup Y_2$, with $Y_1 \in \mathcal{A}_{H_1 \setminus F_1}$ and $Y \in \mathcal{A}_{H_2 \setminus (F_2 \cup H_1)}$.

It yields that

$$\mu_k(Y) \in V_1 + V_2 \qquad \forall\, Y \in \mathcal{A}_{H_2 \setminus F_2},\ \forall\, k \in \mathbb{N}.$$

Suppose now to have already determined $E_1, \cdots, E_n \in \mathcal{G}$ and $H_1, \cdots, H_n \in \mathcal{F}$ such that for every $i \in \{1, \cdots, n\}$ it holds

$$F_i \cup H_{i-1} \subseteq E_i, \qquad \mu_k(Y) \in \sum_{p=1}^{i} V_p \qquad \forall\, Y \in \mathcal{A}_{H_i \setminus F_i},\ \forall\, k \in \mathbb{N},$$

where $H_0 := \emptyset$. Then, from Lemmas 3.1 and 3.2 again, there exist $E_{n+1} \in \mathcal{G}$ and $H_{n+1} \in \mathcal{F}$ such that

$$F_{n+1} \cup H_n \subseteq E_{n+1} \subseteq H_{n+1}, \qquad \mu_k(Y) \in V_{n+1} \qquad \forall\, Y \in \mathcal{A}_{H_{n+1} \setminus (F_{n+1} \cup H_n)},\ \forall\, k \in \mathbb{N}.$$

Since

$$H_{n+1} \setminus F_{n+1} \subseteq \big(H_{n+1} \setminus (F_{n+1} \cup H_n)\big) \cup \big(H_n \setminus F_n\big),$$

it follows that

$$\mu_k(Y) \in \sum_{p=1}^{n+1} V_p \qquad \forall\, Y \in \mathcal{A}_{H_{n+1} \setminus F_{n+1}},\ \forall\, k \in \mathbb{N}.$$

Thus, by induction, there exist two sequences $(E_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{G}$ and in $\mathcal{F}$, respectively, such that for every $n \in \mathbb{N}$ it holds

$$F_n \subseteq E_n \subseteq H_n \subseteq E_{n+1}, \qquad \mu_k(Y) \in \sum_{p=1}^{n} V_p \qquad \forall\, Y \in \mathcal{A}_{H_n \setminus F_n},\ \forall\, k \in \mathbb{N}.$$

Hence for every $n \in \mathbb{N}$, since

$$C_{n+1} \subseteq E_{n+1} \setminus F_n = (E_{n+1} \setminus H_n) \cup (H_n \setminus F_n),$$

there exists $Y_n \in \mathcal{A}_{E_{n+1} \setminus H_n}$ such that

$$\mu_k(C_{n+1}) \in \mu_k(Y_n) + \sum_{p=1}^{n} V_p \qquad \forall\, k \in \mathbb{N},$$

and, in particular,

$$\mu_{n+1}(C_{n+1}) \in \mu_{n+1}(Y_n) + \sum_{p=1}^{n} V_p \qquad \forall\, n \in \mathbb{N}. \tag{3.6}$$

Now let E'_n belong to $\mathcal{G}$ such that

$$Y_n \subseteq E'_n, \qquad \mu_{n+1}(E'_n \setminus Y_n) \in V_{n+1};$$

without less of generality we suppose that $E'_n \subseteq E_{n+1} \setminus H_n$.

Then, from (3.6), it results

$$\mu_{n+1}(C_{n+1}) \in \quad \mu_{n+1}(E'_n) \ + \ \sum_{p=1}^{n+1} V_p \quad \subseteq \quad \mu_{n+1}(E'_n) \ + \ V + V \qquad \forall\, n \in \mathbb{N}.$$

Since $(E_{n+1} \setminus H_n)_{n\in\mathbb{N}}$ is a pairwise disjoint sequence in $\mathcal{G}$, by the uniform $\mathcal{G}$-exhaustivity of $(\mu_n)_{n\in\mathbb{N}}$, there exists $\nu \in \mathbb{N}$ such that for $n > \nu$ one has

$$\mu_k(E'_n) \ \in \ V \quad \forall\, k \in \mathbb{N},$$

therefore

$$\mu_{n+1}(C_{n+1}) \in \quad V + V + V \ \subseteq \ U \qquad \forall\, n > \nu,$$

which contradicts (3.5). $\square$

We are now able to state the main result of this section.

Theorem 3.4. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy $(\mathcal{P})$. If $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{G}$-exhaustive, then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{A}$-exhaustive and uniformly $\mathcal{F}$-regular.*

Proof. Suppose first that $(\mu_n)_{n\in\mathbb{N}}$ is not uniformly $\mathcal{A}$-exhaustive. Then there exist $U \in \mathcal{U}[0]$, a disjoint sequence $(A_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{A}$ and a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that

$$\mu_{n_k}(A_k) \notin \ U \qquad \forall\, k \in \mathbb{N}.$$

Let V be a symmetric element of $\mathcal{U}[0]$ such that $V + V \subseteq \ U$; since every μ_n belongs to $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$, for any $k \in \mathbb{N}$ there exists $F_k \in \mathcal{F}$ such that

$$F_k \subseteq A_k, \qquad \mu_{n_k}(Y) \in \ V \qquad \forall\, Y \in \mathcal{A}_{A_k \setminus F_k}.$$

Hence

$$\mu_{n_k}(A_k) \ \in \ \mu_{n_k}(F_k) \ + \ V \qquad \forall\, k \in \mathbb{N},$$

and so

$$\mu_{n_k}(F_k) \notin \ V \qquad \forall\, k \in \mathbb{N}.$$

This contradicts the uniform $\mathcal{F}$-exhaustivity of $(\mu_n)_{n\in\mathbb{N}}$, that Lemma 3.3 assures.

In order to prove the uniform $\mathcal{F}$-regularity of $(\mu_n)_{n\in\mathbb{N}}$, let $U \in \mathcal{U}[0]$ and let $(V_n)_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of symmetric members of $\mathcal{U}[0]$ such that

$$\sum_{i=0}^{n} V_i \ \subseteq \ U \qquad \forall\, n \in \mathbb{N}, \qquad \sum_{i=n+1}^{n+k} V_i \ \subseteq \ V_n \qquad \forall\, n \in \mathbb{N}, \forall\, k \in \mathbb{N}.$$

Let $A \in \mathcal{A}$. Using the $\mathcal{F}$-regularity of the μ_n, one can determine an increasing sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$ and a decreasing sequence $(G_k)_{k\in\mathbb{N}}$ in $\mathcal{G}$ such that for every $k \in \mathbb{N}$ it holds

$$F_k \subseteq A \subseteq G_k, \qquad \mu_n(Y) \in \ V_k \qquad \forall\, Y \in \mathcal{A}_{G_k \setminus F_k}, \ \forall\, n \in \{1, \cdots, k\}. \tag{3.7}$$

Since $(G_k \setminus F_k)_{k\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{G}$, the above proved uniform $\mathcal{A}$-exhaustivity of $(\mu_n)_{n\in\mathbb{N}}$ implies that for any $Y \in \mathcal{A}$ the sequence

$$(\mu_n(Y \cap (G_k \setminus F_k)))_{k\in\mathbb{N}}$$

is a Cauchy sequence, uniformly with respect to $Y \in \mathcal{A}$ and to $n \in \mathbb{N}$.

Thus there exists $\nu_o \in \mathbb{N}$ such that

$$\Big(\mu_n(Y \cap (G_p \setminus F_p)), \mu_n(Y \cap (G_q \setminus F_q))\Big) \in V_o^* \quad \text{for every } \ p, q \geq \nu_o,\ n \in \mathbb{N},\ Y \in \mathcal{A};$$

in particular

$$\text{(3.8)} \quad \mu_n(Y \cap (G_{\nu_o} \setminus F_{\nu_o})) \in V_o + \mu_n(Y \cap (G_q \setminus F_q)) \quad \text{for every } q \geq \nu_o,\ n \in \mathbb{N},\ Y \in \mathcal{A}.$$

On the other hand, from (3.7) it follows that for any fixed $n \in \mathbb{N}$, denoted as q a natural number bigger than ν_o and n, it results that

$$\text{(3.9)} \qquad \mu_n(Y \cap (G_q \setminus F_q)) \in V_q \subseteq V_1 \qquad \forall\, Y \in \mathcal{A};$$

hence, from (3.8) and (3.9), one obtains that

$$\mu_n(Y \cap (G_{\nu_o} \setminus F_{\nu_o})) \in V_o + V_1 \subseteq U \qquad \forall\, Y \in \mathcal{A},\ \forall\, n \in \mathbb{N},$$

which ends the proof. □

4. Cafiero - Dieudonné theorem

Before stating the main results of the paper, let us recall some definitions.

A ring $\mathcal{R}$ is said to have the *subsequential interpolation property* (SIP) if for every subsequence $(R_{n_k})_{k\in\mathbb{N}}$ of any pairwise disjoint sequence $(R_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{R}$ there exist a subsequence $(R_{n_{k_l}})_{l\in\mathbb{N}}$ and an element $R \in \mathcal{R}$ such that

$$R_{n_{k_l}} \subseteq R \quad \text{for every } l \in \mathbb{N}, \quad \text{and} \quad R_j \cap R = \emptyset \quad \text{for any } \ j \in \mathbb{N} \setminus \{n_{k_l} : l \in \mathbb{N}\}.$$

Moreover a ring $\mathcal{R}$ is said to satisfy the *sequential completeness property* (SCP) whenever every disjoint sequence $(R_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ admits a subsequence $(R_{n_k})_{k\in\mathbb{N}}$ whose union is in $\mathcal{R}$ (we refer the reader to [12], [17], [22], and (P2)-(P1) in [36] for more details).

Then, if $\mathcal{L}$ is a meet semilattice in $\mathcal{A}$, we say that $\mathcal{L}$ is a *SIP* (resp. *SCP*)*-semilattice* if for any pairwise disjoint sequence $(L_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{L}$, there exist a subsequence $(L_{n_k})_{k\in\mathbb{N}}$ and a ring with the SIP (resp. SCP) property, containing all the L_{n_k} and contained in $\mathcal{L}$.

First we determine an extended version of Cafiero's criterion ([8]).

Theorem 4.1 (*Cafiero-Dieudonné*). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{G}$-exhaustive elements of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy $(\mathcal{P})$, and $\mathcal{G}$ is a SIP-semilattice.*

Then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{G}$-exhaustive if and only if the following condition holds

($\star$) *for every pairwise disjoint sequence $(G_k)_{k\in\mathbb{N}}$ in $\mathcal{G}$ and for every $U \in \mathcal{U}[0]$ there exist $k_o, \nu_o \in \mathbb{N}$ such that*

$$\mu_n(G_{k_o}) \in U \qquad \forall\, n \geq \nu_o.$$

Proof. The necessity of the condition ($\star$) is trivial.

For the sufficiency, we argue by contradiction. Let us assume, by passing to a subsequence if necessary, that there exist $U_o \in \mathcal{U}[0]$ and a pairwise disjoint sequence $(G_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{G}$ such that

$$\text{(4.1)} \qquad \mu_k(G_k) \notin U_o \qquad \forall\, k \in \mathbb{N}.$$

Since $\mathcal{G}$ is a SIP-semilattice, there exist a subsequence $(G_{k_l})_{l\in\mathbb{N}}$ and a ring $\mathcal{R}$ with the subsequential interpolation property such that $\mathcal{R} \subseteq \mathcal{G}$ and $G_{k_l} \in \mathcal{R}$, for every $l \in \mathbb{N}$. As the restrictions of the μ_k to $\mathcal{R}$ satisfy the hypotheses of Cafiero Theorem (5.2) in [14] (since, in the Boolean case, (4.3) in [14] holds even for rings, as shown in [17]), then they are uniformly $\mathcal{R}$-exhaustive and, in particular, one has

$$\lim_{l\to+\infty} \mu_k(G_{k_l}) = 0 \qquad \text{uniformly with respect to } k \in \mathbb{N},$$

a contradiction of (4.1). □

Remark 4.2. We stress that, whenever the hypotheses of the previous Theorem are fulfilled, $(\star)$ implies also the uniform $\mathcal{A}$-exhaustivity as well as the uniform $\mathcal{F}$-regularity of the given sequence $(\mu_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$-exhaustive elements of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$, owing to Theorem 3.4.

Remark 4.3. It is worth noting that, in order to prove the uniform $\mathcal{R}$-exhaustivity of the restrictions of the μ_k to $\mathcal{R}$, it is possible to apply also Corollary 4.3 in [36], instead of (5.2) in [14], taking into account the arguments in [36], pp. 272-273.

We point out that by Theorem 4.1 and Remark 4.2, one can fairly easy prove the following result.

Corollary 4.4. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{G}$-exhaustive elements of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy ($\mathcal{P}$), and $\mathcal{G}$ is a SIP-semilattice.*

If $(\mu_n)_{n\in\mathbb{N}}$ converges pointwise to a $\mathcal{G}$-exhaustive set function in $\mathcal{G}$, i.e.

$$\lim_{n\to+\infty} \mu_n(G) = \mu_o(G), \qquad G \in \mathcal{G},$$

then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{A}$-exhaustive and uniformly $\mathcal{F}$-regular on the whole $\mathcal{A}$.

Now we can establish the Dieudonné convergence type theorem.

Theorem 4.5 (*Dieudonné convergence theorem*)**.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{G}$-exhaustive elements of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy ($\mathcal{P}$), and $\mathcal{G}$ is a SIP-semilattice.*

If $(\mu_n)_{n\in\mathbb{N}}$ converges pointwise to a $\mathcal{G}$-exhaustive set function μ_o in $\mathcal{G}$, then $(\mu_n)_{n\in\mathbb{N}}$ converges pointwise in $\mathcal{A}$ to an exhaustive element of $ra_{\mathcal{F}}(\mathcal{A}, \mathcal{S})$.

Proof. Let us consider $A \in \mathcal{A}$ and $U \in \mathcal{U}[0]$ arbitrary fixed. Then let V be a symmetric element of $\mathcal{U}[0]$ such that $V + V + V \subseteq U$.

From an easily deduced uniform version of Proposition 2.1, one can determine the existence of an element G in $\mathcal{G}$ such that

$$A \subseteq G, \qquad \mu_n(Y) \in V \qquad \forall\, Y \in \mathcal{A}_{G\setminus A},\ \forall\, n \in \mathbb{N}.$$

Hence, from the finitely additivity of the μ_n, for every $p, q \in \mathbb{N}$ it holds that

$$\Big(\mu_p(A), \mu_q(A)\Big) = \Big(\mu_p(G), \mu_q(G)\Big) + \Big(\mu_p(A \setminus G), \mu_q(A \setminus G)\Big) \in \Big(\mu_p(G), \mu_q(G)\Big) \circ V^* \circ V^*.$$

Therefore the end of the proof follows from the pointwise convergence of $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$. □

Finally, we show that for group-valued finitely additive regular functions the requirement in Corollary 4.4 and in Theorem 4.5 that the limit function μ_o was $\mathcal{G}$-exhaustive can be eliminated.

The final theorem presented below is, in fact, the "group" version of Theorem 4.5.

Theorem 4.6 (*Dieudonné convergence theorem for group-valued functions*)**.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{G}$-exhaustive elements of $ra_{\mathcal{F}}(\mathcal{A},\Gamma)$, where $\Gamma=(\Gamma,+,0,\tau)$ is an abelian complete Hausdorff topological group. Assume that $\mathcal{F}$ and $\mathcal{G}$ satisfy $(\mathcal{P})$, and $\mathcal{G}$ is a SIP-semilattice.*

If $(\mu_n)_{n\in\mathbb{N}}$ converges pointwise in $\mathcal{G}$, i.e.

$$\lim_{n\to+\infty}\mu_n(G)=\mu_o(G),\qquad G\in\mathcal{G},$$

then

(i) $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{A}$-exhaustive and uniformly $\mathcal{F}$-regular,

(ii) $(\mu_n)_{n\in\mathbb{N}}$ converges pointwise in the whole $\mathcal{A}$ to an exhaustive element of $ra_{\mathcal{F}}(\mathcal{A},\Gamma)$.

Proof. If $\mu_o\equiv 0$ in $\mathcal{G}$, *(i)* easily follows from Theorem 4.1 and Remark 4.2.

In the general case, we argue by contradiction. From Theoren 3.4, then we can assume, by passing to a subsequence if necessary, that there exist a neighbourhood U_o of the neutral element 0 (briefly $U_o\in\mathcal{I}(0)$) and a disjoint sequence $(G_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{G}$ such that

$$\mu_k(G_k)\notin\ U_o\qquad\forall\, k\in\mathbb{N}. \tag{4.2}$$

Let V be a symmetric element of $\mathcal{I}(0)$ such that $V+V\subseteq\ U$; since every μ_k is $\mathcal{G}$-exhaustive, it is possible to construct a subsequence $(\mu_{k_l})_{l\in\mathbb{N}}$ of $(\mu_k)_{k\in\mathbb{N}}$ such that for every $l\in\mathbb{N}$ one has

$$\mu_{k_l}(G_j)\in\ V\qquad\forall\, j\geq k_{l+1}. \tag{4.3}$$

Thus, the sequence $(\mu_{k_{l+1}}-\mu_{k_l})_{l\in\mathbb{N}}$ is a sequence of $\mathcal{G}$-exhaustive elements of $ra_{\mathcal{F}}(\mathcal{A},\Gamma)$ which pointwise converges to 0 in $\mathcal{G}$. Hence, from the starting observation of this proof, it is uniformly $\mathcal{A}$-exhaustive, but, from (4.2) and (4.3), it follows also that for every $l\in\mathbb{N}$

$$\big(\mu_{k_{l+1}}-\mu_{k_l}\big)(G_{k_{l+1}})\notin\ V,$$

a contradiction. This completes the proof of *(i)*.

Assertion *(ii)* can be proved as in Theorem 4.5. □

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