PARAMETER IDENTIFICATION PROBLEMS FOR A CLASS OF STRONGLY DAMPED NONLINEAR WAVE EQUATIONS

S. Nakagiri[1], J-H. Ha and J. Vanualailai

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Abstract. Parameter identification problems of spatially varying coefficients in a class of strongly damped nonlinear wave equations are studied. The problems are formulated by a minimization of quadratic cost functionals by means of distributive and terminal values measurements. The existence of optimal parameters and necessary optimality conditions for the functionals are proved by the continuity and Gâteaux differentiability of solutions on parameters.

1 Introduction Let Ω be an open bounded set of $\mathbf{R}^n$ with the smooth boundary Γ. The inner product of $\mathbf{R}^n$ is denoted by $x \cdot y$ for $x, y \in \mathbf{R}^n$. We put $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\Gamma$ for $T > 0$. We consider the following Dirichlet boundary value problem for the system of strongly damped nonlinear wave equations described by

$$(1.1)\qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \nabla\cdot(a(x)\nabla y + b(x)\nabla\dfrac{\partial y}{\partial t}) = c(x)F(y) + f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \ \text{ in } \Omega, \end{cases}$$

where $a(x)$ is a coefficient of diffusion, $b(x)$ is a coefficient of viscoelasticity, $c(x)$ is an amplifier function on nonlinear activity, f is a forcing function, y_0, y_1 are initial values and $F : \mathbf{R} \to \mathbf{R}$ is a nonlinear activation (or perturbed) function. This type of semilinear strongly damped wave equations appears in the theories of longitudial vibrations with viscous effects, quantum mechanics and others (cf. Fitzgibbon [3], Hale [7], Temam [11]). In this paper we consider the spatially varying parameters identification problems for coefficients $a(x), b(x)$, and $c(x)$ in (1.1). The approach of identification problems has been developed by many reseachers, among them we refer to Ahmed [1] and Omatu and Seinfeld [10] for the theoretical treatments based on the optimal control theory due to Lions [8], and Ha and Nakagiri [5, 6] and Nakagiri and Ha [9] for applications to nonlinear wave equations. It is a physically important problem whether the spatially varying parameters in (1.1) can be estimated or not by the possible observed measurements.

We explain our identification problems precisely as follows. First we replace the positive coefficient $a(x)$ and $b(x)$ in (1.1) by $a^2(x)+a_0$ and by $b^2(x)+b_0$ with $a_0, b_0 > 0$, respectively in order to take the parameter space be a linear space of parameters.

Let $a(x)$, $b(x)$, and $c(x)$ be unknown coefficients. We take the parameter space

$$\mathcal{P} = L^\infty(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega) = L^\infty(\Omega)^3$$

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as the set of L^∞-functions of the parameter $q = (a(\cdot), b(\cdot), c(\cdot))$, or shortly $q = (a, b, c)$. For each $q = (a, b, c) \in \mathcal{P}$, we denote by $y = y(q)$ the solution of (1.1) in which $a(x)$ and $b(x)$ are replaced by $a^2(x) + a_0$ and $b^2(x) + b_0$, respectively. In order to identify the unknown parameter q, we consider the following quadratic cost functional given by the distributive and terminal values mesurements of solutions

$$J(q) = \int_Q |y(q) - z_d|^2 dxdt + \int_\Omega |y(q;T) - z_d^T|^2 dx, \quad \forall q = (a, b, c) \in \mathcal{P}, \tag{1.2}$$

where $z_d \in L^2(Q)$ and $z_d^T \in L^2(\Omega)$ are desired values. Let $\mathcal{P}_{ad} \subset \mathcal{P}$ be an admissible parameter set. The identification problem in view of the cost (1.2) is to find and characterize an element $q^* = (a^*, b^*, c^*) \in \mathcal{P}_{ad}$, called the optimal parameter, such that

$$\inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*). \tag{1.3}$$

We prove the existence of an optimal parameters q^* by using the continuity of solutions on parameters and establish the necessary optimality conditions by introducing some adjoint systems. For this we prove the strong Gâteaux differentiability of the solution mapping $q \to y(q)$ under some restriction on the spatial dimension n. We remark that the weak Gâteaux differentiability of the mapping on constant parameters is proved in [5, 6, 9].

2 Weak solutions of semilinear damped wave equations

In this section we shall give the results on existence, uniqueness and regularity of solutions for the system (1.1) based on the variational formulation due to Dautray and Lions [2].

For the nonlinear function $F : \mathbf{R} \to \mathbf{R}$ in (1.1), we suppose the following assumption:

$(H1)$ there exists a $K_1 > 0$ such that

$$|F(s) - F(r)| \le K_1 |s - r|, \quad \forall s,\ r \in \mathbf{R}.$$

By $(H1)$, we see easily that there exists a $K_2 > 0$ such that

$$|F(s)| \le K_2(1 + |s|), \quad \forall s \in \mathbf{R}. \tag{2.1}$$

For the evolution equation setting of (1.1), we introduce two Hilbert spaces $L^2(\Omega)$ and $H_0^1(\Omega)$. The scalar products and norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted by (ϕ, ψ), $|\phi|$ and $(\phi, \psi)_{H_0^1(\Omega)}$, $\|\phi\|$, respectively. It is well known that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and compact. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ, ψ) and $|\phi|$. Then the scalar product $(\phi, \psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla\phi, \nabla\psi)$ and $\|\phi\| = |(\nabla\phi, \nabla\phi)|^{\frac{1}{2}}$, respectively.

For the problem (1.1) we suppose that $f \in L^2(0, T; H^{-1}(\Omega))$, $y_0 \in H_0^1(\Omega)$, and $y_1 \in L^2(\Omega)$. The solution space $W(0, T)$ of (1.1) is defined by

$$W(0,T) = \{g | g \in L^2(0, T; H_0^1(\Omega)),\ g' \in L^2(0, T; H_0^1(\Omega)),\ g'' \in L^2(0, T; H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g'\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))} \right)^{\frac{1}{2}}.$$

The space $W(0, T)$ is continuously imbedded in $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ (cf. Dautray and Lions [2, p.555]).

Since F satisfies $(H1)$, by (2.1) we can verify that the map $F : L^2(\Omega) \to L^2(\Omega)$ defined by

$$F(\psi)(x) = F(\psi(x)) \quad a.e.\ x \in \Omega, \quad \forall \psi \in L^2(\Omega) \tag{2.2}$$

satisfies

$$|F(\psi)|^2 = \int_\Omega |F(\psi(x))|^2 dx \le 2K_2^2(|\Omega| + |\psi|^2), \quad \forall \psi \in L^2(\Omega), \tag{2.3}$$

and the uniform Lipschitz continuity

$$|F(\phi) - F(\psi)| \le K_1 |\phi - \psi|, \quad \forall \phi,\ \psi \in L^2(\Omega), \tag{2.4}$$

where $|\Omega|$ is the measure of Ω. As in Dautray and Lions [2, p.512], we give the variational formulation of weak solutions of (1.1). A function y is said to be a weak solution of (1.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot), \phi\rangle + (a\nabla y(\cdot) + b\nabla y'(\cdot), \nabla\phi) = (cF(y(\cdot)), \phi) + \langle f(\cdot), \phi\rangle \\ \qquad\qquad \text{for all } \ \phi \in H_0^1(\Omega) \ \text{ in the sense of } \ \mathcal{D}'(0,T), \\ y(0) = y_0 \in H_0^1(\Omega), \quad y'(0) = y_1 \in L^2(\Omega), \end{cases} \tag{2.5}$$

where $\mathcal{D}'(0,T)$ is the space of distributions on $(0,T)$.

For existence and uniqueness of the weak solution for (1.1), we can prove the following theorem by the Galerkin method (cf. Ha and Nakagiri [4]).

Theorem 2.1. *Assume that F satisfies $(H1)$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, $f \in L^2(0,T;H^{-1}(\Omega))$, and $a, b, c \in L^\infty(\Omega)$ with $a(x) \ge a_0 > 0$, $b(x) \ge b_0 > 0$, a.e. $x \in \Omega$. Then the problem (1.1) has a unique weak solution y in $W(0,T)$, which belongs to $C([0,T]; H_0^1(\Omega))$. Further, the solution y satisfies the following inequality*

$$\begin{aligned} &|y'(t)|^2 + |\nabla y(t)|^2 + \int_0^t |\nabla y'(s)|^2 ds \\ &\le \ C(1 + \|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}), \quad \forall t \ \in \ [0,T], \end{aligned} \tag{2.6}$$

where $C > 0$ depends only on $K_1, K_2 > 0$, the L^∞ norms of a, b, c, and the positive constants $a_0, b_0 > 0$.

We will omit writing the integral variables in the definite integral without any confusion. For example, in (2.6) we will write $\int_0^t |\nabla y'|^2 ds$ instead of $\int_0^t |\nabla y'(s)|^2 ds$.

Remark 2.1. We can choose the constant C in (2.6) to be uniformly bounded for each bounded set of a, b, and c in $L^\infty(\Omega)$.

3 Identification problems

In this section we study the identification problems for the parameter $q = (a, b, c) \in \mathcal{P}$ in the problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \nabla \cdot \big((a^2(x) + a_0)\nabla y + (b^2(x) + b_0)\nabla \dfrac{\partial y}{\partial t}\big) = c(x)F(y) + f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where $a_0, b_0 > 0$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $f \in L^2(0,T;H^{-1}(\Omega))$ are fixed. Throughout this section we suppose $(H1)$. The triple of spatially varying coefficients (a, b, c) in (3.1) is an unknown parameter should be identified.

Let $\mathcal{P} = L^\infty(\Omega)^3$ be the space of parameters $q = (a, b, c)$. The norm of $a \in L^\infty(\Omega)$ is denoted simply by $\|a\|_\infty$ and the norm of $\mathcal{P}$ is defined by $\|q\|_{\mathcal{P}} = \|a\|_\infty + \|b\|_\infty + \|c\|_\infty$ for $q = (a, b, c) \in \mathcal{P}$. By Theorem 2.1 we have that for each $q \in \mathcal{P}$ there exists a unique weak solution $y = y(q) \in W(0, T)$ of (3.1).

3.1 Existence of optimal parameters First we shall show the continuous dependence of solutions $y(q)$ on $q = (a, b, c)$.

Theorem 3.1. *The solution map $q \to y(q)$ from $\mathcal{P} = L^\infty(\Omega)^3$ into $W(0, T)$ is continuous.*

Proof. Let $q = (a, b, c)$ be arbitrarily fixed. Suppose $q_m = (a_m, b_m, c_m) \to q = (a, b, c)$ in $\mathcal{P}$. Let $y_m = y(q_m)$ and $y = y(q)$ be the solutions of (3.1) for $q = q_m$ and for q, respectively. Since $\{q_m\}$ is bounded in $\mathcal{P}$, by Theorem 2.1 and Remark 2.1, we see that

$$|y_m'(t)|^2 + |\nabla y_m(t)|^2 + \int_0^t |\nabla y_m'(s)|^2 ds \le C_0 < \infty, \quad \forall t \in [0, T], \tag{3.2}$$

where $C_0 > 0$ is a constant independent of m. We set $z_m = y_m - y$. Then z_m satisfies

(3.3)
$$\begin{cases} \dfrac{\partial^2 z_m}{\partial t^2} - \nabla\cdot\big((a_m^2(x) + a_0)\nabla z_m + (b_m^2(x) + b_0)\nabla \dfrac{\partial z_m}{\partial t}\big) = c_m(x)(F(y_m) - F(y)) \\ -\nabla\cdot\big((a_m^2(x) - a^2(x))\nabla y + (b_m^2(x) - b^2(x))\nabla \dfrac{\partial y}{\partial t}\big) + (c_m(x) - c(x))F(y) \quad \text{in } Q, \\ z_m = 0 \quad \text{on } \Sigma, \\ z_m(0, x) = 0, \quad \dfrac{\partial z_m}{\partial t}(0, x) = 0 \quad \text{in } \Omega \end{cases}$$

in the weak sense. Here, we note that

$$\|q_m\|_{\mathcal{P}} = \|a_m\|_\infty + \|b_m\|_\infty + \|c_m\|_\infty, \ \|q\|_{\mathcal{P}} = \|a\|_\infty + \|b\|_\infty + \|c\|_\infty \le M \tag{3.4}$$

for some $M > 0$ independent of m. Multiply the weak form of (3.3) by z_m' and integrate them over $[0, t]$, then we have

$$\begin{aligned} & |z_m'(t)|^2 + ((a_m^2 + a_0)\nabla z_m(t), \nabla z_m(t)) + 2\int_0^t ((b_m^2 + b_0)\nabla z_m', \nabla z_m')ds \\ = \ & 2\int_0^t (c_m(F(y_m) - F(y)), z_m')ds + 2\int_0^t ((c_m - c)F(y), z_m')ds \\ & + 2\int_0^t ((a_m^2 - a^2)\nabla y + (b_m^2 - b^2)\nabla y', \nabla z_m')ds. \end{aligned} \tag{3.5}$$

If we set

$$\Phi_m(t) = \int_0^t \big((a_m^2 - a^2)\nabla y + (b_m^2 - b^2)\nabla y', \nabla z_m')ds + \int_0^t ((c_m - c)F(y), z_m')ds,$$

then by (3.5) we have

$$\begin{aligned} & |z_m'(t)|^2 + a_0|\nabla z_m(t)|^2 + 2b_0\int_0^t |\nabla z_m'|^2 ds \\ \le \ & 2\Big|\int_0^t (c_m(F(y_m) - F(y)), z_m')ds\Big| + |\Phi_m(t)|. \end{aligned} \tag{3.6}$$

In what follows all necessary constants independent of m are denoted by C for simplicity. We estimate $|\Phi_m(t)|$ in (3.6). Since $|\nabla z_m'| \le |\nabla y'| + |\nabla y_m'|$, we can verify by (3.2) and Schwarz inequality that

$$\begin{aligned}(3.7)\qquad \left|\int_0^t ((b_m^2-b^2)\nabla y', \nabla z_m')ds\right| \;&\le\; C\|b_m^2-b^2\|_\infty \int_0^t |\nabla y'|(|\nabla y'|+|\nabla y_m'|)ds \\ \le\; C\|b_m^2-b^2\|_\infty \int_0^T (|\nabla y'|^2+|\nabla y_m'|^2)ds \;&\le\; C\|b_m^2-b^2\|_\infty.\end{aligned}$$

Similarly, by (3.2), (2.3) and using the Poincare inequality $|\psi| \le c_0|\nabla\psi|$ for $\psi \in H_0^1(\Omega)$, we have

$$(3.8)\qquad \left|\int_0^t ((a_m^2-a^2)\nabla y, \nabla z_m')ds\right| \le C\|a_m^2-a^2\|_\infty,$$

$$(3.9)\qquad \left|\int_0^t ((c_m-c)F(y), z_m')ds\right| \le C\|c_m-c\|_\infty.$$

Since $|a^2(x)-a_m^2(x)| \le |a(x)-a_m(x)|\,(|a(x)|+|a_m(x)|)$, we see that $\|a^2-a_m^2\|_\infty \le C\|a-a_m\|_\infty$ by the boundedness of $\{a_m\}$ in $L^\infty(\Omega)$. Then from (3.7)-(3.9) it follows that

$$(3.10)\quad |\Phi_m(t)| \le C(\|a-a_m\|_\infty + \|b-b_m\|_\infty + \|c-c_m\|_\infty) = C\|q-q_m\|_{\mathcal{P}}, \qquad \forall t \in [0,T].$$

Since $|F(y_m)-F(y)| \le K_1|z_m|$ by (2.4), we have by Schwarz inequality and Poincare inequality that

$$\begin{aligned}(3.11)\qquad 2\left|\int_0^t (c_m(F(y_m)-F(y)), z_m')ds\right| &\le 2K_1\|c_m\|_\infty \int_0^t |z_m||z_m'|ds \\ \le\; C\Big(\int_0^t |z_m|^2ds + \int_0^t |z_m'|^2ds\Big) \;&\le\; C\Big(\int_0^t |z_m'|^2ds + \int_0^t |\nabla z_m|^2ds\Big).\end{aligned}$$

Hence, by (3.6), (3.10), (3.11) and the positivity of a_0, b_0, we can obtain

$$(3.12)\quad |z_m'(t)|^2 + |\nabla z_m(t)|^2 + \int_0^t |\nabla z_m'|^2ds \le C\Big(\|q-q_m\|_{\mathcal{P}} + \int_0^t (|z_m'|^2+|\nabla z_m|^2)ds\Big).$$

Applying Gronwall's inequality to (3.12), we have

$$(3.13)\qquad |z_m'(t)|^2 + |\nabla z_m(t)|^2 + \int_0^t |\nabla z_m'|^2ds \le C\|q-q_m\|_{\mathcal{P}}.$$

Since $q-q_m \to 0$ in $\mathcal{P}$ as $m \to \infty$, we conclude from (3.13) that

$$(3.14)\qquad \begin{cases} z_m \;\to\; 0 \quad \text{in } L^\infty(0,T;H_0^1(\Omega)), \\ z_m' \;\to\; 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \end{cases}$$

In addition we see by (2.3), (2.4), (3.2), and (3.14) that

$$\begin{aligned}\int_0^T |c_mF(y_m)-cF(y)|^2dt \;&\le\; 2\int_0^T |(c_m-c)F(y_m)|^2dt + 2\int_0^T |c(F(y_m)-F(y))|^2dt \\ &\le\; C\|c_m-c\|_\infty\Big(1+\int_0^T |y_m|^2dt\Big) + C\int_0^T |y_m-y|^2dt \\ &\le\; C\|c_m-c\|_\infty + C\|y_m-y\|^2_{L^\infty(0,T;H_0^1(\Omega))},\end{aligned}$$

so that

$$c_m F(y_m) \to cF(y) \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)).$$

This implies $y_m'' \to y''$ strongly in $L^2(0,T;H^{-1}(\Omega))$, and hence

$$y_m(\cdot) \ \to \ y(\cdot) \quad \text{strongly in} \quad W(0,T).$$

This completes the proof.

Let the cost $J = J(q)$ over $\mathcal{P}$ be given by (1.2). Assume that an admissible subset $\mathcal{P}_{ad}$ of $\mathcal{P}$ is convex and closed. If $\mathcal{P}_{ad}$ is compact, then for the minimizing sequence q_m such as $J(q_m) \to J^* = \inf_{q\in\mathcal{P}_{ad}} J(q)$ we can choose a subsequence $\{q_{mj}\}$ of $\{q_m\}$ such that $q_{mj} \to q^* \in \mathcal{P}_{ad}$ and $y(q_{mj}) \to y(q^*)$ strongly in $W(0,T)$ by Theorem 3.1. Due to the continuous embedding $W(0,T) \hookrightarrow C([0,T];L^2(\Omega)) \cap L^2(0,T;L^2(\Omega))$ we have $J^* = J(q^*)$ for the cost (1.2). Then we have the following corollary.

Corollary 3.1. *If $\mathcal{P}_{ad}$ is compact in $\mathcal{P}$, then there exists at least one optimal parameter $q^* \in \mathcal{P}_{ad}$ for the cost J in* (1.2).

Remark 3.1. If we take $p > n$, then the embedding $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ is compact for all $0 \le \alpha < 1 - \frac{n}{p}$ (cf. [11, p.47]). The embedding $C^{0,\alpha}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ is also compact for all $\alpha > 0$ and the embedding $C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$ is continuous. Hence, if $\mathcal{P}_{ad}$ is given by a closed and bounded (not necessarily convex) set of $W^{1,p}(\Omega)^3$ with $p > n$, then $\mathcal{P}_{ad}$ is compact in $\mathcal{P}$.

Remark 3.2. If the cost J is replaced by a stronger one such as

$$J(q) = \|y'(q) - z_d\|^2_{L^2(0,T;H_0^1(\Omega))}$$

with $z_d \in L^2(0,T;H_0^1(\Omega))$ or

$$J(q) = \sum_{i=1}^{k} \|y(q;t_i) - z_d^i\|^2, \quad 0 < t_1 < \cdots < t_k \le T$$

with $z_d^i \in H_0^1(\Omega),\ i = 1, \cdots k$, then the same coclusion in Corollary 3.1 holds.

3.2 Necessary optimality condition Let the addmissible set $\mathcal{P}_{ad}$ be closed and convex in $\mathcal{P}$ and let $q^* = (a^*, b^*, c^*)$ be an optimal parameter on $\mathcal{P}_{ad}$ for the cost J. As is well known the necessary optimality condition for the cost J is given by

$$DJ(q^*)(q - q^*) \ge 0, \qquad \forall q \in \mathcal{P}_{ad}, \tag{3.15}$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$.

In order to prove that the solution map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable, we shall use the Fréchet differentiability of nonlinear map F. Unfortunately, even if $F(s)$ is continuously differentiable, the map $F : L^2(\Omega) \to L^2(\Omega)$ in (2.2) is not Fréchet differentiable in general. To obtain the Fréchet differentiability of F, we have to restrict the domain of F into $H_0^1(\Omega)$ and impose some additional continuity condition on $F'(s)$. That is, we suppose the following assumption (which is stronger than $(H1)$):

$(H2)$ $F(s)$ is continuously differentiable on $\mathbf{R}$, and the derivative $F'(s)$ satisfies the Hölder continuity and the uniform boundedness

$$|F'(s) - F'(r)| \leq K_3|s-r|^{\rho}, \qquad |F'(s)| \leq K_4, \qquad \forall s,\ r \in \mathbf{R}$$

for some $0 < \rho \leq 1$ and $K_3, K_4 > 0$.

Proposition 3.1. *Assume that F satisfies $(H2)$. We define the map $F : H_0^1(\Omega) \to L^2(\Omega)$ same as in (2.2) by*

$$F(\psi)(x) = F(\psi(x)) \quad a.e.\ x \in \Omega, \qquad \forall \psi \in H_0^1(\Omega). \tag{3.16}$$

Assume further that the spatial dimension n satisfies

$$n \leq 2 + \frac{2}{\rho}. \tag{3.17}$$

Then the map $F : H_0^1(\Omega) \to L^2(\Omega)$ is continuously Fréchet differentiable and the Fréchet derivative $\partial_y F(y)$ at $y = \psi \in H_0^1(\Omega)$ is given by the following multiplication operator

$$\partial_y F(\psi)h(x) = F'(\psi(x))h(x) \quad a.e.\ x \in \Omega, \qquad \forall h \in H_0^1(\Omega). \tag{3.18}$$

Further the derivative $\partial_y F(y)$ is Hölder norm continuous in y on $\mathcal{L}(H_0^1(\Omega), L^2(\Omega))$ and satisfies the following inequalities

$$\|\partial_y F(\psi + h) - \partial_y F(\psi)\|_{\mathcal{L}(H_0^1(\Omega),L^2(\Omega))} \leq C\|h\|^{\rho}, \qquad \forall \psi,\ h \in H_0^1(\Omega), \tag{3.19}$$

$$\|\partial_y F(\psi)\|_{\mathcal{L}(H_0^1(\Omega),L^2(\Omega))} \leq C, \qquad \forall \psi \in H_0^1(\Omega), \tag{3.20}$$

where $C > 0$ is a constant independent of $\psi,\ h \in H_0^1(\Omega)$.

Proof. Let $\psi, h \in H_0^1(\Omega)$ be fixed. By the integral mean value theorem for scalar functions we have

$$F(\psi(x) + h(x)) - F(\psi(x)) - F'(\psi(x))h(x) = \int_0^1 \{F'(\psi(x) + \theta h(x)) - F'(\psi(x))\}h(x)d\theta.$$

First we consider the case $n \geq 3$. In this case we recall the continuous embedding $H^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ (cf. [11, p.46]). Applying Hölder inequality for $p = n/2$ and $r = n/(n-2)$ with $p^{-1} + r^{-1} = 1$, we have

$$\begin{aligned}
& |\{F'(\psi + \theta h) - F'(\psi)\}h| \\
= \ & \Big(\int_{\Omega} |F'(\psi(x) + \theta h(x)) - F'(\psi(x))|^2 |h(x)|^2 dx\Big)^{1/2} \\
\leq \ & \Big(\int_{\Omega} |F'(\psi(x) + \theta h(x)) - F'(\psi(x))|^n d\theta\Big)^{1/n} \Big(\int_{\Omega} |h(x))|^{2n/(n-2)} dx\Big)^{(n-2)/2n} \\
\leq \ & \|F'(\psi + \theta h) - F'(\psi)\|_{L^n(\Omega)} \|h\|_{L^{\frac{2n}{n-2}}(\Omega)} \ \leq \ C\|F'(\psi + \theta h) - F'(\psi)\|_{L^n(\Omega)}\|h\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
|F(\psi + h) - F(\psi) - F'(\psi)h| \ &\leq \ \int_0^1 |\{F'(\psi + \theta h) - F'(\psi)\}h| d\theta \\
&\leq \ \|h\| \int_0^1 \|\{F'(\psi + \theta h) - F'(\psi)\}\|_{L^n(\Omega)} d\theta.
\end{aligned} \tag{3.21}$$

By the Hölder continuity of $F'(s)$ in $(H2)$ and $0 \le \theta \le 1$, we can deduce

$$\|\{F'(\psi+\theta h) - F'(\psi)\}\|_{L^n(\Omega)} = \Big(\int_\Omega |F'(\psi(x)+\theta h(x)) - F'(\psi(x))|^n dx\Big)^{1/n} \le K_3\Big(\int_\Omega (\theta^\rho |h(x)|^\rho)^n dx\Big)^{1/n} \le K_3\Big(\int_\Omega |h(x)|^{n\rho} dx\Big)^{1/n} = K_3\|h\|^\rho_{L^{n\rho}(\Omega)}. \tag{3.22}$$

Since Ω is bounded, if $n\rho \le 2n/(n-2)$, i.e., n satisfies (3.17), then the embedding $L^{\frac{2n}{n-2}}(\Omega) \hookrightarrow L^{n\rho}(\Omega)$ is continuous. Hence, by (3.22) we have that

$$\|\{F'(\psi+\theta h) - F'(\psi)\}\|_{L^n(\Omega)} \le C\|h\|^\rho, \tag{3.23}$$

where C is a constant independent of θ. Therefore, it follows ¿from (3.21) and (3.23) that

$$\frac{1}{\|h\|}|F(\psi+h) - F(\psi) - F'(\psi)h| \le C\|h\|^\rho = o(1) \text{ as } \|h\| \to 0. \tag{3.24}$$

This proves that F is Fréchet differentiable and the derivative $\partial_y F(\psi)h$ at $y = \psi$ in the direction h is given by (3.18). Next we shall show the Hölder continuity of $\partial_y F(y)$ on y in the operator norm. By similar calculations as above, under the condition (3.17), we see that for any $\psi, \phi, h \in H_0^1(\Omega)$,

$$|\{\partial_y F(\psi+h) - \partial_y F(\psi)\}\phi| \le C\|F'(\psi+h) - F'(\psi)\|_{L^n(\Omega)}\|\phi\| \le C\|h\|^\rho\|\phi\|. \tag{3.25}$$

This impliess (3.19), i.e., the operator $\partial_y F(y)$ is Hölder norm continuous in the space $\mathcal{L}(H_0^1(\Omega), L^2(\Omega))$. Since $F'(s)$ is bounded in $\mathbf{R}$ by $(H2)$, we see

$$|\partial_y F(\psi)h| = \Big(\int_\Omega |F'(\psi(x))|^2|h(x)|^2 dx\Big)^{1/2} \le K_4\Big(\int_\Omega |h(x)|^2 dx\Big)^{1/2} \le K_4|h| \le C\|h\|,$$

so that the boundedness (3.20) follows. For the case $n = 1, 2$, we have the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \ge 1$. Then by repeating similar calculations as above, in which the constant $\frac{2n}{n-2}$ is replaced by some constant $r > 1$, we have the same conclusion.

Theorem 3.2. *Assume that $(H2)$ is satisfied and the spatial dimension n satisfies $n \le 2 + \frac{2}{\rho}$. Then the map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable and such the Gâteaux derivative of $y(q)$ at $q = q^* = (a^*, b^*, c^*)$ in the direction $\overline{q} = (\overline{a}, \overline{b}, \overline{c}) \in \mathcal{P}$, say $z = D_q y(q^*)\overline{q}$, is a unique weak solution of the following linear problem*

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \nabla\cdot\big((a^{*2}(x)+a_0)\nabla z + (b^{*2}(x)+b_0)\nabla\dfrac{\partial z}{\partial t}\big) \\ \qquad\qquad = c^*(x)\partial_y F(y^*)z + \mathcal{F}(\overline{q}) \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(0,x) = 0, \quad \dfrac{\partial z}{\partial t}(0,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{3.26}$$

where $y^* = y(q^*)$ and

$$\mathcal{F}(\overline{q}) = \nabla\cdot\big(2a^*\overline{a}\nabla y^* + 2b^*\overline{b}\nabla\frac{\partial y^*}{\partial t}\big) + \overline{c}F(y^*). \tag{3.27}$$

Proof. Let $\lambda \in (-1,1)$, and let y_λ and y^* be the weak solutions of (3.1) corresponding to $q^* + \lambda\overline{q}$ and q^*, respectively. Then by the estimate (3.13) we see

(3.28)
$$|y'_\lambda(t) - y^{*\prime}(t)|^2 + |\nabla y_\lambda(t) - \nabla y^*(t)|^2 + \int_0^t |\nabla(y'_\lambda - y^{*\prime})|^2 ds \le C|\lambda|\|\overline{q}\|_{\mathcal{P}}, \qquad \forall t \in [0,T].$$

We set $z_\lambda = \lambda^{-1}(y_\lambda - y^*)$, $\lambda \neq 0$. Then z_λ is a solution of the following problem in the weak sense:

$$\begin{cases} \dfrac{\partial^2 z_\lambda}{\partial t^2} - \nabla\cdot\big((a^{*2}(x)+a_0)\nabla z_\lambda + (b^{*2}(x)+b_0)\nabla\dfrac{\partial z_\lambda}{\partial t}\big) \\ \qquad\qquad\qquad = c^*(x)\dfrac{1}{\lambda}(F(y_\lambda) - F(y^*)) + \mathcal{F}_\lambda(\overline{q}) \quad \text{in } Q, \\ z_\lambda = 0 \quad \text{on } \Sigma, \\ z_\lambda(0,x) = 0, \quad \dfrac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{3.29}$$

where

$$\mathcal{F}_\lambda(\overline{q}) = \nabla\cdot\big(2a^*\overline{a}\nabla y_\lambda + 2b^*\overline{b}\nabla\frac{\partial y_\lambda}{\partial t}\big) + \overline{c}F(y_\lambda). \tag{3.30}$$

We put

$$\begin{aligned} \Psi_\lambda(\overline{q};t) &= \int_0^t \langle \mathcal{F}_\lambda(\overline{q}), z'_\lambda\rangle ds \\ &= \int_0^t (2a^*\overline{a}\nabla y_\lambda + 2b^*\overline{b}\nabla y'_\lambda, \nabla z'_\lambda) ds + \int_0^t (\overline{c}F(y_\lambda), z'_\lambda) ds. \end{aligned} \tag{3.31}$$

Let $\epsilon > 0$ be an arbitrary number. By using Schwarz inequality and (2.3), we have from (3.31) that

$$\begin{aligned} |\Psi_\lambda(\overline{q};t)| &\le C\int_0^t (|\nabla y_\lambda| + |\nabla y'_\lambda|)|\nabla z'_\lambda| ds + C\int_0^t (1+|y_\lambda|)|z'_\lambda| ds \\ &\le \epsilon\int_0^t |\nabla z'_\lambda|^2 ds + C(\epsilon)\int_0^t (|\nabla y_\lambda|^2 + |\nabla y'_\lambda|^2) ds \\ &\quad + C\int_0^t |z'_\lambda|^2 ds + C\int_0^t |y_\lambda|^2 ds + C, \end{aligned} \tag{3.32}$$

where $C > 0$ is a constant depending on $q^*, \overline{q}$ and $C(\epsilon) > 0$ is a constant depending on $q^*, \overline{q}$ and ϵ. Here, we notice that the set $\{y_\lambda, y'_\lambda, \nabla y_\lambda, \nabla y'_\lambda\}_{\lambda\in(-1,1)}$ is bounded in $L^2(0,T;L^2(\Omega))$. Then we can see from (3.32) that

$$|\Psi_\lambda(\overline{q};t)| \le C(\epsilon) + \epsilon\int_0^t |\nabla z'_\lambda|^2 ds + C\int_0^t |z'_\lambda|^2 ds. \tag{3.33}$$

The nonlinear term in (3.29) is estimated as

$$\left| c^*\frac{1}{\lambda}\Big(F(y_\lambda) - F(y^*)\Big)\right| \le K_1\|c^*\|_\infty |z_\lambda| \le C|z_\lambda|,$$

so that by Poincare inequality

$$\begin{aligned}(3.34)\qquad \int_0^t \left|\left(c^*\frac{1}{\lambda}(F(y_\lambda)-F(y^*)), z_\lambda'\right)\right| ds &\le C\int_0^t (|z_\lambda|^2+|z_\lambda'|^2)ds\\ &\le C\int_0^t (|z_\lambda'|^2+|\nabla z_\lambda|^2)ds.\end{aligned}$$

We multiply the weak form of (3.29) by z_λ', use (3.33), (3.34) and set $\epsilon = b_0$ to obatin

$$(3.35)\qquad |\nabla z_\lambda(t)|^2+|z_\lambda'(t)|^2+\int_0^t |\nabla z_\lambda'|^2 ds \le C + C\int_0^t (|\nabla z_\lambda|^2+|z_\lambda'|^2)dt.$$

Applying Gronwall's inequality to (3.35) , we obtain

$$(3.36)\qquad |\nabla z_\lambda(t)|^2+|z_\lambda'(t)|^2+\int_0^t |\nabla z_\lambda'|^2 ds \le C.$$

Therefore, there exists a $z\in W(0,T)$ and a sequence $\{\lambda_k\}\subset(-1,1)$ tending to 0 such that

$$(3.37)\qquad \begin{cases} z_{\lambda_k}\to z \ \text{ weakly star in } \ L^\infty(0,T;H_0^1(\Omega)) \\ \qquad\qquad \text{and weakly in } \ L^2(0,T;H_0^1(\Omega)) \ \text{ as } \ k\to\infty, \\ z_{\lambda_k}'\to z' \ \text{ weakly star in } \ L^\infty(0,T;L^2(\Omega)) \\ \qquad\qquad \text{and weakly in } \ L^2(0,T;H_0^1(\Omega)) \ \text{ as } \ k\to\infty, \\ z(0)=0, \quad z'(0)=0. \end{cases}$$

By Proposition 3.1, $F: H_0^1(\Omega)\to L^2(\Omega)$ is Fréchet differentiable. Then by the integral mean values theorem for Fréchet differentiable functions, we have

$$(3.38)\qquad \frac{F(y_\lambda)-F(y^*)}{\lambda} = \left(\int_0^1 \partial_y F(\theta y_\lambda+(1-\theta)y^*)\, d\theta\right) z_\lambda.$$

For brevity of notations we shall write the space $\mathcal{L}(H_0^1(\Omega),L^2(\Omega))$ simply by $\mathcal{L}$ and the space $\mathcal{L}(L^2(\Omega),H^{-1}(\Omega))$ simply by $\mathcal{L}'$, respectively. First we shall show that

$$(3.39)\qquad \left\|\int_0^1 \partial_y F(\theta y_\lambda+(1-\theta)y^*)\, d\theta - \partial_y F(y^*)\right\|_{L^2(0,T;\mathcal{L})} \to 0 \quad \text{as } \lambda\to 0.$$

By (3.19), we have the following uniform estimate in θ

$$(3.40)\qquad \|\{\partial_y F(\theta y_\lambda+(1-\theta)y^*)-\partial_y F(y^*)\}\|_{\mathcal{L}} \le C\theta^\rho\|y_\lambda-y^*\|^\rho \le C\|y_\lambda-y^*\|^\rho.$$

Hence, by (3.19) and (3.40), we have from (3.28)

$$\begin{aligned} &\left\|\int_0^1 \partial_y F(\theta y_\lambda+(1-\theta)y^*)\, d\theta - \partial_y F(y^*)\right\|^2_{L^2(0,T;\mathcal{L})} \\ =\ &\int_0^T \left\|\int_0^1 \{\partial_y F(\theta y_\lambda+(1-\theta)y^*)-\partial_y F(y^*)\}\, d\theta\right\|^2_{\mathcal{L}} dt \\ \le\ &\int_0^T \left(\int_0^1 \|\{\partial_y F(\theta y_\lambda+(1-\theta)y^*)-\partial_y F(y^*)\}\|_{\mathcal{L}} d\theta\right)^2 dt \\ \le\ & C\int_0^T \|y_\lambda(t)-y^*(t)\|^{2\rho} dt \ \le\ C|\lambda|^\rho\|\overline{q}\|^\rho_{\mathcal{P}}. \end{aligned}$$

Since $0 < \rho \le 1$, the convergence (3.39) follows readily.

We shall show that

$$\partial_y F(y^*)'(c^*\varphi) \in L^2(0,T;H^{-1}(\Omega)), \quad \forall \varphi \in L^2(0,T;L^2(\Omega)), \tag{3.41}$$

where $\partial_y F(\psi)' \in \mathcal{L}' = \mathcal{L}(L^2(\Omega), H^{-1}(\Omega))$ is the adjoint operator of $\partial_y F(\psi)$. We note that $\partial_y F(\psi)'$ is given by $\partial_y F(\psi)' h(x) = F'(\psi(x))h(x)$ *a.e.* $x \in \Omega$ for all $h \in L^2(\Omega)$. It is obvious that $\|\partial_y F(\psi)'\|_{\mathcal{L}'} = \|\partial_y F(\psi)\|_{\mathcal{L}}$. By (3.20), we obtain

$$\begin{aligned}
&\|\partial_y F(\psi)'(c^*\varphi)\|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \|\partial_y F(y^*)'(c^*\varphi)\|^2_{H^{-1}(\Omega)} dt \\
\le\ & \int_0^T \|\partial_y F(y^*)'\|^2_{\mathcal{L}'} |c^*\varphi|^2 dt \\
\le\ & \|c^*\|_\infty \int_0^T \|\partial_y F(y^*)\|^2_{\mathcal{L}} |\varphi|^2 dt \ \le\ C\|\varphi\|^2_{L^2(0,T;L^2(\Omega))} < \infty,
\end{aligned}$$

which proves (3.41). Now we shall estimate the following difference

$$\begin{aligned}
& c^* \frac{F(y_\lambda) - F(y^*)}{\lambda} - c^*\partial_y F(y^*) z_\lambda \\
=\ & c^* \left(\int_0^1 \partial_y F(\theta y_\lambda + (1-\theta)y^*)\, d\theta - \partial_y F(y^*) \right) z_\lambda \\
& + c^*\partial_y F(y^*)(z_\lambda - z) \equiv I_1(\lambda) + I_2(\lambda).
\end{aligned}$$

By (3.36) and (3.40), we deduce that $\{z_\lambda\}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$ and that

$$\begin{aligned}
&\|I_1(\lambda)\|_{L^2(0,T;L^2(\Omega))} \\
\le\ & \|c^*\|_\infty \left\| \int_0^1 \partial_y F(\theta y_\lambda + (1-\theta)y^*) d\theta - \partial_y F(y^*) \right\|_{L^2(0,T;\mathcal{L})} \|z_\lambda\|_{L^\infty(0,T;H_0^1(\Omega))} \\
\le\ & C \left\| \int_0^1 \{\partial_y F(\theta y_\lambda + (1-\theta)y^*) - \partial_y F(y^*)\} \right\|_{L^2(0,T;\mathcal{L})} \to 0 \quad \text{as } \lambda \to 0.
\end{aligned} \tag{3.42}$$

Let $\varphi \in L^2(0,T;L^2(\Omega))$ be fixed. Then

$$\int_0^T (I_2(\lambda), \varphi) dt \ = \int_0^T \langle z_\lambda - z, , \partial_y F(y^*)'(c^*\varphi) \rangle dt \tag{3.43}$$

and by (3.41), $\partial_y F(\psi)'(c^*\varphi) \in L^2(0,T;H^{-1}(\Omega))$. Since $z_{\lambda_k} - z \to 0$ weakly in $L^2(0,T;H_0^1(\Omega))$ by (3.37), from (3.43) it follows that $I_2(\lambda_k) \to 0$ weakly in $L^2(0,T;L^2(\Omega))$. This together with (3.42) implies that

$$c^* \frac{F(y_{\lambda_k}) - F(y^*)}{\lambda} \to c^*\partial_y F(y^*) z \quad \text{weakly in } \ L^2(0,T;L^2(\Omega)) \tag{3.44}$$

as $\lambda_k \to 0$. Next we shall prove that

$$\mathcal{F}_\lambda(\overline{q}) \to \mathcal{F}(\overline{q}) \ \text{ strongly in } \ L^2(0,T;H^{-1}(\Omega)). \tag{3.45}$$

Let $\varphi \in L^2(0,T;H_0^1(\Omega))$ be fixed. By Schwarz inequality and (3.28),

$$\begin{aligned}
&\int_0^T \langle \mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q}), \varphi\rangle dt \\
= & \int_0^T \big(\big(2a^*\overline{a}\nabla(y_\lambda - y^*) + 2b^*\overline{b}\nabla(y_\lambda' - y^{*\prime})\big), \nabla\varphi\big)dt + \int_0^T (\overline{c}(F(y_\lambda) - F(y^*)), \varphi)dt \\
\leq & \; 2\|a^*\|_\infty\|\overline{a}\|_\infty \int_0^T |\nabla(y_\lambda - y^*)||\nabla\varphi|dt \; + \; 2\|b^*\|_\infty\|\overline{b}\|_\infty \int_0^T |\nabla(y_\lambda' - y^{*\prime})||\nabla\varphi|dt \\
& + K_1\|\overline{c}\|_\infty \int_0^T |F(y_\lambda) - F(y^*)||\varphi|dt \\
\leq & \; C\Big(\int_0^T (|y_\lambda - y^*|^2 + |\nabla(y_\lambda - y^*)|^2 + |\nabla(y_\lambda' - y^{*\prime})|^2)dt\Big)^{1/2}\Big(\int_0^T |\nabla\varphi|^2 dt\Big)^{1/2} \\
\leq & \; C(|\lambda|\|\overline{q}\|_{\mathcal{P}})^{1/2}\|\varphi\|_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}$$

This shows (3.45). Consequently from (3.44), (3.45) and (3.37), we see that z is a unique weak solution of (3.26). Hence, by the uniqueness of solutions of (3.26), $z_\lambda \to z$ weakly in $W(0,T)$. This proves that the map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is weakly Gâteaux differentiable at $q = q^*$ and the Gâteaux derivative of $y(q)$ at $q = q^*$ in the direction $\overline{q} = (\overline{a}, \overline{b}, \overline{c}) \in \mathcal{P}$ is given by the unique solution z of (3.26). Further, we can prove the strong convergence $z_\lambda \to z$ in $W(0,T)$. For this we set $\varphi_\lambda = z_\lambda - z$. Then φ_λ satisfies the following problem

$$\text{(3.46)} \quad \begin{cases}
\dfrac{\partial^2\varphi_\lambda}{\partial t^2} - \nabla\cdot\big((a^{*2}(x) + a_0)\nabla\varphi_\lambda + (b^{*2}(x) + b_0)\nabla\dfrac{\partial\varphi_\lambda}{\partial t}\big) \\
\qquad = c^*(x)\partial_y F(y^*)\varphi_\lambda + c^*(x)\left(\displaystyle\int_0^1 \partial_y F(\theta y_\lambda + (1-\theta)y^*)\, d\theta - \partial_y F(y^*)\right) z_\lambda \\
\qquad\quad + \mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q}) \quad \text{ in } Q, \\
\varphi_\lambda = 0 \quad \text{ on } \Sigma, \\
\varphi_\lambda(0,x) = 0, \quad \dfrac{\partial\varphi_\lambda}{\partial t}(0,x) = 0 \quad \text{ in } \Omega
\end{cases}$$

in the weak sense. We multilpy φ_λ' to the weak form of (3.46). For simplicity of calculations we set

$$\text{(3.47)} \qquad \delta_\lambda = \left(\int_0^1 \partial_y F(\theta y_\lambda + (1-\theta)y^*)\, d\theta - \partial_y F(y^*)\right),$$

$$\text{(3.48)} \qquad \Theta_\lambda(t) = \int_0^t (c^*\delta_\lambda z_\lambda, \varphi_\lambda')ds + \int_0^t \langle \mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q}), \varphi_\lambda'\rangle ds.$$

By Schwarz inequality we have that

$$\begin{aligned}
\text{(3.49)} \qquad |\Theta_\lambda(t)| \leq & \int_0^t \|\mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q})\|_{H^{-1}(\Omega)}\|\varphi_\lambda'\|ds + \|c^*\|_\infty \int_0^t |\delta_\lambda z_\lambda||\varphi_\lambda'|ds \\
\leq & \; \epsilon \int_0^t \|\varphi_\lambda'\|^2 ds + \frac{1}{4\epsilon}\int_0^t \|\mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q})\|_{H^{-1}(\Omega)}^2 ds \\
& + \frac{1}{2}\|c^*\|_\infty\Big(\int_0^t |\varphi_\lambda'|^2 ds + \|z_\lambda\|_{L^\infty(0,T;H_0^1(\Omega))}\int_0^t \|\delta_\lambda\|_{\mathcal{L}}^2 ds\Big) \\
\leq & \; \epsilon \int_0^t \|\varphi_\lambda'\| ds + C\int_0^t |\varphi_\lambda'|^2 ds \\
& + (4\epsilon)^{-1}\|\mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q})\|_{L^2(0,T;H^{-1}(\Omega))} \; + \; C\|\delta_\lambda\|_{L^2(0,T;\mathcal{L})},
\end{aligned}$$

where C is a positive constant independent of λ. Further we can verify by (3.20) that

$$\begin{aligned}(3.50)\qquad &\Big|\int_0^t (c^*\partial_y F(y^*)\varphi_\lambda, \varphi'_\lambda)ds\Big| \\ &\le \|c^*\|_\infty \int_0^t \|\partial_y F(y^*)\|_{\mathcal{L}}\|\varphi_\lambda\||\varphi'_\lambda|ds \;\le\; C\int_0^t \|\varphi_\lambda\||\varphi'_\lambda|ds \\ &\le C\Big(\int_0^t |\varphi'_\lambda|^2 ds + \int_0^t \|\varphi_\lambda\|^2 ds\Big).\end{aligned}$$

If we put $\epsilon = b_0$ in the multiplied equation by φ'_λ, then by (3.49), (3.50) and by repeating similar calculations as in Theorem 3.1, we obtain that

$$\begin{aligned}(3.51)\quad &|\varphi'_\lambda(t)|^2 + |\nabla\varphi_\lambda(t)|^2 + \int_0^t |\nabla\varphi'_\lambda|^2 ds \\ &\le C\big(\|\mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q})\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\delta_\lambda\|^2_{L^2(0,T;\mathcal{L})}\big) \;+\; C\int_0^t (|\nabla\varphi_\lambda|^2 + |\varphi'_\lambda|^2)ds.\end{aligned}$$

Applying Gronwall inequality to (3.51), we deduce by (3.39) and (3.45) that

(3.52)

$$|\varphi'_\lambda(t)|^2 + |\nabla\varphi_\lambda(t)|^2 + \int_0^t |\nabla\varphi'_\lambda|^2 ds \;\le\; C\big(\|\mathcal{F}_\lambda(\overline{q}) - \mathcal{F}(\overline{q})\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\delta_\lambda\|^2_{L^2(0,T;\mathcal{L})}\big) \to 0$$

as $\lambda \to 0$ for all $t \in [0, T]$. As in the proof of Theorem 3.1, by using (3.38) and (3.39), we can verify that

$$c^*\frac{F(y_\lambda) - F(y^*)}{\lambda} \to c^*\partial_y F(y^*)z \quad \text{strongly in } \; L^2(0,T;L^2(\Omega)).$$

This and (3.52) imply that $\varphi_\lambda \to 0$ in $W(0,T)$, that is

$$(3.53)\qquad z_\lambda \;\to\; z \;\text{ strongly in }\; W(0,T).$$

This completes the proof.

Now we can characterize the necessary optimality condition for the given cost functional J in (1.2) in terms of the propoer adjoint system by using Theorem 3.1. Cost J is represented by

$$(3.54)\qquad J(q) = \int_0^T |y(q;t) - z_d(t)|^2 dt + |y(q;T) - z_d^T|^2, \quad \forall q \in \mathcal{P},$$

where $z_d \in L^2(0,T;L^2(\Omega))$ and $z_d^T \in L^2(\Omega)$. Then it is easily verified that the optimality condition (3.15) is rewritten as

$$(3.55)\qquad \int_0^T (y(q^*;t) - z_d(t), z(t))dt + (y(q^*;T) - z_d^T, z(T)) \ge 0, \quad \forall q \in \mathcal{P}_{ad},$$

where $z = D_q y(q^*)(q - q^*)$ is the weak solution of (3.26) with $\overline{q} = q - q^*$.

Theorem 3.3. *The optimal parameter $q^* = (a^*, b^*, c^*)$ for (3.54) is characterized by the following system of equations and inequality.*

$$(3.56) \quad \begin{cases} \dfrac{\partial^2 y^*}{\partial t^2} - \nabla \cdot \big((a^{*2}(x) + a_0)\nabla y^* + (b^{*2}(x) + b_0)\dfrac{\partial y^*}{\partial t}\big) = c^*(x)F(y^*) + f \quad \text{in } Q, \\ y^* = 0 \quad \text{on } \Sigma, \\ y^*(0,x) = y_0(x), \quad \dfrac{\partial y^*}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega. \end{cases}$$

$$(3.57) \quad \begin{cases} \dfrac{\partial^2 p}{\partial t^2} - \nabla \cdot \big((a^{*2}(x) + a_0)\nabla p + (b^{*2}(x) + b_0)\dfrac{\partial p}{\partial t}\big) \\ \qquad\qquad = \partial_y F(y^*)'(c^*(x)p) + y^* - z_d \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(T,x) = 0, \quad \dfrac{\partial p}{\partial t}(T,x) = -(y^*(T,x) - z_d^T(x)) \quad \text{in } \Omega. \end{cases}$$

$$(3.58) \quad 2\int_Q \nabla p \cdot \big(a^*(a - a^*)\nabla y^* + b^*(b - b^*)\nabla \frac{\partial y^*}{\partial t}\big)dxdt - \int_Q p(c - c^*)F(y^*)dxdt \le 0, \quad \forall q = (a,b,c) \in \mathcal{P}_{ad}.$$

Proof. First we need to explain about the term $\partial_y F(y^*)'(c^*p)$ in the adjoint system (3.57). Since $F'(s)$ satisfies $(H2)$, we have $\partial_y F(y^*)' \in L^\infty(0,T; \mathcal{L}(H^{-1}(\Omega), L^2(\Omega)))$. Therefore, we can easily verify that the adjoint system (3.57) has a unique weak solution p if we consider the reversed flow in time. Then multiplying both sides of the weak form of the equation in (3.57) by $z(t)$ and integrating over $[0,T]$, we have

$$\int_0^T \langle p'', z\rangle dt - \int_0^T ((a^{*2} + a_0)\nabla p + (b^{*2} + b_0)\nabla p', \nabla z)dt$$
$$= \int_0^T \langle \partial_y F(y^*)'(c^*p), z\rangle dt + \int_0^T (y^* - z_d, z)dt.$$

Hence, by using integration by parts twice and noting that $p(T) = 0$ and $z(0) = 0,\ z'(0) = 0$, we obtain

$$(3.59) \quad (p'(T), z(T)) + \int_0^T \langle p, z''\rangle dt + \int_0^T (\nabla p, (a^{*2} + a_0)\nabla z + (b^{*2} + b_0)\nabla z')dt$$
$$= \int_0^T (p, c^*\partial_y F(y^*)z)dt + \int_0^T (y^* - z_d, z)dt.$$

By the weak form of equation (3.26) for z with $\overline{q} = q - q^*$, we have

$$(3.60) \quad \int_0^T \langle z'', p\rangle dt + \int_0^T ((a^{*2} + a_0)\nabla z + (b^{*2} + b_0)\nabla z', \nabla p)dt$$
$$= \int_0^T (c^*\partial_y F(y^*)z, p)dt - \int_0^T \langle \nabla p, 2a^*(a - a^*)\nabla y^* + 2b^*(b - b^*)\nabla y^{*\prime}\rangle dt$$
$$+ \int_0^T (p, (c - c^*)F(y^*))dt.$$

If we use the condition $p'(T) = -(y^*(T) - z_d^T)$ in (3.57), then by (3.59) and (3.60) we obtain

$$(3.61) \quad \int_0^T (y^* - z_d, z)dt + (y^*(T) - z_d^T, z(T))$$
$$= -2\int_0^T (\nabla p, a^*(a - a^*)\nabla y^* + b^*(b - b^*)\nabla y^{*\prime})dt + \int_0^T (p, (c - c^*)F(y^*))dt.$$

Therefore, by (3.61) the necessary optimality condition (3.55) is represented by (3.58). This completes the proof of Theorem 3.3.

3.3 Bang-bang property Let us deduce the bang-bang principle from Theorem 3.3. Suppose that

$$\mathcal{P}_{ad} = \mathcal{P}^a_{ad} \times \mathcal{P}^b_{ad} \times \mathcal{P}^c_{ad}$$

and $\mathcal{P}^a_{ad}$, $\mathcal{P}^b_{ad}$, $\mathcal{P}^c_{ad} \subset L^\infty(\Omega)$. In this case the necessary condition in Theorem 3.3 is equivalent to

$$\int_Q \nabla p \cdot \big(a^*(a^* - a)\nabla y^*\big)dxdt \geq 0, \quad \forall a \in \mathcal{P}^a_{ad}. \tag{3.62}$$

$$\int_Q \nabla p \cdot \big(b^*(b^* - b)\nabla y^{*\prime}\big)dxdt \geq 0, \quad \forall b \in \mathcal{P}^b_{ad}, \tag{3.63}$$

$$\int_Q (c - c^*)pF(y^*)dxdt \geq 0, \quad \forall c \in \mathcal{P}^c_{ad}. \tag{3.64}$$

For example we shall characterize the condition (3.64). For this sake, we set

$$\mathcal{P}^c_{ad} = \{c(x) : \gamma_0(x) \leq \ c(x) \leq \gamma_1(x) \ \ a.e.\ x \in \Omega\},$$

where $\gamma_0, \gamma_1 \in L^\infty(\Omega)$. By the Lebesgue convergence theorem, we have from (3.64)

$$\int_\Omega (c(x) - c^*(x))p(t,x)F(y^*(t,x))dx \geq 0, \quad a.e.\ t \in [0,T], \quad \forall c \in \mathcal{P}^c_{ad}. \tag{3.65}$$

Then we can deduce the following property of c^* from (3.65). We fix t such that (3.65) holds. If $p(t,x)F(y^*(t,x)) > 0$, then $c^*(x) = \gamma_0(x)$ and if $p(t,x)F(y^*(t,x)) < 0$, then $c^*(x) = \gamma_1(x)$ holds $a.e.$ $x \in \Omega$. This is the bang-bang property of optimal parameter c^* in this case (cf. Lions [8]). The similar bang-bang property of optimal parameters a^*, b^* can be derived from (3.62) and (3.63).

4 Conclusions In this paper, we have investigated the parameter identification problem of spatially varying coefficients in a class of strongly damped nonlinear wave equations. We have formulated the problem by a minimization problem of quadratic cost functionals by means of distributive and terminal values measurements. By showing the continuity and Gâteaux differentiability of solutions on parameters, we have established the existence of optimal parameters and necessary optimality conditions for the functionals. The main Theorem 3.3 as well as the bang-bang property can be used to find the optimal parameters. Further extension of results for the quasilinear visco-elastic wave equation is a subject of future study.

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Department of Applied Mathematics, Faculty of Engineering, Kobe University, Japan

School of Liberal Arts, Korea University of Technology and Education, Korea

Department of Mathematics and Computing Science, University of the South Pacific, Fiji