STRONG CONVERGENCE THEOREMS FOR THREE-STEP ITERATIONS WITH ERRORS UNDER A MODIFIED SENTER AND DOTSON'S CONDITION

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ABSTRACT. Let C be a closed convex subset of a Banach space. In this paper, we construct a three-step iteration process $\{x_n\}$ with errors for three nonlinear mappings $S, T, U\colon C \to C$ given by: $x_1 \in C$, $z_n = \alpha''_n x_n + \beta''_n U^n x_n + \gamma''_n w_n$, $y_n = \alpha'_n x_n + \beta'_n T^n z_n + \gamma'_n v_n$, $x_{n+1} = \alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n$, for all $n \geq 1$ and prove a strong convergence theorem under a modified Senter and Dotson's condition (A). This improves Xu and Noor's result in 2002. Further, we generalize a Khan and Fukhar-ud-dim's result in 2005.

1. INTRODUCTION

Let C be a closed convex subset of a Banach space and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \boldsymbol{N}$, where $\boldsymbol{N}$ denotes the set of all positive integers. In particular, if $k_n = 1$ for all $n \in \boldsymbol{N}$, T is said to be *nonexpansive*. The weaker definition (cf. Kirk [8]) requires that

$$\overline{\lim}_{n\to\infty} \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

for each $x \in C$, and that T^N is continuous for some $N \in \boldsymbol{N}$. T is said to be *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\overline{\lim}_{n\to\infty} \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

On the other hand, a mapping $T\colon C \to C$ with a fixed point is said to satisfy *condition* (A) [11] if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that

$$\|x - Tx\| \geq f(d(x, F))$$

for all $x \in C$, where F is the set of all fixed points of T and $d(x, F) = \inf_{z \in F} \|x - z\|$. We modify this condition for three mappings $S, T, U\colon C \to C$ as follows. Three mappings $S, T, U\colon C \to C$ with a common fixed point is said to satisfy *condition* (A') if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that

$$\|x - Sx\| + \|x - Tx\| + \|x - Ux\| \geq f(d(x, F))$$

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for all $x \in C$, where F is the set of all common fixed points of S, T, U.

Recently, for a mapping $T\colon C \to C$, Xu and Noor [15] constructed a three-step iteration process in C and motivated by their work, we [2] construct a three-step iteration process $\{x_n\}$ with errors for two nonlinear mappings $S, T\colon C \to C$ given by:

$$\begin{cases} x_1 \in C, \\ z_n = \alpha''_n x_n + \beta''_n T^n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n x_n + \beta'_n S^n z_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$, $\{\gamma''_n\}$ are sequences in $[0,1]$ such that

$$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1 \text{ for all } n \in \boldsymbol{N}, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \ \sum_{n=1}^{\infty} \gamma'_n < \infty \text{ and } \sum_{n=1}^{\infty} \gamma''_n < \infty \tag{1.3}$$

and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C. This scheme is named as Noor's iteration with errors. We proved weak and strong theorems in [7] when $\alpha''_n = 1$ in (1.1). The following result was proved in [2].

Theorem 1.1 ([2])**.** *Let C be a bounded closed convex subset of a uniformly convex Banach space satisfying Opial's condition. Let $S, T\colon C \to C$ be asymptotically nonexpansive mappings in the intermediate sense such that S and T have a common fixed point. Put*

$$r_n = \sup_{x,y \in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be generated by (1.1)*, where the sequences $\{\beta_n\}$ and $\{\beta'_n\}$ satisfy the additional restrictions:* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} \beta'_n \leq \limsup_{n\to\infty} \beta'_n < 1$*. Then $\{x_n\}, \{y_n\}, \{z_n\}$ converge weakly to the same common fixed point of S and T.*

On the other hand, for convergence theorems under the condition (A), see [11] and [14] for example. Khan and Fukhar-ud-din [5] modified the condition (A) and they proved a convergence theorem (see Corollary 3.4) for two mappings $S, T\colon C \to C$ under such condition.

In this paper, we construct a three-step iteration process $\{x_n\}$ with errors for three nonlinear mappings $S, T, U\colon C \to C$ given by:

$$\begin{cases} x_1 \in C, \\ z_n = \alpha''_n x_n + \beta''_n U^n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n x_n + \beta'_n T^n z_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$, $\{\gamma''_n\}$ are sequences in $[0,1]$ satisfying (1.2) and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C. We prove a stong convergence theorem for this iteration under the modified condition (A$'$) in the section 2. Further, we study a three-step iteration process $\{x_n\}$ with errors for three nonlinear mappings

$S, T, U\colon C \to C$ given by:

$$
(1.5) \qquad \begin{cases} x_1 \in C, \\ z_n = \alpha''_n x_n + \beta''_n U x_n + \gamma''_n w_n, \\ y_n = \alpha'_n x_n + \beta'_n T z_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$, $\{\gamma''_n\}$ are sequences in $[0,1]$ satisfying (1.2) and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C in the section 3. We proved weak and strong theorems in [6] when $\alpha''_n = 1$ and $S = T$ in (1.5) and weak and strong theorems in [9] when $\alpha''_n = 1$ in (1.5). We prove a strong convergence theorem for the iteration (1.5) under the modified condition (A$'$) which improves Khan and Fukhar-ud-din's result.

2. Main Results

Recall that a Banach space X is said to be *uniformly convex* if the modulus of convexity $\delta_X = \delta_X(\varepsilon)$ $(0 < \varepsilon \leq 2)$ of X defined by

$$
\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \;\middle|\; x, y \in X,\ \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon \right\}
$$

satisfies the inequality $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. Now, we state some lemmas used in the following.

Lemma 2.1 ([14])**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and*

$$
a_{n+1} \leq a_n + b_n
$$

for all $n \in \boldsymbol{N}$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2 ([12])**.** *Let X be a uniformly convex Banach space, let $0 < b \leq t_n \leq c < 1$ for all $n \in \boldsymbol{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\overline{\lim}_{n\to\infty} \|x_n\| \leq a$, $\overline{\lim}_{n\to\infty} \|y_n\| \leq a$ and $\overline{\lim}_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = a$ for some $a \geq 0$. Then, it holds that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

In this paper, the iterations defined by (1.4) and (1.5) are always assumed that $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0,1]$ satisfying (1.2) and (1.3) and $\{u_n\}$, $\{v_n\}, \{w_n\}$ are bounded sequences in C. Denote by $F(T)$, the set of fixed points of a mapping T.

Lemma 2.3. *Let C be a convex subset of a Banach space and let $S, T, U\colon C \to C$ be mappings satisfying $F(S) \cap F(T) \cap F(U) \neq \emptyset$. Put*

$$
\begin{aligned} r_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x-y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \\ \vee \sup_{x,y\in C} (\|U^n x - U^n y\| - \|x-y\|) \vee 0 \end{aligned}
$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$ generated by the (1.4). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S) \cap F(T) \cap F(U)$.

Proof. Take p in $F(S) \cap F(T) \cap F(U)$ and set

$$
M = \sup_{n\in\boldsymbol{N}} \|u_n - p\| \vee \sup_{n\in\boldsymbol{N}} \|v_n - p\| \vee \sup_{n\in\boldsymbol{N}} \|w_n - p\|.
$$

Since

$$\|z_n - p\| \le \alpha''_n \|x_n - p\| + \beta''_n \|U^n x_n - p\| + \gamma''_n \|w_n - p\| \le \alpha''_n \|x_n - p\| + \beta''_n (\|x_n - p\| + r_n) + \gamma''_n M \le \|x_n - p\| + r_n + \gamma''_n M, \tag{2.1}$$

we have

$$\|y_n - p\| \le \alpha'_n \|x_n - p\| + \beta'_n \|T^n z_n - p\| + \gamma'_n \|v_n - p\| \le \alpha'_n \|x_n - p\| + \beta'_n (\|z_n - p\| + r_n) + \gamma'_n M \le \|x_n - p\| + 2r_n + (\gamma''_n + \gamma'_n)M \tag{2.2}$$

and so

$$\|x_{n+1} - p\| \le \alpha_n \|x_n - p\| + \beta_n \|S^n y_n - p\| + \gamma_n \|u_n - p\| \le \alpha_n \|x_n - p\| + \beta_n (\|y_n - p\| + r_n) + \gamma_n M \le \|x_n - p\| + 3r_n + (\gamma''_n + \gamma'_n + \gamma_n)M. \tag{2.3}$$

The conclusion follows from Lemma 2.1. □

Lemma 2.4. *Let C be a closed convex subset of a uniformly convex Banach space and $S, T, U\colon C \to C$ be asymptotically nonexpansive mappings in the intermediate sense such that $F(S) \cap F(T) \cap F(U) \neq \emptyset$. Let r_n be as in Lemma 2.3 and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$ be generated by* (1.4).

(1) *If there exists δ such that $0 < \delta \le \beta_n, \beta'_n, \beta''_n \le 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then*

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Ux_n\| = 0. \tag{2.4}$$

(2) *If $U = S$ and there exists δ such that $0 < \delta \le \beta_n, \beta'_n \le 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then* (2.4) *holds.*

Proof. We prove the only case (1). Take p in $F(S) \cap F(T) \cap F(U)$ and set $r = \lim_{n\to\infty} \|x_n - p\|$ which exists by Lemma 2.3. By (2.2), we have $\overline{\lim}_{n\to\infty} \|y_n - p\| \le r$. By

$$\|S^n y_n - p + \gamma_n(u_n - x_n)\| \le \|y_n - p\| + r_n + \gamma_n \|u_n - x_n\|$$

and the boundedness of $\{u_n - x_n\}$, we have

$$\overline{\lim}_{n\to\infty} \|S^n y_n - p + \gamma_n(u_n - x_n)\| \le r.$$

Similarly, we have $\overline{\lim}_{n\to\infty} \|x_n - p + \gamma_n(u_n - x_n)\| \le r$. Further, since

$$\begin{aligned} x_{n+1} - p &= \alpha_n(x_n - p) + \beta_n(S^n y_n - p) + \gamma_n(u_n - p) \\ &= \beta_n(S^n y_n - p + \gamma_n(u_n - p - x_n + p)) \\ &\quad + (1 - \beta_n)(x_n - p + \gamma_n(u_n - p - x_n + p)) \\ &= \beta_n(S^n y_n - p + \gamma_n(u_n - x_n)) + (1 - \beta_n)(x_n - p + \gamma_n(u_n - x_n)) \end{aligned}$$

and $\lim_{n\to\infty} \|x_{n+1} - p\| = r$, we have

$$\lim_{n\to\infty} \|\beta_n(S^n y_n - p + \gamma_n(u_n - x_n)) + (1 - \beta_n)(x_n - p + \gamma_n(u_n - x_n))\| = r.$$

Therefore, by Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{2.5}$$

Since

$$\begin{aligned}\|x_n - p\| &\le \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\le \|x_n - S^n y_n\| + \|y_n - p\| + r_n,\end{aligned}$$

we have $r \le \underline{\lim}_{n\to\infty} \|y_n - p\| \le \overline{\lim}_{n\to\infty} \|y_n - p\| \le r$ and so

$$\lim_{n\to\infty} \|y_n - p\| = r.$$

Using this and

$$\begin{aligned}y_n - p &= \alpha'_n(x_n - p) + \beta'_n(T^n z_n - p) + \gamma'_n(v_n - p) \\ &= \beta'_n(T^n z_n - p + \gamma'_n(v_n - p - x_n + p)) \\ &\quad + (1-\beta'_n)(x_n - p + \gamma'_n(v_n - p - x_n + p)) \\ &= \beta'_n(T^n z_n - p + \gamma'_n(v_n - x_n)) + (1-\beta'_n)(x_n - p + \gamma'_n(v_n - x_n)),\end{aligned}$$

we have

$$\lim_{n\to\infty} \|\beta'_n(T^n z_n - p + \gamma'_n(v_n - x_n)) + (1-\beta'_n)(x_n - p + \gamma'_n(v_n - x_n))\| = r.$$

On the other hand, by similar way with the argument above, we have

$$\overline{\lim}_{n\to\infty} \|T^n z_n - p + \gamma'_n(v_n - x_n)\| \le r$$

and

$$\overline{\lim}_{n\to\infty} \|x_n - p + \gamma'_n(v_n - x_n)\| \le r.$$

Therefore, by Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{2.6}$$

Since

$$\begin{aligned}\|x_n - S^n x_n\| &\le \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ &\le \|x_n - y_n\| + r_n + \|S^n y_n - x_n\| \\ &\le \beta'_n \|x_n - T^n z_n\| + \gamma'_n \|v_n - x_n\| + r_n + \|S^n y_n - x_n\|,\end{aligned}$$

using (2.5) and (2.6), we have

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0. \tag{2.7}$$

Since

$$\begin{aligned}\|x_n - p\| &\le \|x_n - T^n z_n\| + \|T^n z_n - p\| \\ &\le \|x_n - T^n z_n\| + \|z_n - p\| + r_n,\end{aligned}$$

we have $r \le \underline{\lim}_{n\to\infty} \|z_n - p\| \le \overline{\lim}_{n\to\infty} \|z_n - p\| \le r$ by (2.1) and so

$$\lim_{n\to\infty} \|z_n - p\| = r.$$

Thus, by the similar calculation above, we have

$$\lim_{n\to\infty} \|\beta''_n(U^n x_n - p + \gamma''_n(w_n - x_n)) + (1-\beta''_n)(x_n - p + \gamma''_n(w_n - x_n))\| = r.$$

On the other hand, by similar way with the argument above, we have

$$\overline{\lim}_{n\to\infty} \|U^n x_n - p + \gamma''_n(w_n - x_n)\| \le r$$

and

$$\overline{\lim}_{n\to\infty} \|x_n - p + \gamma''_n(w_n - x_n)\| \le r.$$

Therefore, by Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|U^n x_n - x_n\| = 0. \tag{2.8}$$

Since

$$\begin{aligned}
\|x_n - T^n x_n\| &\le \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\| \\
&\le \|x_n - z_n\| + r_n + \|T^n z_n - x_n\| \\
&\le \beta_n'' \|x_n - U^n x_n\| + \gamma_n'' \|w_n - x_n\| + r_n + \|T^n z_n - x_n\|,
\end{aligned}$$

using (2.6) and (2.8), we have

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0.$$

Since

$$\begin{aligned}
\|x_n - x_{n+1}\| &\le \|x_n - S^n x_n\| + \|S^n x_n - x_{n+1}\| \\
&\le (1+\alpha_n)\|x_n - S^n x_n\| + \beta_n(\|x_n - y_n\| + r_n) \\
&\qquad + \gamma_n \|S^n x_n - u_n\| \\
&\le 2\|x_n - S^n x_n\| + \beta_n' \|T^n z_n - x_n\| + \gamma_n' \|v_n - x_n\| + r_n \\
&\qquad + \gamma_n \|S^n x_n - u_n\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$$

by virtue of the boundedness of $\{S^n x_n\}$. Since

$$\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + \left\|S^{n+1}x_{n+1} - S^{n+1}x_n\right\| \\
&\qquad + \left\|S^{n+1}x_n - Sx_n\right\| \\
&\le 2\|x_n - x_{n+1}\| + r_{n+1} + \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| \\
&\qquad + \left\|S^{n+1}x_n - Sx_n\right\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0$$

by virtue of the uniform continuity of S. Similarly, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Ux_n\| = 0.$$

To prove the case (2), we may use (2.7) in stead of (2.8) and so the proof is complete. □

The proof of Lemma 2.4 is simpler than Lemma 5 of [2]. Now, we state our main result of this section.

Theorem 2.5. *Let C be a closed convex subset of a uniformly convex Banach space X. Let $S, T, U\colon C \to C$ be asymptotically nonexpansive mappings in the intermediate sense such that $F(S)\cap F(T)\cap F(U) \ne \emptyset$ and satisfy the condition (A′). Set $F = F(S)\cap F(T)\cap F(U)$. Put*

$$r_n = \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x-y\|) \vee \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be generated by (1.4).

(1) *If there exists δ such that $0 < \delta \le \beta_n, \beta_n', \beta_n'' \le 1-\delta < 1$ for all $n \in \boldsymbol{N}$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge to the same point of F.*

(2) *If $U = S$ and there exists δ such that $0 < \delta \leq \beta_n, \beta_n' \leq 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge to the same point of F.*

Proof. Let $r = \lim_{n\to\infty} \|x_n - p\|$ which exists for a fixed $p \in F$ by Lemma 2.3. If $r = 0$, the conclusion is clear. So, we assume $r > 0$. By the condition (A$'$), we have

$$\|x_n - Sx_n\| + \|x_n - Tx_n\| + \|x_n - Ux_n\| \geq f(d(x_n, F)) \geq 0.$$

So, using Lemma 2.4, we obtain $\lim_{n\to\infty} d(x_n, F) = 0$. Then, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{s_n\}$ in F such that $\|x_{n_j} - s_n\| < 2^{-j}$. Using (2.3), we have

$$\begin{aligned}
\|s_{j+1} - s_j\| &\leq \|s_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - s_j\| \\
&\leq 2^{-(j+1)} + \|x_{n_j} - s_j\| + \sum_{k=n_j}^{n_{j+1}-1} (3r_k + (\gamma_k'' + \gamma_k' + \gamma_k)M) \\
&\leq 2^{-(j+1)} + 2^{-j} + \sum_{k=n_j}^{n_{j+1}-1} (3r_k + (\gamma_k'' + \gamma_k' + \gamma_k)M),
\end{aligned}$$

where $M = \sup_{n\in\boldsymbol{N}} \|u_n - p\| \vee \sup_{n\in\boldsymbol{N}} \|v_n - p\| \vee \sup_{n\in\boldsymbol{N}} \|w_n - p\|$. So, we can easily see that $\{s_n\}$ is a Cauchy sequence in F. Therefore, there exits s in F such that $s_n \to s$ and so $x_{n_j} \to s$. Further, we obtain $x_n \to s$. □

3. Further Results

In this section, we discuss about the convergence of the sequence $\{x_n\}$ generated by (1.5). By similar ways with Lemma 2.3, Lemma 2.4 and Theorem 2.5, we can prove the following Lemma 3.1, Lemma 3.2 and Theorem 3.3, respectively.

Lemma 3.1. *Let C be a convex subset of a Banach space and let $S, T, U\colon C \to C$ be nonexpansive mappings satisfying $F(S) \cap F(T) \cap F(U) \neq \emptyset$. Then for the sequence $\{x_n\}$ generated by* (1.5), $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in F(S) \cap F(T) \cap F(U)$.*

Lemma 3.2. *Let C be a closed convex subset of a uniformly convex Banach space and $S, T, U\colon C \to C$ be nonexpansive mappings such that $F(S) \cap F(T) \cap F(U) \neq \emptyset$. Let $\{x_n\}$ be generated by* (1.5).

(1) *If there exists δ such that $0 < \delta \leq \beta_n, \beta_n', \beta_n'' \leq 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then*

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Ux_n\| = 0. \tag{3.1}$$

(2) *If $U = S$ and there exists δ such that $0 < \delta \leq \beta_n, \beta_n' \leq 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then* (3.1) *holds.*

Theorem 3.3. *Let C be a closed convex subset of a uniformly convex Banach space. Let $S, T, U\colon C \to C$ be nonexpansive mappings such that $F(S) \cap F(T) \cap F(U) \neq \emptyset$ and satisfy the condition (A$'$). Let $\{x_n\}, \{y_n\}, \{z_n\}$ be generated by* (1.5).

(1) *If there exists δ such that $0 < \delta \leq \beta_n, \beta_n', \beta_n'' \leq 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge to the same point of $F(S) \cap F(T) \cap F(U)$.*

(2) *If $U = S$ and there exists δ such that $0 < \delta \leq \beta_n, \beta_n' \leq 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge to the same point of $F(S) \cap F(T) \cap F(U)$.*

As a direct consequence, taking $U = S$ and $\alpha_n'' = 1$ for all $n \in \boldsymbol{N}$ in Theorem 3.3, we have the following result which is obtained in [5].

Corollary 3.4. *Let C be a closed convex subset of a uniformly convex Banach space. Let $S, T\colon C \to C$ be nonexpansive mappings such that $F(S) \cap F(T) \neq \emptyset$. Assume that there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that*

$$\frac{1}{2}\left(\|x - Sx\| + \|x - Tx\|\right) \geq f(d(x, F))$$

for all $x \in C$, where $F = F(S) \cap F(T)$. Let $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C, \\ y_n = \alpha'_n x_n + \beta'_n T z_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases}$$

If there exists δ such that $0 < \delta \leq \beta_n, \beta'_n \leq 1 - \delta < 1$ for all $n \in \boldsymbol{N}$, we have $\{x_n\}$ converges to the same point of F.

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