STATIONARY SOLUTIONS FOR FOREST KINEMATIC MODEL UNDER DIRICHLET CONDITIONS

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Abstract. We continue the study of a mathematical model for the forest ecosystem which has been presented by Kuzunetsov et al. [5] and which is equipped with the Dirichlet boundary conditions. In the preceding two papers [7, 8], we constructed a dynamical system and investigated asymptotic behavior of trajectories of the dynamical system. This paper is then devoted to studying the stationary problem. Stability and instability of the zero stationary solution is investigated. When the mortality of trees is suitably small, it is shown by the numerical methods that the system possesses a unique inhomogeneous solution which has a clear gap which indicates discontinuity of the density of trees.

1 Introduction This paper is devoted to studying the stationary problem for a forest kinematic model presented by Kuzunetsov et al. [5]:

$$
(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty), \\ w = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in} \quad \Omega. \end{cases}
$$

Here, Ω is a two-dimensional bounded domain. The unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and time $t \in [0,\infty)$. The third unknown function $w = w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates, respectively. On the w, the Dirichlet boundary conditions are imposed. While the first and second equations describe the growth of young and old trees, respectively. The constant $0 < \delta \leqslant 1$ is an establishment rate of seeds, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density v and is expected to hit a minimum at a certain optimal value of v, say $b > 0$, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees.

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In the preceding two papers [7, 8], we studied the evolutionary problem of (1.1). In [7], we constructed global solutions and a dynamical system $(S(t), K, X)$ determined from (1.1). As the underlying space X, we set a space of the form

$$X = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix};\ u \in L^\infty(\Omega),\, v \in L^\infty(\Omega),\, w \in L^2(\Omega) \right\}. \tag{1.2}$$

The phase space K consists of triplets of nonnegative functions of X, i.e.,

$$K = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix};\ 0 \le u \in L^\infty(\Omega),\, 0 \le v \in L^\infty(\Omega),\, 0 \le w \in L^2(\Omega) \right\}. \tag{1.3}$$

The nonlinear semigroup $S(t)$ acts on K for $0 \le t < \infty$. In [8], we found a Lyapunov function and investigated asymptotic behavior of trajectories $S(t)U_0$, $U_0 \in K$. Since some $S(t)U_0$ possibly converges to discontinuous stationary solution even if the initial value $U_0 \in K$ consists of smooth functions (see the numerical results presented in Section 5) and since if so the trajectory $S(t)U_0$ must have an empty ω-limit set in X, the dynamical system $(S(t), K, X)$ never enjoys a compact attractor in general. By this reason we have introduced three kinds of ω-limit sets for $U_0 \in K$, i.e., $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \mathrm{w}^*\text{-}\omega(U_0) \neq \emptyset$, here $\omega(U_0)$ denotes the usual one (see [10, 13]), $L^2\text{-}\omega(U_0)$ is an ω-limit set with respect to the L^2 topology and $\mathrm{w}^*\text{-}\omega(U_0)$ is that with respect to the weak* topology of $L^\infty(\Omega)$. And we proved by utilizing the Lyapunov function that $L^2\text{-}\omega(U_0)$ consists of stationary solutions only. So, roughly speaking, every trajectory $S(t)U_0$, $U_0 \in K$, converges asymptotically to some stationary solution of (1.1).

In this paper, we intend to study the stationary problem of (1.1). We shall first study homogeneous stationary solutions. It is easy to see that the Dirichlet boundary conditions imply a fact that the zero solution is a unique homogeneous solution of (1.1), cf. the Neumann condition case [4]. The zero solution will then be shown to be unstable if $0 < h < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$ and to be stable if $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \infty$, where a, b and c are positive constants contained in $\gamma(v)$ (see (1.4) below) and where $\mu_0 > 0$ denotes the minimal eigenvalue of the Laplace operator $-\Delta$ in $L^2(\Omega)$ equipped with the Dirichlet boundary conditions. Furthermore, it will be shown that the zero solution is globally stable, namely, every solution tends to zero as $t \to \infty$, if $\big(\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} <\big) \frac{f\alpha\delta}{c+f} < h < \infty$. Secondly, we shall seek inhomogeneous stationary solutions by the numerical methods. In fact, when $0 < h < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$, some numerical computations show that (1.1) possesses a unique inhomogeneous solution. On the contrary, when $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \frac{f\alpha\delta}{c+f}$, some numerical computations suggest that (1.1) possesses a large number of inhomogeneous solutions. As mentioned, when $\frac{f\alpha\delta}{c+f} < h < \infty$, it is rigorously shown that the zero solution is a globally stable stationary solution, and hence no other (inhomogeneous) stationary solution can exist.

Throughout the paper, Ω is a convex or $\mathcal{C}^2$, bounded domain in $\mathbb{R}^2$. We assume as in the paper [5] that the function $\gamma(v)$ is given by a quadratic function

$$\gamma(v) = a(v-b)^2 + c, \tag{1.4}$$

where a, b, $c > 0$ are all positive constants.

2 Homogeneous stationary solution Consider a homogeneous stationary solution $(\overline{u},\overline{v},\overline{w})$ to (1.1). Obviously, $(\overline{u},\overline{v},\overline{w})$ is a solution to the algebraic equations

$$\begin{cases} \beta\delta\overline{w}-\gamma(\overline{v})\overline{u}-f\overline{u}=0, \\ f\overline{u}-h\overline{v}=0, \\ -\beta\overline{w}+\alpha\overline{v}=0 \end{cases} \tag{2.1}$$

for $\overline{U}=(\overline{u},\overline{v},\overline{w})\in\mathbb{R}^3$ with $\overline{u}\geq 0,\ \overline{v}\geq 0,\ \overline{w}\geq 0$. In the meantime, we have $\overline{w}=0$ because of the Dirichlet boundary conditions on the unknown w. Therefore, it follows that $\overline{u}=\overline{v}=0$. Hence, the zero solution $O=(0,0,0)$ is a unique homogeneous stationary solution of (1.1).

3 Stability and instability of O We are then interested in investigating stability and instability of O. We will localize the problem (1.1) in a neighborhood of O and will extend the dynamical system $(S(t),\mathcal{X},X)$ determines from (1.1) in [7] to the complex-valued functions in the neighborhood of O in order to apply the linearized principle for nonlinear evolution equations, see [10, 13] (cf. also [1]).

Let $\chi(\lambda)$ be a cutoff function defined on the complex plane $\mathbb{C}$ such that $\chi(\lambda)=\lambda$ for $\lambda:|\lambda|<1$, $\chi(\lambda)$ vanishes for $\lambda:|\lambda|>2$, and $\chi(\lambda)$ is a smooth function in the real variables λ' and λ'' such that $\lambda=\lambda'+i\lambda''$.

The localized problem is then written in the form

$$\begin{cases} \dfrac{\partial u}{\partial t}=\beta\delta\chi(w)-\gamma(\chi(v))\chi(u)-fu & \text{in}\quad \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial t}=f\chi(u)-hv & \text{in}\quad \Omega\times(0,\infty), \\ \dfrac{\partial w}{\partial t}=d\Delta w-\beta w+\alpha\chi(v) & \text{in}\quad \Omega\times(0,\infty), \\ w=0 & \text{on}\quad \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x) & \text{in}\quad \Omega. \end{cases} \tag{3.1}$$

We can handle this localized problem in a quite analogous way as for the original one (see [7]) by applying the general method presented in [6]. In fact, the problem (3.1) is formulated as the initial value problem for an abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt}+AU=\widetilde{F}(U), \qquad 0<t\leq\infty, \\ U(0)=U_0 \end{cases} \tag{3.2}$$

in function space X. Here, the linear operator A is defined by

$$A=\begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A)=\left\{\begin{pmatrix} u \\ v \\ w \end{pmatrix};\ u,v\in L^\infty(\Omega)\ \text{ and }\ w\in H^2_D(\Omega)\right\},$$

where Λ is a realization of the Laplace operator $-d\Delta+\beta$ in $L^2(\Omega)$ under the Dirichlet boundary conditions on the boundary $\partial\Omega$ (see [11, Chap. VI]). According to [9] (in particular, in the case when Ω is merely convex), it is known that Λ is a nonnegative self-adjoint operator of $L^2(\Omega)$ with

$$\mathcal{D}(\Lambda^\theta)=\begin{cases} H^{2\theta}(\Omega) & \text{if } 0\leq\theta<\frac14, \\ H^{2\theta}_D(\Omega)=\{w\in H^{2\theta}(\Omega);\ w=0\ \text{ on }\ \partial\Omega\} & \text{if } \frac14<\theta\leq 1,\ \theta\neq\frac34. \end{cases}$$

It is clear that A is a sectorial operator with angle less than $\frac{\pi}{2}$. Moreover, for $0 \le \theta \le 1$, we have

$$A^\theta = \begin{pmatrix} f^\theta & 0 & 0 \\ 0 & h^\theta & 0 \\ 0 & 0 & \varLambda^\theta \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A^\theta) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix};\ u, v \in L^\infty(\Omega) \ \text{ and } \ w \in \mathcal{D}(\varLambda^\theta) \right\}.$$

The nonlinear operator $\widetilde{F}$ is given by

$$\widetilde{F}(U) = \begin{pmatrix} \beta\delta\chi(w) - \gamma\big(\chi(v)\big)\chi(u) \\ f\chi(u) \\ \alpha\chi(v) \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(A^\eta),$$

where η is an arbitrarily fixed exponent in such a way that $\frac{1}{2} < \eta < 1$. Initial value U_0 is taken from $\mathcal{D}(A^\theta)$ with $0 \le \theta < \eta$.

Since $\chi(u)$, $\chi(v)$ and $\chi(w)$ are uniformly bounded, we can use the same arguments as in [7] to construct local solutions and global solutions for all initial values from $\mathcal{D}(A^\theta)$ in the function space

$$U \in \mathcal{C}([0,\infty); \mathcal{D}(A^\theta)) \cap \mathcal{C}^1((0,\infty); X) \cap \mathcal{C}((0,\infty); \mathcal{D}(A)).$$

Therefore, the localized problem (3.1) defines a semigroup $\widetilde{S}(t)$ acting on $\mathcal{D}_\theta = \mathcal{D}(A^\theta)$, $0 \le \theta < 1$. By the similar argument as for verifying the Lipschitz continuity of solutions of (1.1) ([7, Proposition 5.3]), $\widetilde{S}(t)$ is also shown to enjoy the same Lipschitz conditions. In this way, Problem (3.1) defines a dynamical system $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$.

From now we fix θ in such a way that $\frac{1}{2} < \theta < 1$ in order to have $\mathcal{D}_\theta \subset \mathbb{L}^\infty(\Omega) \equiv L^\infty(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$. Then, in a suitable neighborhood of O in $\mathcal{D}_\theta$, any solution of the original problem (i.e., (1.1)) is a solution of (3.1), too. Therefore, in such a neighborhood, any trajectory of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$ is that of $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$, where $\mathcal{K}_\theta = K \subset \mathcal{D}_\theta$. Clearly, O is an equilibrium of $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$, too. Furthermore, we notice that, if O is stable as an equilibrium of $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$, then it is the same as that of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$. However, we cannot say that, if O is unstable in $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$, then it is the same as that of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$. Nevertheless, instability of O in $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$ provides crucial information concerning the behavior of trajectories of the original system $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$ in the neighborhood of O.

Let us verify the Fréchet differentiability of $\widetilde{S}(t)$ in the neighborhood of O. We know that $\widetilde{S}(t)$ is determined by the Cauchy problem for a semilinear equation of the form (3.2). In a neighborhood of O in $\mathcal{D}(A^\eta) \subset \mathcal{D}(A^\theta) \subset \mathbb{L}^\infty(\Omega)$, $\widetilde{F}$ is Fréchet differentiable with the derivative

$$\widetilde{F}'(U) = \begin{pmatrix} -\gamma(v) & -\gamma'(v)u & \beta\delta \\ f & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}, \qquad U \in B^{\mathcal{D}(A^\eta)}(O; r).$$

We can then repeat the same arguments as in [1] to conclude that $\widetilde{S}(t)\colon \mathcal{D}_\theta \to \mathcal{D}_\theta$ is of class $\mathcal{C}^{1,1}$ in a neighborhood of O in $\mathcal{D}_\theta$ for $0 \le t \le T$, where $T > 0$. In particular, $\widetilde{S}(t)$ is Fréchet differentiable at O for any $0 \le t < \infty$ with the derivative $[\widetilde{S}(t)]'O = e^{-t\overline{A}}$, where $e^{-t\overline{A}}$ is an analytic semigroup on X generated by

$$\overline{A} = A - \widetilde{F}'(O) = \begin{pmatrix} m & 0 & -\beta\delta \\ -f & h & 0 \\ 0 & -\alpha & \varLambda \end{pmatrix}, \tag{3.3}$$

where $m = \gamma(0) + f = ab^2 + c + f$.

We shall next verify the hyperbolicity of O, i.e., $\sigma([\widetilde{S}(t)]'0) \cap \{\lambda \in \mathbb{C};\ |\lambda| = 1\} = \emptyset$, as an equilibrium of the dynamical system $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$. But for this end we know that it suffices to verify that

$$\sigma(\overline{A}) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda = 0\} = \emptyset.$$

Let us find $\lambda \in \mathbb{C}$ such that, for any $H \in X$, the vector equation

$$(\lambda - \overline{A})U = H, \quad H = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \in X,\ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(\overline{A}),$$

or the system

$$\begin{cases} (\lambda - m)u + \beta\delta w &= p, \\ fu + (\lambda - h)v &= q, \\ \alpha v + (\lambda - \Lambda)w &= r \end{cases}$$

has a unique solution $U \in \mathcal{D}(\overline{A})$. From these equations, it then follows that

$$\{(\lambda - m)(\lambda - h)(\lambda - \Lambda) + f\alpha\beta\delta\}w = f\alpha p - \alpha(\lambda - m)q + (\lambda - m)(\lambda - h)r.$$

Therefore, if λ is a solution to the quadratic equation $(\lambda - m)(\lambda - h) = 0$, i.e., $\lambda = m$ or h, then w cannot belong to $H^2(\Omega)$ in general, i.e., $m,\ h \in \sigma(\overline{A})$. Now, let $\lambda \neq m,\ h$, then $\lambda \in \sigma(\overline{A})$ if and only if $\lambda + \frac{f\alpha\beta\delta}{(\lambda-m)(\lambda-h)} \in \sigma(\Lambda)$. In other words, $\lambda \in \sigma(\overline{A})$ if and only if λ is a solution to one of the following infinite number of cubic equations

$$(\lambda - m)(\lambda - h)(\lambda - d\mu_n - \beta) + f\alpha\beta\delta = 0, \tag{3.4}$$

where $0 < \mu_0 < \mu_1 \leq \mu_2 \leq \dots$ are the eigenvalues of the Laplace operator $-\Delta$ in $L^2(\Omega)$ equipped with the Dirichlet boundary conditions.

Theorem 3.1. *The zero solution O is a hyperbolic equilibrium if and only if $(ab^2 + c + f)h(d\mu_n + \beta) - f\alpha\beta\delta \neq 0$ for every $n = 0, 1, 2, \dots$.*

Proof. Necessity is trivial because if $mh(d\mu_n + \beta) - f\alpha\beta\delta = 0$ with some μ_n then $\lambda = 0$ is clearly in $\sigma(\overline{A})$.

Now, let $mh(d\mu_n + \beta) - f\alpha\beta\delta \neq 0$ for every $n = 0, 1, 2, \dots$. It is easy to see that (3.4) has no imaginary solutions. Indeed, assume that $\lambda = iy,\ y \in \mathbb{R}$ is a solution of (3.4) for some $\mu_n = \mu_{n_0}$. Then, by directly calculating, we observe that $y \neq 0$ and

$$\begin{cases} y^2 = (m + h)(d\mu_{n_0} + \beta) + mh, \\ y^2(m + h + d\mu_{n_0} + \beta) = mh(d\mu_{n_0} + \beta) - f\alpha\beta\delta. \end{cases}$$

But it is impossible that these two equalities are valid at the same time. ∎

Theorem 3.2. *i) Let $0 < h < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$ and let the condition*

$$\mu_n \neq \frac{\beta\{f\alpha\delta - (ab^2 + c + f)h\}}{(ab^2 + c + f)hd} \qquad \textit{for every} \quad n = 0, 1, 2, \dots$$

be satisfied. Then, O is an unstable equilibrium of $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$.

ii) Let $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \infty$. Then, O is an exponentially stable equilibrium of $(\widetilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$.

Proof. *i)* By using Theorem 3.1, we obtain that O is a hyperbolic equilibrium. It is now suffices to verify that $[\widetilde{S}(t)]'O$ has a spectral set such that $\sigma([\widetilde{S}(t)]'O)\cap\{\lambda\in\mathbb{C};|\lambda|>1\}\neq\emptyset$ or equivalently $\sigma(\overline{A})\cap\{\lambda\in\mathbb{C};\mathrm{Re}\lambda<0\}\neq\emptyset$. The equation of (3.4) is rewritten by

$$\lambda^3-\{m+h+(d\mu_n+\beta)\}\lambda^2+\{mh+(m+h)(d\mu_n+\beta)\}\lambda + f\alpha\beta\delta-mh(d\mu_n+\beta)=0.$$

By virtue of the Routh-Hurwitz theorem, we verify that, for μ_n's satisfying

$$\mu_n>\frac{\beta\{f\alpha\delta-(ab^2+c+f)h\}}{(ab^2+c+f)hd},$$

the equations of (3.4) have all their solutions in the region $\{\lambda\in\mathbb{C};\mathrm{Re}\lambda>0\}$. On the other hand, for μ_n's satisfying

$$\mu_0\le\mu_n<\frac{\beta\{f\alpha\delta-(ab^2+c+f)h\}}{(ab^2+c+f)hd},$$

the equations of (3.4) have a negative real solution λ_n. Therefore, we have $\lambda_n\in\sigma_-(\overline{A})=\sigma(\overline{A})\cap\{\lambda\in\mathbb{C};\mathrm{Re}\,\lambda<0\}$. Consequently, there exists a smooth unstable manifold $\mathcal{M}_+(O)$ with $\dim X_-$ which is tangential to the subspace X_- at O, where X_- denotes the subspace of $\mathcal{D}_\theta$ corresponding the spectral set $\sigma_-(\overline{A})$.

More precisely, $\sigma_-(\overline{A})$ consists of a finite number of eigenvalues and the space X_- corresponding to $\sigma_-(\overline{A})$ is a finite-dimensional subspace spanned by vectors of the form

$$\begin{pmatrix}\beta\delta(h-\lambda_n)\\ f\beta\delta\\ (ab^2+c+f-\lambda_n)(h-\lambda_n)\end{pmatrix}\phi_n,\quad \mu_0\le\mu_n<\frac{\beta\{f\alpha\delta-(ab^2+c+f)h\}}{(ab^2+c+f)hd},$$

where ϕ_n denote the eigenfunction of $-\varDelta$ corresponding to the eigenvalue μ_n.

ii) In this case, we verify by the Routh-Hurwitz theorem that the equations of (3.4) have all their solutions in the region $\{\lambda\in\mathbb{C};\mathrm{Re}\lambda>0\}$. Consequently, O is exponentially stable in $(\widetilde{S}(t),\mathcal{D}_\theta,\mathcal{D}_\theta)$. □

4 Global stability of O By Theorem 3.2 we know that, if h is large in such a way that $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}<h<\infty$, then the zero solution O is a stable equilibrium of $(S(t),\mathcal{K}_\theta,\mathcal{D}_\theta)$. Moreover, by [8, Proposition 2.3], we can see that, if $\frac{f\alpha\delta}{c+f}<h<\infty$, then O is globally stable.

Theorem 4.1. *Let $\frac{f\alpha\delta}{c+f}<h<\infty$. Then, the stationary O is a globally stable equilibrium of $(S(t),\mathcal{K}_\theta,\mathcal{D}_\theta)$.*

Proof. For $U_0\in K$, let $U(t)={}^t(u(t),v(t),w(t))=S(t)U_0$. According to [8, Proposition 2.3], it is already known that, as $t\to\infty$, $u(t)\to0$ and $v(t)\to0$ in $L^\infty(\Omega)$ and $w(t)\to0$ in $L^2(\Omega)$. Using that

$$\|\varLambda^\theta w(t)\|_{L^2}\le C\|\varLambda w(t)\|_{L^2}^\theta\|w(t)\|_{L^2}^{1-\theta},\qquad 0<t<\infty,$$

we then observe that $\|\varLambda^\theta w(t)\|_{L^2}\to0$, i.e., $w(t)\to0$ in $\mathcal{D}(\varLambda^\theta)$, too. Hence, as $t\to\infty$, $S(t)U_0\to O$ in the topology of $\mathcal{D}_\theta$. □

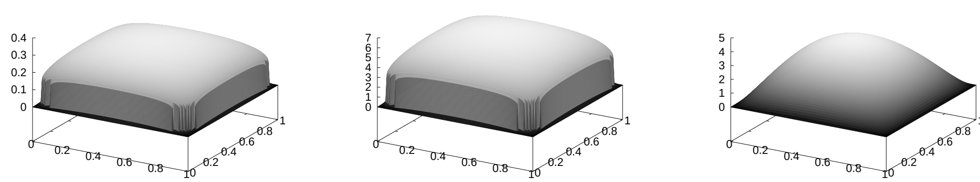
5 Numerical examples This section is devoted to presenting numerical results. We consider the following problem:

$$
(5.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = 1.0 \cdot 1.0w - [1.0(v-3.0)^2 + 0.2]u - 1.0u & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = 1.0u - hv & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = 0.05\Delta w - 1.0w + 1.0v & \text{in} \quad \Omega \times (0,\infty), \\ w = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in} \quad \Omega \end{cases}
$$

in a quadratic domain $\Omega = [0,1] \times [0.1]$. All the parameters except h are fixed as in (5.1), but the h is varied as a control parameter. In this case, the minimal eigenvalue of the realization of $-\Delta$ under the Dirichlet conditions is given by $2.0\pi^2$.

5.1 Case when $0 < h < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$**.** Let us take $h = 0.0488 < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$. According to Theorem 3.2, this means that the zero solution O is unstable.

We also take continuous initial functions u_0, v_0, w_0 which are of the shape of circular truncated cone and are put on the center of Ω. In the initial stage, the graphs of the numerical solution (u,v,w), especially that of u, change very rapidly. Gradually, the graphs stabilize and tend (at least in the numerical sense) to graphs of a non-homogeneous stationary solution $(\overline{u},\overline{v},\overline{w})$, see Fig. 1. The value of Lyapunov function given by [8, (3.4)] also stabilizes as $t \to \infty$ along this trajectory.

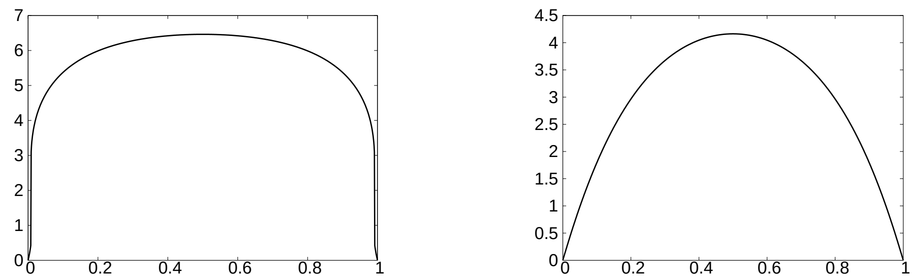


(a) Graph of $\overline{u}$ (b) Graph of $\overline{v}$ (c) Graph of $\overline{w}$

Fig. 1: Case when $0 < h < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$

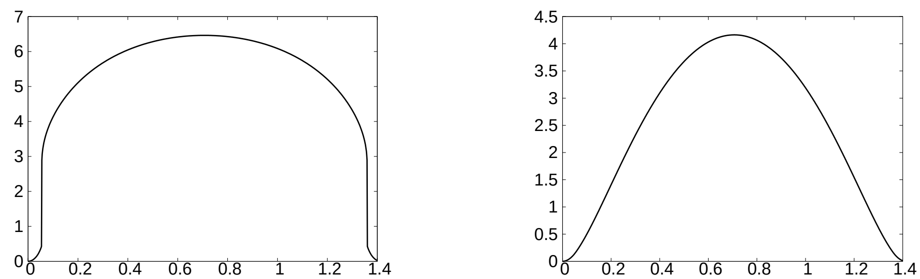
As we can see from Fig. 2 and Fig. 3, the graphs of u and v have a clear gapping curve near the boundary $\partial\Omega$ on which the densities of young age class trees and of old age class trees are discontinuous.

Fixing h as 0.0488 but taking other initial functions, we have performed several numerical computations to find out that every trajectory tends as $t \to \infty$ to the same stationary solution $(\overline{u},\overline{v},\overline{w})$ illustrated by Fig. 1. This suggests that, when $0 < h < \frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)}$, Problem (1.1) possesses a unique stable stationary solution. Consequently, the gapping curve of u and v are also determined in a unique way by the parameters given in (1.1).



(a) Section of $\overline{v}$ (b) Section of $\overline{w}$

Fig. 2: Sections of graphs $\overline{v}$, $\overline{w}$ by the plane $x = \frac{1}{2}$



(a) Section of $\overline{v}$ (b) Section of $\overline{w}$

Fig. 3: Section of graphs $\overline{v}$, $\overline{w}$ by the plane $y = x$

5.2 Case when $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \frac{f\alpha\delta}{c+f}$. Let us now take $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h = 0.167 < \frac{f\alpha\delta}{c+f}$. Theorem 3.2 asserts in this case that the zero solution O is stable.

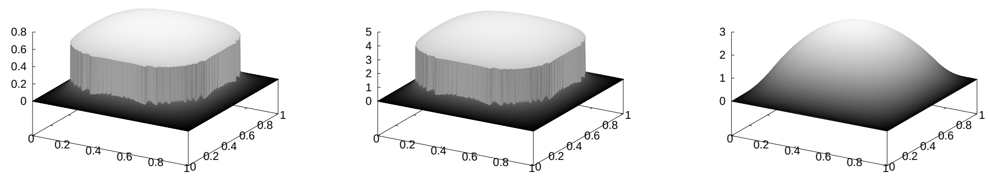
We have performed several numerical computations by different initial functions. In every case, the trajectory tends (at least in the numerical sense) to a non-homogeneous stationary solution $(\overline{u}, \overline{v}, \overline{w})$ which depends however on initial functions (u_0, v_0, w_0). Fig. 4 shows the limit stationary solution when the trajectory starts from a continuous initial functions of the shape of circular truncated cone put on the center of Ω.

Fig. 5 shows the the limit stationary solution when the trajectory starts from a discontinuous initial functions of the shape of circular cylinder.

Fig. 6 shows the limit stationary solution when the trajectory starts from a discontinuous initial functions of the shape of cubic put on the center of Ω.

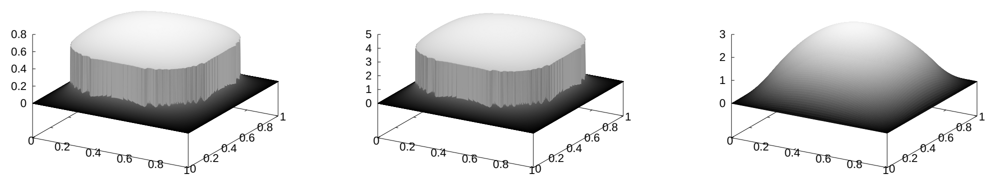
We can know that the three stationary solutions illustrated by Fig. 4, Fig. 5 and Fig. 6 are different each other by the fact that they have different values of Lyapunov function. Similarly, they have different gapping curves on which the values of u and v are discontinuous.

These numerical results then suggest, when $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \frac{f\alpha\delta}{c+f}$, that there exist not only the zero solution O but also many other stable stationary solutions.



(a) Graph of $\overline{u}$ (b) Graph of $\overline{v}$ (c) Graph of $\overline{w}$

Fig. 4: Case when $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \frac{f\alpha\delta}{c+f}$, I



(a) Graph of $\overline{u}$ (b) Graph of $\overline{v}$ (c) Graph of $\overline{w}$

Fig. 5: Case when $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \frac{f\alpha\delta}{c+f}$, II

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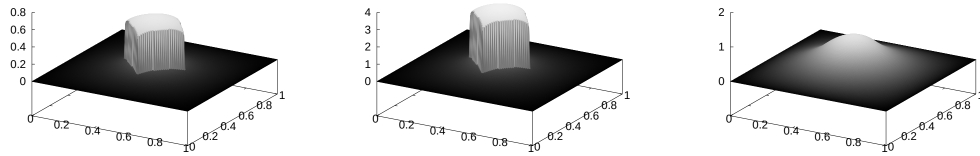

(a) Graph of $\overline{u}$ (b) Graph of $\overline{v}$ (c) Graph of $\overline{w}$

Fig. 6: Case when $\frac{f\alpha\beta\delta}{(ab^2+c+f)(d\mu_0+\beta)} < h < \frac{f\alpha\delta}{c+f}$, III

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