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p-QUASI-λ-NUCLEARITY

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Abstract. In this paper we give a positive answer for the following problem: Assume that $2 < r < s < +\infty$. Is it possible to find a sequence space λ which is proper subset of ℓ_1 and a linear map T between normed spaces E and F such that T is s-quasi-λ-nuclear which is not r-quasi-λ-nuclear? [2]

1 Basic Concepts.

For two sequences of scalars $x = (x_n)$ and $y = (y_n)$, we write $x_n = O(y_n)$ if there is $\rho > 0$ such that $x_n \leq \rho y_n$ for all $n \in \mathbf{N}$.

For two sequence spaces λ_1 and λ_2, we define the sequence space $\lambda_1 * \lambda_2$ to be $\{(x_n) : (x_{2n}) \in \lambda_2, (x_{2n-1}) \in \lambda_1\}$.

A set A of sequences of non-negative real numbers is called a **Köthe set**, if it satisfies the following conditions:

1. For each pair of elements $a, b \in A$ there is $c \in A$ with $a_n = O(c_n)$ and $b_n = O(c_n)$.

2. For every integer $r \in N$ there exists $a \in A$ with $a_r > 0$.

The space of all sequences $x = (x_n)$ such that

$$p_a(x) := \sum_n |x_n| \, a_n < +\infty$$

for all $a \in A$, is called the **Köthe space**, $\lambda(A)$, generated by A.

Let $\alpha = (\alpha_n)$ be an unbounded non-decreasing sequence of positive real numbers. Then $P_\infty = \{(k^{\alpha_n}) : \, k \in \mathbf{N}\}$ is countable Köthe set. The corresponding Köthe space $\Lambda_\infty(\alpha) = \lambda(P_\infty)$ is called the **power series of infinite type** [1].

By the sequence e_n we mean the sequence whose nth term is 1 and all other terms are 0.

Definition 1.1 *For $0 < p < +\infty$, a linear map T of a normed space E into a normed space F is called a* **p-quasi-λ-nuclear map** *if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that*

$$||Tx|| \leq \left(\sum\nolimits_n |\alpha_n| \, |\langle x, a_n\rangle|^p\right)^{1/p},$$

for all x in E [2].

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2 Main results. For $2 < r < s < +\infty$ with $\frac{1}{r} + \frac{1}{r'} = 1$, we introduce a sequence space $\lambda = \ell_1 * \Lambda_\infty(n)$ which is a proper subset of ℓ_1, two normed spaces $E = \ell_{r'}$ and $F = \ell_s$, and a linear map T between normed spaces E and F such that T is s-quasi-λ-nuclear which is not r-quasi-λ-nuclear.

Example 2.1 *Let $2 < r < s < +\infty$. Define a map*

$$T : \ell_{r'} \to \ell_s$$

by

$$Tx = \left(\frac{x_1}{1}, \frac{x_2}{2^2}, \frac{x_3}{3^{\frac{1}{r}}}, \frac{x_4}{4^4}, \frac{x_5}{5^{\frac{1}{r}}}, \ldots\right),$$

where r and r' satisfy

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

The map T is s-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear which is not r-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear

Proof. Note that

$$||Tx||_s^s = \sum_n \left|\frac{x_{2n-1}}{(2n-1)^{\frac{1}{r}}}\right|^s + \sum_n \left|\frac{x_{2n}}{(2n)^{2n}}\right|^s.$$

Let $a_n = e_n$ and

$$(\beta_n) = \left(1, \frac{1}{2^{2s}}, \frac{1}{3^{\frac{s}{r}}}, \frac{1}{4^{4s}}, \frac{1}{5^{\frac{s}{r}}}, \ldots\right).$$

Since (e_n) is bounded sequence in $\ell_{r'}$, $(\beta_n) \in \ell_1 * \Lambda_\infty(n)$ and

$$||Tx||_s^s = \sum_n |\beta_n||\langle x, e_n\rangle|^s,$$

we have T is s-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear. To prove that T is not r-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear. Assume the contrary, then there exist a sequence (α_n) in $\ell_1 * \Lambda_\infty(n)$ and a bounded sequence $(a_n := (a_1^n, a_2^n, \ldots))_{n=1}^\infty$ in ℓ_r such that

$$||Tx||^r \le \sum_n |\alpha_n||\langle x, a_n\rangle|^r \quad \text{for all } x \in \ell_{r'}.$$

For $m \in \mathbf{N}$, we have

$$||Te_{2m-1}||^r \le \sum_n |\alpha_n||\langle e_{2m-1}, a_n\rangle|^r$$

and

$$||Te_{2m}||^r \le \sum_n |\alpha_n||\langle e_{2m}, a_n\rangle|^r.$$

Since

$$\langle e_{2m-1}, a_n\rangle = a_{2m-1}^n \quad \text{and} \quad \langle e_{2m}, a_n\rangle = a_{2m}^n,$$

we have

$$\frac{1}{2m-1} \le \sum_n |\alpha_n||a_{2m-1}^n|^r$$

and

$$\frac{1}{(2m)^{2mr}} \le \sum_n |\alpha_n||a_{2m}^n|^r.$$

Using the above two inequalities, we have

$$\begin{aligned}\sum_m \left(\frac{1}{2m-1} + \frac{1}{(2m)^{2mr}}\right) &\leq \sum_m \sum_n |\alpha_n||a_m^n|^r \\ &= \sum_n |\alpha_n| \sum_m |a_m^n|^r.\end{aligned}$$

Thus we have

$$\sum_{m=1}^{\infty} \left(\frac{1}{2m-1} + \frac{1}{(2m)^{2mr}}\right) \leq \sum_{n=1}^{\infty} |\alpha_n|||a_n||_r^r$$

But

$$\sum_{m=1}^{\infty} \left(\frac{1}{2m-1} + \frac{1}{(2m)^{2mr}}\right)$$

diverges while

$$\sum_{n=1}^{\infty} |\alpha_n|||a_n||_r^r$$

converges. So T is not r-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear. ■

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