ON DOUBLING ALGEBRAS

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Abstract. If $(X;*)$ and $(X;\circ)$ are binary systems then $(X;*) \Longrightarrow (X;\circ)$ if $(x*y)\circ z = (x*z)*(y*z)$ where $(X;\circ)$ is the doubling algebra of the source algebra $(X;*)$. Obviously there are many mutual influences on the types of $(X;*)$ and $(X;\circ)$. In this paper we investigate several of these mutual influences, including when $(X;*)$ is a group, B-algebra, a cancellative semigroup with identity.

Given a set X, let $V(X)$ denote the collection of all binary algebras (or equivalently, groupoids) on X, i.e., $V(X) = \{(X;*) \mid * : \text{binary operation on} X\}$. An algebra $(X;*)$ is said to be a *source algebra* of an algebra $(X;\circ)$ if $(x*y)\circ z = (x*z)*(y*z)$, for any $x,y,z \in X$, and denoted by $(X;*) \Longrightarrow (X;\circ)$. In this case, we say $(X;\circ)$ the *doubling algebra* of $(X;*)$.

Example 1. Every quandle $(X;\triangleright,\triangleright^{-1})$ (see [2]) is a doubling algebra as well as a source algebra, i.e., $(x\triangleright y)\triangleright z = (x\triangleright z)\triangleright(y\triangleright z)$, for any $x,y,z \in X$.

Example 2. Every positive implicative BCK-algebra is a doubling algebra of itself.

Example 3. Every right distributive semigroup (see [1]) is a doubling algebra of itself.

We illustrate a construction of many doubling algebras from any abelian group.

Theorem 4. Let $(X;+)$ be an abelian group and let $\{a_n\}$ be a sequence defined by $a_1 = 1, a_{n+1} = a_n(a_n+1)$, $n \in \mathbf{N}$. Define binary operation "$*_n$" on X by $x *_n y := x + a_n y$, for any $x,y \in X$, $n \in \mathbf{N}$. Then the algebra $(X;*_{n+1})$ is a doubling algebra of the algebra $(X;*_n)$.

Proof. Straightforward. □

Let $I(X;*)$ be the set of all idempotent elements of an algebra $(X;*)$.

Proposition 5. *If* $(X;*) \Longrightarrow (X;\circ)$, *then* $I(X;*) \subseteq I(X;\circ)$.

Proof. Straightforward. □

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An algebra $(X;*)$ is said to be *of type* (a,b) if $a*x=b$ for any $x\in X$.

Example 6. Let $X:=\{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	0
3	0	0	2	3

Then the semigroup $(X;*)$ is both of type $(0,0)$ and of type $(2,0)$.

Proposition 7. *If $(X;*)\Longrightarrow(X;\circ)$ and $(X;*)$ is of type (a,b), then $b\circ x=b*b$ for all $x\in X$.*

Proof. We have $b\circ x=(a*y)\circ x=(a*x)*(y*x)=b*(y*x)$, and $y=a$ yields $b\circ x=b*(a*x)=b*b$ for all $x\in X$. □

Corollary 8. *If $(X;*)\Longrightarrow(X;\circ)$ and $(X;*)$ is of type (a,b), then $(X;\circ)$ is of type (b,c) where $c=b*b$.*

An algebra $(X;*)$ is said to be *of type* $[a,b]$ if $x*a=b$ for any $x\in X$. In Example 6, $(X;*)$ is both of type $[0,0]$ and of type $[1,0]$.

Proposition 9. *Let $(X;*)\Longrightarrow(X;\circ)$ and $X*X=X$. If $(X;*)$ is of type $[a,b]$ then $x\circ a=b*b$ for all $x\in X$.*

Proof. Let $x=u*v$. Then $x\circ a=(u*v)\circ a=(u*a)*(v*a)=b*b$ for all $x\in X$. □

Corollary 10. *Let $(X;*)\Longrightarrow(X;\circ)$ and $X*X=X$. If $(X;*)$ is of type $[a,b]$ then $(X;\circ)$ is of type $[a,c]$ where $c=b*b$.*

Proposition 11. *If $(X;*)\Longrightarrow(X;\circ)$ and $x*x=x$ for all $x\in X$, then $x\circ y=x*y$ for all $x,y\in X$, i.e., $(X;*)$ and $(X;\circ)$ are identical.*

Proof. Since $(X;*)\Longrightarrow(X;\circ)$, we have $x\circ y=(x*x)\circ y=(x*y)*(x*y)=x*y$ as well. □

Theorem 12. *If a cancellative semigroup $(X;*)$ with identity e is a source algebra of any algebra $(X;\circ)$, then it is commutative.*

Proof. Since $(X;*)\Longrightarrow(X;\circ)$, we have

$$(x*y)\circ z=(x*z)*(y*z) \tag{1}$$

for any $x,y,z\in X$. If we put $x:=e$ and $y:=e$ in (1) respectively, then

$$y\circ z=z*(y*z), \tag{2}$$

and

$$x\circ z=(x*z)*z. \tag{3}$$

Hence, by applying (3) and (2),

$$\begin{aligned}(x * z) * z &= x \circ z && \text{[by (3)]}\\ &= z * (x * z) && \text{[by (2)]}\\ &= (z * x) * z. && \text{[associativity]}\end{aligned}$$

Since $(X; *)$ is cancellative, we obtain $x * z = z * x$, for any $x, z \in X$, proving the theorem. □

Remark. If $(X; \circ)$ is a group in Theorem 12, then it is a trivial group, i.e., $X = \{e\}$.

J. Neggers and H. S. Kim ([4]) defined the notion of B-algebra. An algebra $(X; *, 0)$ is said to be a *B-algebra* if (I) $x * x = 0$; (II) $x * 0 = x$; (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$. (see [3, 4, 5, 6, 7] for details)

Proposition 13. *The doubling algebra $(X; \circ)$ of a B-algebra $(X; *, 0)$ is a left zero semigroup.*

Proof. If $(X; *) \Longrightarrow (X; \circ)$, then

$$(x * y) \circ z = (x * z) * (y * z) \tag{4}$$

for any $x, y, z \in X$. If we let $y := 0$ in (4), then

$$\begin{aligned}x \circ z &= (x * 0) \circ z\\ &= (x * z) * (0 * z)\\ &= x * ((0 * z) * (0 * z))\\ &= x * 0\\ &= x,\end{aligned}$$

for any $x, z \in X$. This means that $(X; \circ)$ is a left zero semigroup. □

Proposition 14. *If a group $(X; *)$ is a source algebra of an algebra $(X; \circ)$, then*

(i) *$(X; *)$ is abelian;*

(ii) *the doubling algebra $(X; \circ)$ should be defined by $x \circ y := x * y^2$, for any $x, y \in X$.*

Proof. Since $(X; *) \Longrightarrow (X; \circ)$, we have

$$(x * y) \circ z = (x * z) * (y * z) \tag{5}$$

for any $x, y, z \in X$. If we let $y := e$ in (5), where e is the identity of the group $(X; *)$, then we have

$$\begin{aligned}x \circ z &= (x * e) \circ z\\ &= (x * z) * (e * z)\\ &= (x * z) * z\\ &= x * z^2 && \cdots\cdots\cdots \quad (6)\end{aligned}$$

Moreover, if $x := e$ in (5), then

$$\begin{aligned} y \circ z &= (e * y) \circ z \\ &= (e * z) * (y * z) \\ &= z * (y * z) \end{aligned} \qquad \cdots\cdots\cdots \qquad (7)$$

If we let $y := x$ in (7), then

$$x \circ z = z * (x * z) \tag{8}$$

Combining (6) with (8) we obtain $(x * z) * z = z * (x * z)$. Since $(X; *)$ is a group, we conclude that $x * z = z * x$, for any $x, z \in X$. Also, $x \circ y = y * (x * y) = x * y^2$ in that case. □

In view of Proposition 14 the operation $x \circ y = x * y^2$ defines the doubled operation. Given this situation, if we write additively $x * y = x + y$, then $x \circ y = x + 2y = x + a_2 y$, $a_2 = 2$, whence "redoubling" provides for $(x*y) \circ z = (x*z)*(y*z) = (x+a_2 z)+a_2(y+a_2 z) = (x+a_2 y)+a_3 z$, where $a_3 = a_2(a_2+1)$, whence $y = 0$ implies $x \circ z = x + a_3 z$. Accordingly, we may redouble to obtain a "factorial-like-sequence", $a_1 = 1, a_2 = 2$, and $a_{n+1} = a_n(a_n+1)$. For example, in $Z/(23)$, the successive doublings yield $x + y \Longrightarrow x + 2y \Longrightarrow x + 6y \Longrightarrow x + 42y = x + 19y \Longrightarrow x + 380y = x + 12y \Longrightarrow x + 18y \Longrightarrow x + 20y \Longrightarrow x + 6y \Longrightarrow \cdots$. Thus, $x + 6y$ is of period 5 with respect to redoubling in this setting, while $x + y$ cannot be returned to by redoubling.

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