WAVELET CHARACTERIZATIONS OF WEIGHTED HERZ SPACES

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Abstract. We characterize the homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and the non-homogeneous weighted Herz space $K_q^{\alpha,p}(w_1,w_2)$ using wavelets in $C^1(\mathbb{R}^n)$ with compact support. Applying the characterizations, we prove that the wavelet basis forms an unconditional basis in $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and in $K_q^{\alpha,p}(w_1,w_2)$.

1 Introduction The wavelet characterizations of various function spaces are studied (cf. [HW, HWY, Me, W]). In this paper, we consider wavelet characterizations of the homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and the non-homogeneous weighted Herz space $K_q^{\alpha,p}(w_1,w_2)$. Hernández, Weiss and Yang used compactly supported wavelets in $C^1(\mathbb{R}^n)$, and established the characterizations of non-weighted Herz spaces by means of a local version of the discrete tent spaces at the origin ([HWY]). We follow a different way in order to obtain the characterizations. Our method is due to the boundedness of sublinear operators on weighted Herz spaces ([LY]), the duality ([HY]), and the result on density ([NTY]). As an application of the wavelet characterizations, we also give a construction of unconditional bases in $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and in $K_q^{\alpha,p}(w_1,w_2)$ using wavelets.

Let us explain the outline of this article. In Section 2, we explain wavelets briefly. We define the homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and the non-homogeneous weighted Herz space $K_q^{\alpha,p}(w_1,w_2)$ in Section 3. We define two classes of weights A_p and A_1 in Section 4. Section 5 consists of some important lemmas. We show the wavelet characterizations of $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and $K_q^{\alpha,p}(w_1,w_2)$ in Section 6. Lastly, in Section 7, we construct the unconditional bases in $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and in $K_q^{\alpha,p}(w_1,w_2)$ in terms of wavelets.

2 Wavelets First let us recall the definition of wavelet ([Me], [W]).

Definition 2.1 *Let $\{\psi^e : e = 1,2,\cdots,2^n-1\}$ be a set of functions belonging to $L^2(\mathbb{R}^n)$. Define*

$$\psi^e_{j,k}(x) := 2^{jn/2}\psi^e(2^jx-k) = 2^{jn/2}\psi^e(2^jx_1-k_1,\cdots,2^jx_n-k_n) \quad (x=(x_1,\cdots,x_n)\in\mathbb{R}^n)$$

for each $e = 1,2,\cdots,2^n-1$, $j\in\mathbb{Z}$ and $k=(k_1,\cdots,k_n)\in\mathbb{Z}^n$. The sequence $\{\psi^e : e = 1,2,\cdots,2^n-1\}$ is a wavelet set if $\{\psi^e_{j,k} : e=1,2,\cdots,2^n-1,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$. Then $\{\psi^e_{j,k} : e=1,2,\cdots,2^n-1,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$ is a wavelet basis in $L^2(\mathbb{R}^n)$ and each ψ^e is a wavelet.

We generally need suitable smoothness or decay on wavelets in order to obtain wavelet characterizations of function spaces. In this paper, we use a wavelet set $\{\psi^e : e = 1,2,\cdots,2^n-1\}$ satisfying that each wavelet is compactly supported and in $C^1(\mathbb{R}^n)$. Actually there exists a wavelet set $\{\psi^e : e = 1,2,\cdots,2^n-1\}$ which consists of wavelets in $C^1(\mathbb{R}^n)$ with compact support. We can construct it by means of a multiresolution analysis and tensor products ([Da1], [Da2], [Me], [W]).

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3 Weighted Herz spaces

We use the following notation to define weighted Herz spaces.

Notation 3.1
(a) χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^n$.
(b) $B_l := \{x \in \mathbb{R}^n : |x| \leq 2^l\}$ and $R_l := B_l \setminus B_{l-1}$ for $l \in \mathbb{Z}$.
(c) We define the set of functions $\{\tilde{\chi}_l\}_{l=0}^{\infty}$ by $\tilde{\chi}_0 := \chi_{B_0}$ and $\tilde{\chi}_l := \chi_{R_l}$ if $l \geq 1$.
(d) For a $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a compact set $F \subset \mathbb{R}^n$, we write $w(F) := \int_F w(x)dx$.

Definition 3.2 *Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, and $w_1, w_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w_1, w_2 > 0$ a.e..*
(a) *The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1, w_2)$ is defined by*

$$\dot{K}_q^{\alpha,p}(w_1, w_2) := \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}\ ,\ w_2(x)dx) : \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} := \left\| \left\{ w_1(B_l)^{\alpha/n} \|f\chi_{R_l}\|_{L^q(w_2)} \right\}_{l=-\infty}^{\infty} \right\|_{l^p(\mathbb{Z})}.$$

(b) *The non-homogeneous weighted Herz space $K_q^{\alpha,p}(w_1, w_2)$ is defined by*

$$K_q^{\alpha,p}(w_1, w_2) := \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n, w_2(x)dx) : \|f\|_{K_q^{\alpha,p}(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(w_1,w_2)} := \left\| \left\{ w_1(B_l)^{\alpha/n} \|f\tilde{\chi}_l\|_{L^q(w_2)} \right\}_{l=0}^{\infty} \right\|_{l^p(\mathbb{Z}_+)},$$

and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Remark 3.3 Let $0 < p \leq \infty$ and $w_1, w_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w_1, w_2 > 0$ a.e.. Then we see that $\dot{K}_p^{0,p}(w_1, w_2) = K_p^{0,p}(w_1, w_2) = L^p(w_2)$ and $\|f\|_{\dot{K}_p^{0,p}(w_1,w_2)} = \|f\|_{K_p^{0,p}(w_1,w_2)} = \|f\|_{L^p(w_2)}$.

4 A_p weights and A_1 weights

Definition 4.1
(a) *Let $1 < p < \infty$, and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w > 0$ a.e. and $w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The class of weights A_p consists of all w satisfying*

$$A_p(w) := \sup_{B:\mathrm{ball}} \frac{1}{|B|} w(B) \left(\frac{1}{|B|} \int_B w(y)^{-1/(p-1)} dy \right)^{p-1} < \infty,$$

and each $w \in A_p$ is an A_p weight, where $|B|$ means the Lebesgue measure of B.
(b) *Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w > 0$ a.e.. The class of weights A_1 consists of all w satisfying*

$$A_1(w) := \sup_{B:\mathrm{ball}} \frac{1}{|B|} w(B) \left\| w^{-1} \right\|_{L^\infty(B)} < \infty,$$

and each $w \in A_1$ is an A_1 weight.

We have the inclusion relation $A_p \subset A_q$ for $1 \leq p \leq q < \infty$ by Hölder's inequality. In the case of $1 < p < \infty$, we also see that $w \in A_p$ if and only if $w^{-1/(p-1)} \in A_{p'}$. In fact, it clearly follows that $A_p(w) = A_{p'}(w^{-1/(p-1)})^{p-1}$. Here p' means the conjugate exponent of p, i.e., p' satisfies $1/p + 1/p' = 1$. Additionally we describe some properties of A_p weight. We refer to [Du, p.133–140] in order to describe the following property.

Lemma 4.2 *Let $1 \le p < \infty$ and $w \in A_p$. Then there exist three constants $C_1, C_2 > 0$ and $0 < \delta < 1$ depending only on n, p, $A_p(w)$ such that for every ball $B \subset \mathbb{R}^n$ and measurable set $E \subset B$,*

$$\frac{w(E)}{w(B)} \le C_1 \left(\frac{|E|}{|B|}\right)^{\delta} \tag{1}$$

and

$$\frac{w(B)}{w(E)} \le C_2 \left(\frac{|B|}{|E|}\right)^{p}.$$

Muckenhoupt proved the next weak (p,p) inequality for the Hardy-Littlewood maximal function M with respect to $w(x)dx$ ([Mu]). Here we recall the definition of M. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $B(0,r) := \{y \in \mathbb{R}^n : |y| < r\}$ for $r > 0$. The Hardy-Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-y)|dy \quad (x \in \mathbb{R}^n).$$

Lemma 4.3 *Let $1 \le p < \infty$ and $w \in A_p$. Then there exists a constant $C_{n,p} \ge 1$ depending only on n and p such that for all $\lambda > 0$ and $f \in L^p(w)$,*

$$\lambda^p w\left(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\right) \le C_{n,p} A_p(w) \|f\|^p_{L^p(w)}. \tag{2}$$

The estimate of the constant in Lemma 4.3 follows by [Du].

Remark 4.4
(a) Let $1 \le p < \infty$ and $w \in A_p$. Following [Du], the constant $0 < \delta < 1$ appearing in (1) is determined as follows. Let $0 < a < 1$, and

$$0 < \varepsilon < \log \frac{C_{n,p}A_p(w)}{C_{n,p}A_p(w) - (1-a)^p} \cdot \left(\log(2^n a^{-1})\right)^{-1},$$

where $C_{n,p} \ge 1$ is the constant appearing in (2). Then $\delta := \varepsilon/(\varepsilon+1)$ is the desired constant. Let us give a concrete example of δ. If we take $a = 1/2$ and

$$\varepsilon = \log \frac{C_{n,p}A_p(w)}{C_{n,p}A_p(w) - 2^{-p}} \cdot \left((n+2)\log 2\right)^{-1},$$

then we obtain

$$\delta = \log \frac{C_{n,p}A_p(w)}{C_{n,p}A_p(w) - 2^{-p}} \left(\log \frac{2^{n+2}C_{n,p}A_p(w)}{C_{n,p}A_p(w) - 2^{-p}}\right)^{-1}.$$

(b) We introduce a special version of (1). Let $1 < q < \infty$, $1 \le r \le q$ and $w \in A_r$. Denote $v := w^{-1/(q-1)}$ and $\tilde{\delta} := (q-r)/(q-1)$. Then there exists a constant $C > 0$ depending only on n, q, r, $A_q(w)$ and $A_r(w)$ such that for all $l, m \in \mathbb{Z}$ with $l \ge m$,

$$\frac{v(B_m)}{v(B_l)} \le C \left(\frac{|B_m|}{|B_l|}\right)^{\tilde{\delta}}. \tag{3}$$

Now we show (3) applying Lemma 4.2. Since $w \in A_r$, there exists a constant $C_2 > 0$ depending only on n, r, $A_r(w)$ such that

$$\frac{w(B_l)}{w(B_m)} \le C_2 \left(\frac{|B_l|}{|B_m|}\right)^{r}.$$

On the other hand, following Hölder's inequality and $w \in A_q$, we have that

$$1 \leq \frac{1}{|B|} w(B) \left(\frac{1}{|B|} v(B) \right)^{q-1} \leq A_q(w)$$

for any ball B. Namely it follows that

$$v(B_m) \leq A_q(w)^{1/(q-1)} |B_m|^{q/(q-1)} w(B_m)^{-1/(q-1)} \quad \text{and} \quad v(B_l) \geq |B_l|^{q/(q-1)} w(B_l)^{-1/(q-1)}.$$

Consequently we obtain

$$\frac{v(B_m)}{v(B_l)} \leq A_q(w)^{1/(q-1)} \left(\frac{|B_m|}{|B_l|} \right)^{q/(q-1)} \left(\frac{w(B_l)}{w(B_m)} \right)^{1/(q-1)} \leq A_q(w)^{1/(q-1)} C_2^{1/(q-1)} \left(\frac{|B_m|}{|B_l|} \right)^{\tilde{\delta}}.$$

(c) Let $1 < p < \infty$ and $w \in A_p$. From [Du, Corollary 7.6 (1)], we can take a constant $0 < \gamma < p-1$ depending only on n, p, $A_p(w)$ so that $w \in A_{p-\gamma}$. Following [Du], the constant γ is determined as follows. Let $0 < a < 1$, and

$$0 < \tilde{\varepsilon} < \log \frac{C_{n,p'} A_{p'}(w^{-1/(p-1)})}{C_{n,p'} A_{p'}(w^{-1/(p-1)}) - (1-a)^{p'}} \cdot \left(\log(2^n a^{-1}) \right)^{-1},$$

where $C_{n,p'} \geq 1$ is a constant depending only on n and p, and satisfies

$$\lambda^{p'} w^{-1/(p-1)} \left(\{ x \in \mathbb{R}^n : Mf(x) > \lambda \} \right) \leq C_{n,p'} A_{p'}(w^{-1/(p-1)}) \|f\|^{p'}_{L^{p'}(w^{-1/(p-1)})}$$

for all $\lambda > 0$ and $f \in L^{p'}(w^{-1/(p-1)})$. Now we take $\gamma := \tilde{\varepsilon}(p-1)/(\tilde{\varepsilon}+1)$. Then γ is the desired constant.

(d) Let $1 < p < \infty$, $w \in A_p$, then $w^{-1/(p-1)} \in A_{p'}$. Let δ be the constant appearing in (1), and denote $\tilde{\gamma} := \delta(p'-1)$. Then we obtain $w^{-1/(p-1)} \in A_{p'-\tilde{\gamma}}$ by Remark 4.4 (a) and (c).

5 Lemmas To begin with, we introduce the known wavelet characterizations of the weighted L^p space. Lemarié-Rieusset gave characterizations of $L^p(w)$ with $w \in A_p$ by compactly supported and Hölder continuous wavelets. Although he proved it in the case of one-variable, it is true in the case of several-variables with obvious modifications. We need further notation in order to describe his result. We define a dyadic cube

$$Q_{j,k} := \prod_{i=1}^{n} \left[2^{-j} k_i, 2^{-j}(k_i+1) \right)$$

and denote $\chi_{j,k} := 2^{jn/2} \chi_{Q_{j,k}}$ for $j \in \mathbb{Z}$ and $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$. Given a wavelet set $\{\psi^e : e = 1, 2, \cdots, 2^n - 1\}$, we use the following two square functions in order to obtain the wavelet characterizations of function spaces:

$$Vf := \left(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| < f, \psi^e_{j,k} > \psi^e_{j,k} \right|^2 \right)^{1/2} \text{ and } Wf := \left(\sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| < f, \psi^e_{j,k} > \chi_{j,k} \right|^2 \right)^{1/2}$$

Here $< \cdot, \cdot >$ means the L^2-inner product.

Lemma 5.1 (cf. [L]). *Let $1 < p < \infty$, $w \in A_p$ and $\{\psi^e : e = 1, 2, \cdots, 2^n - 1\}$ be a wavelet set such that each ψ^e is compactly supported and Hölder continuous. Then there exist constants $0 < c, c', C, C' < \infty$ depending only on n, p, $A_p(w)$ and $\{\psi^e\}_e$ such that for every $f \in L^p(w)$,*

$$c \|f\|_{L^p(w)} \leq \|Vf\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad \text{and} \quad c' \|f\|_{L^p(w)} \leq \|Wf\|_{L^p(w)} \leq C' \|f\|_{L^p(w)}.$$

The wavelet characterizations stated later are generalizations of Lemma 5.1. We will use Khintchine's inequality described below (cf. [Z]) following the argument by Meyer ([Me]).

Lemma 5.2 *Let Ω be the product set $\{-1,1\}^\Lambda$ and $d\mu(\varepsilon)$ be the Bernoulli probability measure on Ω for $\varepsilon = \left\{\{\varepsilon(\lambda)\}_{\lambda\in\Lambda} : \varepsilon(\lambda) = \pm 1\right\} \in \Omega$, obtained by taking the product of the measures on each factor which give a mass of $1/2$ to each of the points -1 and 1. Then, for all $1 < p < \infty$, there exist two constants $0 < c \le C < \infty$ depending only on p such that for all $\{\alpha(\lambda)\}_{\lambda\in\Lambda} \subset l^2(\Lambda)$,*

$$c\left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2\right)^{1/2} \le \left(\int_\Omega\left|\sum_{\lambda\in\Lambda}\alpha(\lambda)\varepsilon(\lambda)\right|^p d\mu(\varepsilon)\right)^{1/p} \le C\left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2\right)^{1/2}.$$

We shall introduce further important lemmas. The following boundedness of sublinear operators on weighted Herz spaces is proved by Lu, Yabuta and Yang ([LYY]).

Lemma 5.3 *Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$, $1 < q < \infty$, $1 \le q_1 < \infty$, $1 \le q_2 \le q$, $w_1 \in A_{q_1}$, $w_2 \in A_{q_2}$, and T be a sublinear operator satisfying that for all $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,*

$$|Tf(x)| \le C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy,$$

where $C > 0$ is a constant independent of f and x. Suppose the following (4) or (5):

$$w_1 = w_2,\ q_1 = q_2,\ \text{and}\ -\frac{n}{q} < \alpha < n\left(\frac{1}{q_1} - \frac{1}{q}\right), \tag{4}$$

$$-\frac{\delta_2 n}{q_1 q} < \alpha < \frac{n}{q_1}\left(1 - \frac{q_2}{q}\right). \tag{5}$$

Here δ_2 is a constant in (1) for w_2. If T is bounded on $L^q(w_2)$, then T is also bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and on $K_q^{\alpha,p}(w_1,w_2)$.

Remark 5.4 We can take $\delta_2 \in (0,1)$ such that

$$\frac{w_2(B_m)}{w_2(B_l)} \le C\left(\frac{|B_m|}{|B_l|}\right)^{\delta_2}, \tag{6}$$

for some constant $C > 0$ and for all $l, m \in \mathbb{Z}$ with $l \ge m$. We remark that our condition (6) is weaker than (1).

Lu, Yabuta and Yang assumed the condition (4) or the following (7):

$$0 < \alpha < \frac{n}{q_1}\left(1 - \frac{q_2}{q}\right), \tag{7}$$

and gave the result above. Noting Lemma 4.2 and following their proof again, we can modify (7) as (5). We also remark that the conditions (4) and (5) ensure the boundedness for the vector-valued case ([TY]), although it seems that there is a mistake in the condition of Tang and Yang's result.

Next we introduce the result on density. Nakai, Tomita and Yabuta proved it by applying the preceding lemma ([NTY]). Although they give the general result on the weighted Herz-Sobolev spaces, we have only to state the simple case.

Lemma 5.5 *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$, $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, $w_1 \in A_{q_1}$ and $w_2 \in A_{q_2}$. Suppose* (4) *or* (5) *in Lemma 5.3. Then the set of all infinitely differentiable functions with compact support is dense in $\dot{K}_q^{\alpha,p}(w_1, w_2)$ and in $K_q^{\alpha,p}(w_1, w_2)$.*

Finally we state the duality of Herz spaces by Hernández and Yang ([HY]). They give the result for non-weighted case. We obtain the following duality for the weighted case by the same argument as their proof. Let X^* denote the dual space of a Banach space X.

Lemma 5.6 *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$, $w_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w_1 > 0$ a.e., and $w_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w_2 > 0$ a.e. and $w_2^{-1/(q-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then it follows that*

$$\dot{K}_q^{\alpha,p}(w_1, w_2)^* = \dot{K}_{q'}^{-\alpha,p'}(w_1, w_2^{-1/(q-1)})$$

and

$$K_q^{\alpha,p}(w_1, w_2)^* = K_{q'}^{-\alpha,p'}(w_1, w_2^{-1/(q-1)}).$$

Here p' means ∞ if $0 < p \leq 1$.

6 Wavelet characterizations

Theorem 6.1 *Let $\alpha \in \mathbb{R}$, $1 < q < \infty$, $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, $w_1 \in A_{q_1}$, $w_2 \in A_{q_2}$, and $\{\psi^e : e = 1, 2, \cdots, 2^n - 1\}$ be a wavelet set such that each ψ^e is compactly supported and in $C^1(\mathbb{R}^n)$. Then the following* (A) *and* (B) *hold:*
(A) *Let $0 < p \leq \infty$ and suppose* (4) *or* (5) *in Lemma 5.3. Then there exist two constants $0 < C, C' < \infty$ such that for every $f \in K_q^{\alpha,p}(w_1, w_2)$,*

$$\|Vf\|_{K_q^{\alpha,p}(w_1,w_2)} \leq C\,\|f\|_{K_q^{\alpha,p}(w_1,w_2)} \quad \text{and} \quad \|Wf\|_{K_q^{\alpha,p}(w_1,w_2)} \leq C'\,\|f\|_{K_q^{\alpha,p}(w_1,w_2)}.$$

(B) *Let $1 < p < \infty$ and suppose* (5) *in Lemma 5.3. Then there exist two constants $0 < c, c' < \infty$ such that for every $f \in K_q^{\alpha,p}(w_1, w_2)$,*

$$c\,\|f\|_{K_q^{\alpha,p}(w_1,w_2)} \leq \|Vf\|_{K_q^{\alpha,p}(w_1,w_2)} \quad \text{and} \quad c'\,\|f\|_{K_q^{\alpha,p}(w_1,w_2)} \leq \|Wf\|_{K_q^{\alpha,p}(w_1,w_2)}.$$

The same results as (A) *and* (B) *are also true for $\dot{K}_q^{\alpha,p}(w_1, w_2)$.*

Remark 6.2 Here we have to check that the L^2-inner products $\{< f, \psi^e_{j,k} >\}_{j,k}$ are well-defined in Theorem 6.1. The non-homogeneous case is easy. In fact, by $K_q^{\alpha,p}(w_1, w_2) \subset L^q_{\mathrm{loc}}(\mathbb{R}^n, w_2(x)dx)$ and Hölder's inequality, we can easily show that the L^2-inner products are well-defined. Next we consider the homogeneous case. Under the assumption (4) or (5), Tomita proved that $\dot{K}_q^{\alpha,p}(w_1, w_2) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ ([T, Proof of Theorem 2]). Thus we see that the statement is also true for the homogeneous case.

Remark 6.3 Hernández, Weiss and Yang gave the wavelet characterizations for non-weighted Herz spaces with $0 < p < \infty$, $1 < q < \infty$ and $0 < \alpha < n(1 - 1/q)$ by a different method ([HWY]).

Proof of Theorem 6.1 It suffices to prove the theorem for the non-homogeneous case because the homogeneous case follows by the essentially same proof.

We have only to estimate $\|Wf\|_{K_q^{\alpha,p}(w_1,w_2)}$. The estimate of $\|Vf\|_{K_q^{\alpha,p}(w_1,w_2)}$ is proved by the same arguments below.

We prove (A) first. Let $0 < p \le \infty$ and suppose (4) or (5). It suffices to show that the operator W satisfies the conditions of Lemma 5.3. It obviously follows that W is sublinear. We also see that W is bounded on $L^q(w_2)$ by Lemma 5.1. On the other hand, let

$$\Omega := \left\{\varepsilon = \{\varepsilon^e_{j,k} : e = 1, 2, \cdots, 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\} : \varepsilon^e_{j,k} = \pm 1\right\}$$

and $d\mu(\varepsilon)$ be the Bernoulli probability measure on Ω. By Khintchine's inequality, there exists a constant $C_1 > 0$ depending only on q such that for all $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$Wf(x) \le C_1 \left(\int_\Omega |T_\varepsilon f(x)|^q \, d\mu(\varepsilon) \right)^{1/q},$$

where

$$T_\varepsilon f := \sum_{e=1}^{2^n-1} \sum_{j=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} \varepsilon^e_{j,k} < f, \psi^e_{j,k} > \chi_{j,k}.$$

From [Da2, Proof of Lemma 9.1.5], there exists a constant $C_2 > 0$ independent of f, x and ε such that

$$|T_\varepsilon f(x)| \le C_2 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy.$$

Hence we have that

$$Wf(x) \le C_1 C_2 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy.$$

Therefore W is bounded on $K^{\alpha,p}_q(w_1, w_2)$ by Lemma 5.3, that is, (A) is proved.

Next we show (B) applying a duality argument (cf. [HW, Chapter 6]). Let $1 < p < \infty$ and suppose (5). Now we denote $\tilde{\delta}_2 := (q - q_2)/(q - 1)$, $\tilde{\gamma}_2 := \delta_2(q' - 1)$ and $v := w_2^{-1/(q-1)}$. Then it clearly follows that $v \in A_{q'}$. As mentioned in Remark 4.4 (d), we also see that $\tilde{\gamma}_2$ satisfies $v \in A_{q'-\tilde{\gamma}_2}$. By Remark 4.4 (b), the constant $\tilde{\delta}_2$ satisfies that for all $l, m \in \mathbb{Z}$ with $l \ge m$,

$$\frac{v(B_m)}{v(B_l)} \le C_3 \left(\frac{|B_m|}{|B_l|} \right)^{\tilde{\delta}_2},$$

where $C_3 > 0$ is a constant which depends only on n, q, q_2, $A_q(w_2)$ and $A_{q_2}(w_2)$. On the other hand, we get

$$1 < p', q' < \infty \quad \text{and} \quad -\frac{\tilde{\delta}_2 n}{q_1 q'} < -\alpha < \frac{n}{q_1} \left(1 - \frac{q' - \tilde{\gamma}_2}{q'} \right). \tag{8}$$

By Lemma 5.6, it follows that for all $f \in K^{\alpha,p}_q(w_1, w_2)$,

$$\|f\|_{K^{\alpha,p}_q(w_1,w_2)} = \sup \left\{ \left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| : \|g\|_{K^{-\alpha,p'}_{q'}(w_1,v)} \le 1 \right\}.$$

In addition, by Lemma 5.5 and the condition (8), we see that $K^{\alpha,p}_q(w_1, w_2) \cap L^2(\mathbb{R}^n)$ is dense in $K^{\alpha,p}_q(w_1, w_2)$, and that $K^{-\alpha,p'}_{q'}(w_1, v) \cap L^2(\mathbb{R}^n)$ is dense in $K^{-\alpha,p'}_{q'}(w_1, v)$. Thus we have only to show that

$$\left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| \le C \, \|Wf\|_{K^{\alpha,p}_q(w_1,w_2)}$$

for every $f \in K_q^{\alpha,p}(w_1,w_2)\cap L^2(\mathbb{R}^n)$ and $g \in K_{q'}^{-\alpha,p'}(w_1,v)\cap L^2(\mathbb{R}^n)$ with $\|g\|_{K_{q'}^{-\alpha,p'}(w_1,v)} \le 1$, where $C>0$ is a constant independent of f and g. Because the wavelet basis $\{\psi_{j,k}^e : e = 1,2,\cdots,2^n-1,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$, it follows that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \\
&= \left|\int_{\mathbb{R}^n}\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi_{j,k}^e > \psi_{j,k}^e(x)\cdot \sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < g,\overline{\psi_{j,k}^e} > \overline{\psi_{j,k}^e(x)}dx\right| \\
&= \left|\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi_{j,k}^e >< g,\overline{\psi_{j,k}^e} >\right| \\
&= \left|\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n} < f,\psi_{j,k}^e >< g,\overline{\psi_{j,k}^e} > \cdot \int_{\mathbb{R}^n}\chi_{j,k}(x)^2 dx\right| \\
&\le \int_{\mathbb{R}^n}\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|< f,\psi_{j,k}^e > \chi_{j,k}(x)\cdot < g,\overline{\psi_{j,k}^e} > \chi_{j,k}(x)\right| dx.
\end{aligned}$$

Now we define

$$\widetilde{W}g := \left(\sum_{e=1}^{2^n-1}\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|< g,\overline{\psi_{j,k}^e} > \chi_{j,k}\right|^2\right)^{1/2}.$$

Then $\widetilde{W}$ is sublinear and bounded on $L^{q'}(v)$. Therefore by (8) and Lemma 5.3, there exists a constant $C_4>0$ independent of f and g such that

$$\left\|\widetilde{W}g\right\|_{K_{q'}^{-\alpha,p'}(w_1,v)} \le C_4\|g\|_{K_{q'}^{-\alpha,p'}(w_1,v)} \le C_4.$$

By the Cauchy-Schwarz inequality and Hölder's inequality, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| &\le \int_{\mathbb{R}^n} Wf(x)\cdot\widetilde{W}g(x)dx \\
&= \sum_{l=0}^{\infty}\int_{\mathbb{R}^n} Wf(x)\tilde{\chi}_l(x)w_2(x)^{1/q}\cdot\widetilde{W}g(x)\tilde{\chi}_l(x)w_2(x)^{-1/q}dx \\
&\le \sum_{l=0}^{\infty}\|(Wf)\tilde{\chi}_l\|_{L^q(w_2)}\left\|(\widetilde{W}g)\tilde{\chi}_l\right\|_{L^{q'}(v)} \\
&= \sum_{l=0}^{\infty} w_1(B_l)^{\alpha/n}\|(Wf)\tilde{\chi}_l\|_{L^q(w_2)}\cdot w_1(B_l)^{-\alpha/n}\left\|(\widetilde{W}g)\tilde{\chi}_l\right\|_{L^{q'}(v)} \\
&\le \|Wf\|_{K_q^{\alpha,p}(w_1,w_2)}\left\|\widetilde{W}g\right\|_{K_{q'}^{-\alpha,p'}(w_1,v)} \\
&\le C_4\|Wf\|_{K_q^{\alpha,p}(w_1,w_2)}.
\end{aligned}$$

Consequently we have proved the desired result. $\square$

7 Unconditional bases

First we recall the definition of unconditional basis ([W]).

Definition 7.1 *Let X be a Banach space, A be a countable index set, $\{x_m\}_{m\in A} \subset X$ and $\{\tilde{x}_k\}_{k\in A} \subset X^*$. $\{x_m, \tilde{x}_m\}_{m\in A}$ is said to be an unconditional basis in X if the following three conditions are satisfied:*
(i) *$\{x_m, \tilde{x}_m\}_{m\in A}$ is a biorthogonal system, i.e., $\tilde{x}_k(x_m) = \delta_{m,k}$. Here $\delta_{m,k}$ means Kronecker's delta, that is, $\delta_{m,m} = 1$ and $\delta_{m,k} = 0$ if $m \neq k$.*
(ii) *$\mathrm{span}\{x_m\}_{m\in A}$ is dense in X, where $\mathrm{span}\{x_m\}_{m\in A}$ means the set of finite linear combinations of elements in $\{x_m\}_{m\in A}$.*
(iii) *There exists a constant $C > 0$ such that $\left\| \sum_{m\in B} \tilde{x}_m(x) x_m \right\|_X \leq C\|x\|_X$ for every $x \in X$ and every finite subset $B \subset A$.*

Remark 7.2 Let $\{x_m, \tilde{x}_m\}_{m\in A}$ be an unconditional basis in a Banach space X. We see that the functionals $\{\tilde{x}_k\}_{k\in A} \subset X^*$ are determined uniquely by the vectors $\{x_m\}_{m\in A} \subset X$ from two conditions (i) and (ii) in Definition 7.1. Thus we often say that $\{x_m\}_{m\in A}$ is an unconditional basis in X.

Applying Theorem 6.1, we have the following result.

Theorem 7.3 *Let $\alpha \in \mathbb{R}$, $1 < q < \infty$, $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, $1 < p < \infty$, $w_1 \in A_{q_1}$, $w_2 \in A_{q_2}$, and $\{\psi^e : e = 1, 2, \cdots, 2^n - 1\}$ be a wavelet set such that each ψ^e is compactly supported and in $C^1(\mathbb{R}^n)$. Suppose (5) in Lemma 5.3. Then the wavelet basis $\{\psi^e_{j,k} : e = 1, 2, \cdots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an unconditional basis in $\dot{K}^{\alpha,p}_q(w_1, w_2)$ and in $K^{\alpha,p}_q(w_1, w_2)$.*

We need the next lemma in order to prove Theorem 7.3. The lemma is the dominated convergence theorem for Banach function spaces with absolutely continuous norm ([BS, Proposition 3.6 in Chapter 1]). Here a Banach function space X is said to be have an absolutely continuous norm $\|\cdot\|$ if $\lim_{j\to\infty} \left\|f\chi_{E_j}\right\| = 0$ for all $f \in X$ and all sequences of measurable sets $\{E_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} E_j = \emptyset$.

Lemma 7.4 *Let $(X, \|\cdot\|)$ be a Banach function space with absolutely continuous norm, $f \in X$ and $\{f_j\}_{j=1}^\infty \subset X$. Suppose that $\lim_{j\to\infty} f_j = f$ a.e. and there exists a positive function $g \in X$ such that $|f_j| \leq g$ a.e. for all $j \in \mathbb{N}$. Then we have $\lim_{j\to\infty} \|f_j - f\| = 0$.*

Proof of Theorem 7.3 For convenience, we denote $\Lambda := \{1, 2, \cdots, 2^n - 1\} \times \mathbb{Z} \times \mathbb{Z}^n$, and $T_A f := \sum_{(e,j,k)\in A} < f, \psi^e_{j,k} > \psi^e_{j,k}$ for $A \subset \Lambda$. We prove for the case of $K^{\alpha,p}_q(w_1, w_2)$. It suffices to check the following two conditions:
(I) There exists a constant $C > 0$ such that $\|T_A f\|_{K^{\alpha,p}_q(w_1,w_2)} \leq C\,\|f\|_{K^{\alpha,p}_q(w_1,w_2)}$ for all $A \subset \Lambda$ and all $f \in K^{\alpha,p}_q(w_1, w_2)$.
(II) $\mathrm{span}\left\{\psi^e_{j,k} : (e, j, k) \in \Lambda\right\}$ is dense in $K^{\alpha,p}_q(w_1, w_2)$.

First we check (I). By Theorem 6.1 and the orthonormality, it follows that for all $f \in K^{\alpha,p}_q(w_1, w_2)$,

$$(9)\quad \|T_A f\|_{K^{\alpha,p}_q(w_1,w_2)} \leq C_0\,\|V(T_A f)\|_{K^{\alpha,p}_q(w_1,w_2)} \leq C_0\,\|Vf\|_{K^{\alpha,p}_q(w_1,w_2)} \leq C_0 C_1\,\|f\|_{K^{\alpha,p}_q(w_1,w_2)},$$

where $C_0, C_1 > 0$ are constants independent of f. This completes (I).

Next we check (II). It suffices to show $\lim_{A \to \Lambda} \|f - T_A f\|_{K_q^{\alpha,p}(w_1,w_2)} = 0$. We see that $V(f - T_A f) \le Vf$ and $\|Vf\|_{K_q^{\alpha,p}(w_1,w_2)} \le C_1 \|f\|_{K_q^{\alpha,p}(w_1,w_2)}$ by (9). Because $K_q^{\alpha,p}(w_1,w_2)$ is a Banach function space with absolutely continuous norm $\|\cdot\|_{K_q^{\alpha,p}(w_1,w_2)}$, Lemma 7.4 gives $\lim_{A \to \Lambda} \|V(f - T_A f)\|_{K_q^{\alpha,p}(w_1,w_2)} = 0$. On the other hand, (9) implies $\|f - T_A f\|_{K_q^{\alpha,p}(w_1,w_2)} \le C_0 \|V(f - T_A f)\|_{K_q^{\alpha,p}(w_1,w_2)}$. Namely we obtain $\lim_{A \to \Lambda} \|f - T_A f\|_{K_q^{\alpha,p}(w_1,w_2)} = 0$.

Consequently we have proved Theorem 7.3 for the case of $K_q^{\alpha,p}(w_1,w_2)$. The same proof is valid for $\dot{K}_q^{\alpha,p}(w_1,w_2)$. $\square$

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