HEAVISIDE CABLE, THOMSON CABLE AND THE BEST CONSTANT OF A SOBOLEV-TYPE INEQUALITY

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Abstract. A simple boundary value problem for a n-th order linear ordinary differential equation which appears typically in the theory of Heaviside cable and Thomson cable is treated. Output-input voltage relation is investigated. We found the best constant of Sobolev-type inequality, which estimates the square of supremum of absolute value of output voltage from above by the power of input voltage. The best constant is a rational function of the characteristic roots and also a rational function of the characteristic coefficients. The second formula for the best constant is very important because even for small number of n it is difficult to obtain the exact value of characteristic roots but in some cases it is easy to know the coefficients of characteristic polynomial. Giambelli's formula which appears in the theory of representation of finite groups plays an important role.

1 Introduction

We consider the following boundary value problem for a n-th order linear ordinary differential operator $P(d/dt)$.

BVP

$$\begin{cases} P(d/dt)\, u \,=\, f(t) \qquad (-\infty < t < \infty) & (1.1) \\ u^{(i)}(t) \,\in\, L^2(-\infty,\infty) \qquad (0 \le i \le n) & (1.2) \end{cases}$$

The characteristic polynomial with real coefficients

$$P(z) \,=\, \prod_{j=0}^{n-1} (z + a_j) \,=\, \sum_{j=0}^{n} p_j\, z^{n-j} \qquad (p_0 = 1) \tag{1.3}$$

is assumed to be a Hurwitz polynomial [1] with distinct characteristic roots $a_0, \cdots, a_{n-1}$. That is to say, we impose one of the following three equivalent assumptions.

Assumption 1.1

$$a_i \,\neq\, a_j \quad (0 \le i < j \le n-1), \qquad \mathrm{Re}\, a_j \,>\, 0 \quad (0 \le j \le n-1)$$

Assumption 1.2 *Suppose that* $l, m = 0, 1, 2, \cdots,\ n = l + 2m \ge 1$

$$\begin{aligned} &a_i \,\neq\, a_j \quad (0 \le i < j \le n-1), \qquad a_j \,>\, 0 \quad (0 \le j \le l-1) \\ &a_{l+m+j} \,=\, \overline{a}_{l+j}, \quad \mathrm{Re}\, a_{l+j} \,>\, 0, \quad \mathrm{Im}\, a_{l+j} \,>\, 0 \qquad (0 \le j \le m-1) \end{aligned}$$

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Assumption 1.3

$$\text{G.C.D.}(P(z), P'(z)) = 1, \qquad D_k = \left| \; p_{-i+2j+1} \; \right|_{0 \le i,j \le k-1} > 0 \qquad (k = 1, 2, \cdots, n)$$

where $p_k = 0$ ($k < 0$ *or* $k > n$).

Through the Fourier transform

$$f(t) \quad \xrightarrow{\widehat{\ }} \quad \widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\,\omega t} f(t)\, dt \tag{1.4}$$

BVP is converted to the following problem.

BVP$\widehat{\ }$

$$\begin{cases} P(z)\, \widehat{u}(\omega) = \widehat{f}(\omega) \qquad (-\infty < \omega < \infty) & (1.5) \\ (\, 1 + |\omega|\,)^n\, \widehat{u}(\omega) \in L^2(-\infty, \infty) & (1.6) \end{cases}$$

Hereafter we use the following abbreviation.

$$z = \sqrt{-1}\, \omega \tag{1.7}$$

The solution $\widehat{u}(\omega)$ to BVP$\widehat{\ }$ is given as follows.

$$\widehat{u}(\omega) = \widehat{G}(\omega)\, \widehat{f}(\omega), \quad \widehat{G}(\omega) = 1 \,/\, P(z) \qquad (-\infty < \omega < \infty) \tag{1.8}$$

The one and only one solution of BVP is given by

$$u(t) = \int_{-\infty}^{\infty} G(t, s)\, f(s)\, ds = \int_{-\infty}^{\infty} G(t - s)\, f(s)\, ds \qquad (-\infty < t < \infty) \tag{1.9}$$

where $G(t, s) = G(t - s)$ is a Green function. $G(t)$ is defined by the inverse Fourier transform of $\widehat{G}(\omega)$.

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sqrt{-1}\, t\omega}\, \widehat{G}(\omega)\, d\omega \tag{1.10}$$

Since we have

$$| \, P(z) \, |^2 =$$
$$\prod_{j=0}^{l-1} (\omega^2 + a_j^2) \prod_{j=0}^{m-1} \Big((\omega + \mathrm{Im}\, a_{l+j})^2 + (\mathrm{Re}\, a_{l+j})^2 \Big) \Big((\omega - \mathrm{Im}\, a_{l+j})^2 + (\mathrm{Re}\, a_{l+j})^2 \Big)$$

there exist positive constants δ_1 and δ_2 such that the following inequalities hold.

$$\delta_1 \, (\, 1 + |\omega|\,)^n \le | \, P\,(z) \, | \le \delta_2 \, (\, 1 + |\omega|\,)^n$$

We define Green operator $\mathcal{G}$ by the following formula.

$$(\mathcal{G} f)(t) = \int_{-\infty}^{\infty} G(t - s)\, f(s)\, ds \qquad (-\infty < t < \infty) \tag{1.11}$$

$\mathcal{G}$ is a bounded and linear operator from a Hilbert space $L^2(-\infty, \infty)$ to a Hilbert space $W^{2,n}(-\infty, \infty)$. $\mathcal{G}$ is also bounded and linear from $L^2(-\infty, \infty)$ to a Banach space $L^\infty(-\infty, \infty)$. The conclusion of this paper is as follows.

Theorem 1.1 *The operator norm of Green operator*

$$\mathcal{G} \;:\; L^2(-\infty,\infty) \qquad \longrightarrow \qquad L^\infty(-\infty,\infty) \tag{1.12}$$

is given as follows.

$$\|\,\mathcal{G}\,\| \;=\; \|\,G\,\| \;=\; \sqrt{\int_{-\infty}^{\infty} |\,G(t)\,|^2 dt} \tag{1.13}$$

From the viewpoint of the Sobolev inequality [4] [5] [7], the above theorem is equivalently rewritten as follows.

Theorem 1.2 *For any function $u(t)$ which satisfies $u^{(i)}(t) \in L^2(-\infty,\infty)$ $(0 \le i \le n)$, there exists a positive constant C which is independent of $u(t)$ such that the following Sobolev-type inequality holds.*

$$\left(\sup_{-\infty<s<\infty} |\,u(s)\,|\right)^2 \;\le\; C\int_{-\infty}^{\infty} |\,P(d/dt)u(t)\,|^2\; dt \tag{1.14}$$

Among such C the best constant $C(n)$ is given as follows.

$$C(n) \;=\; \|\,G\,\|^2 \tag{1.15}$$

Let us choose a special solution $u(t) = U(t)$ of BVP *for a special function $f(t) = G(-t)$ $(-\infty < t < \infty)$. If we replace C by $C(n)$ in* (1.14)*, the equality holds for*

$$u(t) \;=\; c\,U(t-t_0) \qquad (-\infty < t < \infty) \tag{1.16}$$

where t_0 is an arbitrary real number and c is an arbitrary complex number.

The best constant $C(n)$ can be expressed as a rational function of the characteristic roots $a_0,\cdots,a_{n-1}$.

Theorem 1.3

$$(1)\quad C(n) \;=\; (-1)^{n+1}\,\frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{a_j\displaystyle\prod_{k=0,\ k\neq j}^{n-1}\left(a_j^2-a_k^2\right)} \tag{1.17}$$

$$(2)\quad C(n) \;=\; \frac{(-1)^{n+1}}{2a_0\cdots a_{n-1}}\left|\begin{array}{c} a_j^{2i+1} \\ \hline \cdots \quad 1 \quad \cdots \end{array}\right| \Big/ \left|\begin{array}{c} a_j^{2i} \end{array}\right| \tag{1.18}$$

In the numerator we have $0 \le i \le n-2,\ 0 \le j \le n-1$ and in the denominator $0 \le i,j \le n-1$.

$C(n)$ is a symmetric rational function of $a_0,\cdots,a_{n-1}$. It can also be expressed as a rational function of the characteristic coefficients p_j $(0 \le j \le n)$.

Theorem 1.4

$$C(n) \;=\; \frac{1}{2p_n}\left|\; p_{n-2-2i+j} \;\right|_{0\le i,j\le n-3} \Big/ \left|\; p_{n-1-2i+j} \;\right|_{0\le i,j\le n-2} \tag{1.19}$$

Here we assume that $p_j \;=\; 0 \quad (j<0 \text{ or } n<j)$.

For small number of n, we list concrete forms of $C(n)$.

$$C(1) = \frac{1}{2a_0} = \frac{1}{2p_1}$$

$$C(2) = -\frac{1}{2a_0a_1}\begin{vmatrix} a_0 & a_1 \\ 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 \\ a_0^2 & a_1^2 \end{vmatrix} = \frac{1}{2a_0a_1(a_0+a_1)} = \frac{1}{2p_1p_2}$$

$$C(3) = \frac{1}{2a_0a_1a_2}\begin{vmatrix} a_0 & a_1 & a_2 \\ a_0^3 & a_1^3 & a_2^3 \\ 1 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 \\ a_0^4 & a_1^4 & a_2^4 \end{vmatrix} =$$

$$\frac{a_0+a_1+a_2}{2a_0a_1a_2(a_0+a_1)(a_0+a_2)(a_1+a_2)} = \frac{1}{2p_3}p_1 \Big/ \begin{vmatrix} p_2 & p_3 \\ p_0 & p_1 \end{vmatrix} = \frac{p_1}{2p_3(p_1p_2-p_3)}$$

$$C(4) = -\frac{1}{2a_0a_1a_2a_3}\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^5 & a_1^5 & a_2^5 & a_3^5 \\ 1 & 1 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 \end{vmatrix} =$$

$$\frac{1}{2p_4}\begin{vmatrix} p_2 & p_3 \\ p_0 & p_1 \end{vmatrix} \Big/ \begin{vmatrix} p_3 & p_4 & 0 \\ p_1 & p_2 & p_3 \\ 0 & p_0 & p_1 \end{vmatrix} = \frac{p_1p_2-p_3}{2p_4(p_1p_2p_3-p_3^2-p_1^2p_4)}$$

$$C(5) =$$

$$\frac{1}{2a_0a_1a_2a_3a_4}\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 \\ a_0^7 & a_1^7 & a_2^7 & a_3^7 & a_4^7 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 & a_4^6 \\ a_0^8 & a_1^8 & a_2^8 & a_3^8 & a_4^8 \end{vmatrix} =$$

$$\frac{1}{2p_5}\begin{vmatrix} p_3 & p_4 & p_5 \\ p_1 & p_2 & p_3 \\ 0 & p_0 & p_1 \end{vmatrix} \Big/ \begin{vmatrix} p_4 & p_5 & 0 & 0 \\ p_2 & p_3 & p_4 & p_5 \\ p_0 & p_1 & p_2 & p_3 \\ 0 & 0 & p_0 & p_1 \end{vmatrix} =$$

$$\frac{p_1p_2p_3-p_3^2-p_1^2p_4+p_1p_5}{2p_5(p_1p_2p_3p_4-p_3^2p_4-p_1^2p_4^2-p_1p_2^2p_5+p_2p_3p_5+2p_1p_4p_5-p_5^2)}$$

This paper consists mainly of two parts. Section 2,3,4,5 is devoted to the proof of the above main theorems. In particular, Giambelli's formula plays an important role in the expression of the best constant. Section 6,7,8 presents one interesting application of the obtained results to the theory of electric circuit. We also calculate the best constants in some special cases.

2 Green function

In this section, we first obtain the concrete expression of Green function $G(t)$. In the second place, we calculate its L^2 norm $\| G \|$.

We first introduce Heaviside step function.

$$Y(t) = \begin{cases} 1 & (0 \le t < \infty) \\ 0 & (-\infty < t < 0) \end{cases} \tag{2.1}$$

For any complex number a with $\mathrm{Re}\, a > 0$, we have the following relation.

$$Y(t)\, e^{-at} \qquad \stackrel{\frown}{\longrightarrow} \qquad (z+a)^{-1} \tag{2.2}$$

We also use the abbreviation $z = \sqrt{-1}\,\omega$. From the expansion of $1/P(z)$ into partial fractions

$$\frac{1}{P(z)} = \sum_{j=0}^{n-1} \frac{1}{P'(-a_j)}\, \frac{1}{z+a_j} = \left| \begin{array}{c} (-a_j)^i \\ \hline \cdots \quad (z+a_j)^{-1} \quad \cdots \end{array} \right| \Big/ \left| \quad (-a_j)^i \quad \right| \tag{2.3}$$

we can easily conclude the following theorem.

Theorem 2.1 *By using functions*

$$G_j(t) = Y(t)\, e^{-a_j t} \qquad (-\infty < t < \infty, \quad 0 \le j \le n-1) \tag{2.4}$$

Green function $G(t)$ can be expressed as

$$(1) \quad G(t) = \sum_{j=0}^{n-1} \frac{1}{P'(-a_j)}\, G_j(t) \tag{2.5}$$

$$(2) \quad G(t) = (-1)^{n+1} \left| \begin{array}{c} a_j^i \\ \hline \cdots \quad G_j(t) \quad \cdots \end{array} \right| \Big/ \left| \quad a_j^i \quad \right| \tag{2.6}$$

$$(3) \quad G(t) = (\, G_0 * \cdots * G_{n-1}\,)(t) \tag{2.7}$$

where

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-s)\, g(s)\, ds \qquad (-\infty < t < \infty) \tag{2.8}$$

is a convolution of $f(t)$ and $g(t)$.

From (3), in the case of $a_j > 0$ $(0 \le j \le n-1)$, we can conclude that the inequality

$$G(t) \begin{cases} > 0 & (0 \le t < \infty) \\ = 0 & (-\infty < t < 0) \end{cases} \tag{2.9}$$

holds.

3 Sobolev-type inequality

In this section, we give a proof of Theorem 1.2, from which Theorem 1.1 is derived simultaneously.

Proof of Theorem 1.2 For any function $u(t)$ satisfying $u^{(i)}(t) \in L^2(-\infty,\infty)$ $(0 \le i \le n)$, we define $f(t) \in L^2(-\infty,\infty)$ by the following relation.

$$f(t) = P(d/dt)\, u(t) \qquad (-\infty < t < \infty) \tag{3.1}$$

The inequality

$$|\, u(s)\,|^2 \le \int_{-\infty}^{\infty} |\, G(s-t)\,|^2\, dt \int_{-\infty}^{\infty} |\, f(t)\,|^2\, dt \tag{3.2}$$

is obtained by applying Schwarz inequality to (1.9). Considering that the relation

$$\int_{-\infty}^{\infty} |\, G(s-t)\,|^2 \, dt \;=\; \int_{-\infty}^{\infty} |\, G(t)\,|^2 \, dt \;=\; \|\, G\,\|^2 \tag{3.3}$$

holds and that the right hand side of (3.2) does not depend on s, we have the following Sobolev-type inequality.

$$\Big(\sup_{-\infty<s<\infty} |\, u(s)\,| \Big)^2 \;\le\; \|\, G\,\|^2 \int_{-\infty}^{\infty} |\, f(t)\,|^2 \, dt \tag{3.4}$$

Taking a special solution $u(t) = U(t)$ of BVP with a special function $f(t) = G(-t)$ $(-\infty < t < \infty)$, we have the following relation.

$$U(s) \;=\; \int_{-\infty}^{\infty} G(s-t)\, G(-t)\, dt \qquad (-\infty < s < \infty) \tag{3.5}$$

In particular, we have

$$U(0) \;=\; \int_{-\infty}^{\infty} |\, G(-t)\,|^2 \, dt \;=\; \|\, G\,\|^2 \tag{3.6}$$

by putting $s = 0$ in (3.5). We also have

$$\begin{aligned} &\|\, G\,\|^4 \;=\; \Big(U(0) \Big)^2 \;\le\; \Big(\sup_{-\infty<s<\infty} |\, U(s)\,| \Big)^2 \;\le \\ &\|\, G\,\|^2 \int_{-\infty}^{\infty} |\, P(d/dt)\, U(t)\,|^2 \, dt \;=\; \|\, G\,\|^2 \int_{-\infty}^{\infty} |\, G(-t)\,|^2 \, dt \;=\; \|\, G\,\|^4 \end{aligned} \tag{3.7}$$

from (3.4) and (3.6). This means that

$$\Big(\sup_{-\infty<s<\infty} |\, U(s)\,| \Big)^2 \;=\; \|\, G\,\|^2 \int_{-\infty}^{\infty} |\, P(d/dt)\, U(t)\,|^2 \, dt \tag{3.8}$$

which completes the proof of Theorem 1.2. ■

The concrete form of the best function $U(t)$ is shown later in section 5.

4 The best constant

We here prove the Theorem 1.3 and 1.4 concerning the best constant $C(n) = ||G||^2$.

Proof of Theorem 1.3 We start with the following Parseval's identity.

$$\begin{aligned} &\|\, G\,\|^2 \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big|\, \widehat{G}(\omega) \Big|^2 \, d\omega \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\big|\, P\big(\sqrt{-1}\,\omega\big)\,\big|^2} \, d\omega \;= \\ &\frac{1}{2\pi\sqrt{-1}} \int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} \frac{dz}{P(z)\, P(-z)} \end{aligned} \tag{4.1}$$

Since the integrand $1 \big/ \big(\, P(z)P(-z)\,\big)$ has n simple poles $z = -a_j$ $(0 \le j \le n-1)$ in the left half plane $\mathrm{Re}\, z < 0$, we have

$$\begin{aligned} &\|\, G\,\|^2 \;=\; \sum_{j=0}^{n-1} \operatorname*{Res}_{z\,=\,a_j} \left(\frac{1}{P(z)\, P(-z)} \right) \;=\; \sum_{j=0}^{n-1} \frac{1}{P'(-a_j)\, P(a_j)} \;= \\ &(-1)^{n+1} \sum_{j=0}^{n-1} \frac{1}{2a_j \displaystyle\prod_{k=0,\ k\ne j}^{n-1} (a_j^2 - a_k^2)} \end{aligned} \tag{4.2}$$

from the residue theorem. Now we introduce a new polynomial

$$R(z) \;=\; \prod_{k=0}^{n-1} (z-\alpha_k) \tag{4.3}$$

where $\alpha_k = a_k^2$ $(0 \le k \le n-1)$. From the relation

$$R'(\alpha_j) \;=\; \prod_{k=0,\; k\neq j}^{n-1} (\alpha_j-\alpha_k) \;=\; \prod_{k=0,\; k\neq j}^{n-1} \left(a_j^2-a_k^2\right) \tag{4.4}$$

and (2.5), we have

$$\begin{aligned}
\|\, G \,\|^2 \;&=\; (-1)^{n+1}\frac{1}{2}\sum_{j=0}^{n-1} \frac{1}{R'(\alpha_j)}\,\frac{1}{a_j} \;= \\
&(-1)^{n+1}\frac{1}{2}\left|\begin{array}{c} \alpha_j^i \\ \hline \cdots \quad a_j^{-1} \quad \cdots \end{array}\right| \Bigg/ \left| \quad \alpha_j^i \quad \right| \;= \\
&(-1)^{n+1}\frac{1}{2}\left|\begin{array}{c} a_j^{2i} \\ \hline \cdots \quad a_j^{-1} \quad \cdots \end{array}\right| \Bigg/ \left| \quad a_j^{2i} \quad \right| \;= \\
&(-1)^{n+1}\frac{1}{2a_0\cdots a_{n-1}}\left|\begin{array}{c} a_j^{2i+1} \\ \hline \cdots \quad 1 \quad \cdots \end{array}\right| \Bigg/ \left| \quad a_j^{2i} \quad \right|
\end{aligned} \tag{4.5}$$

which completes the proof of Theorem 1.3. ■

Before going into the proof of Theorem 1.4, we show that the best constant is rewritten equivalently as a ratio of Schur polynomials. From Theorem 1.3 (2), it is easy to see that the relation

$$C(n) \;=\; \|\, G\,\|^2 \;=\; \frac{1}{2p_n}\left|\begin{array}{c} a_j^{2(n-1-i)-1} \\ \hline \cdots \quad 1 \quad \cdots \end{array}\right| \Bigg/ \left| \quad a_j^{2(n-1-i)} \quad \right| \tag{4.6}$$

holds. Here we introduce two partitions of natural numbers

$$\lambda \;=\; (\lambda_0, \lambda_1, \cdots, \lambda_{n-1}) \quad \text{and} \quad \mu \;=\; (\mu_0, \mu_1, \cdots, \mu_{n-1}) \tag{4.7}$$

where λ_i and μ_i are given as follows.

$$\lambda_i \;=\; n-1-i \qquad (0 \le i \le n-1) \tag{4.8}$$

$$\mu_i \;=\; \begin{cases} \lambda_i \,-\, 1 & (0 \le i \le n-2) \\ 0 & (i = n-1) \end{cases} \tag{4.9}$$

Using these notations λ and μ, $C(n)$ is rewritten as follows.

$$C(n) \;=\; \frac{1}{2p_n}\left| \; a_j^{n-1-i+\mu_i} \; \right| \Bigg/ \left| \; a_j^{n-1-i+\lambda_i} \; \right| \tag{4.10}$$

For a given partition $Y = (Y_0, Y_1, \cdots, Y_{n-1})$, $Y_0 \geq Y_1 \geq \cdots \geq Y_{n-1} \geq 0$, we define Schur polynomials by the following relation.

$$S_Y(a) = S_Y(a_0, \cdots, a_{n-1}) = \left| a_j^{n-1-i+Y_i} \right| \Big/ \left| a_j^{n-1-i} \right| \tag{4.11}$$

Then the best constant is rewritten in the Schur polynomial representation.

$$C(n) = \frac{1}{2p_n} S_\mu(a) \Big/ S_\lambda(a) \tag{4.12}$$

Proof of Theorem 1.4 We start with the following Giambelli's formula.

Lemma 4.1 (Giambelli [6]) *For a partition*

$$Y = (Y_0, Y_1, \cdots, Y_{n-1}) \qquad (Y_0 \geq Y_1 \geq \cdots \geq Y_{n-1} \geq 0) \tag{4.13}$$

of a natural number, let $\widehat{Y}$ *be a conjugate of* Y *defined by* $\widehat{Y} = (\widehat{Y}_0, \widehat{Y}_1, \cdots, \widehat{Y}_{n-1})$, $\widehat{Y}_i = \#\{\, j \mid Y_j \geq i+1 \,\}$. *Then we have*

$$S_Y(a) = \left| p_{j-i+\widehat{Y}_i} \right|_{0 \leq i,j \leq n-1} \tag{4.14}$$

where p_j $(1 \leq j \leq n)$ *is the* j*-th fundamental symmetric polynomial of* $a = (a_0, \cdots, a_{n-1})$. *We also assume that* $p_0 = 1$ *and* $p_j = 0$ *for* $j < 0$ *or* $j > n$.

Applying the Giambelli's formula to (4.12) and considering that $\widehat{\lambda}_i = \lambda_i$ and $\widehat{\mu}_i = \mu_i$ holds, we have the following equality.

$$C(n) = \frac{1}{2p_n} \left| p_{j-i+\widehat{\mu}_i} \right|_{0 \leq i,j \leq n-1} \Big/ \left| p_{j-i+\widehat{\lambda}_i} \right|_{0 \leq i,j \leq n-1} =$$

$$\frac{1}{2p_n} \begin{vmatrix} p_{n-2} & p_{n-1} & \cdots & p_{2n-4} & p_{2n-3} \\ p_{n-4} & p_{n-3} & \cdots & p_{2n-6} & p_{2n-5} \\ \vdots & \vdots & & \vdots & \vdots \\ p_{-n+2} & p_{-n+3} & \cdots & p_0 & p_1 \\ p_{-n+1} & p_{-n+2} & \cdots & p_{-1} & p_0 \end{vmatrix} \Bigg/$$

$$\begin{vmatrix} p_{n-1} & p_n & \cdots & p_{2n-3} & p_{2n-2} \\ p_{n-3} & p_{n-2} & \cdots & p_{2n-5} & p_{2n-4} \\ \vdots & \vdots & & \vdots & \vdots \\ p_{-n+3} & p_{-n+4} & \cdots & p_1 & p_2 \\ p_{-n+1} & p_{-n+2} & \cdots & p_{-1} & p_0 \end{vmatrix}$$

By using the fact $p_0 = 1$ and $p_j = 0$ $(j < 0)$, $C(n)$ is reduced to

$$C(n) = \frac{1}{2p_n} \begin{vmatrix} p_{n-2} & p_{n-1} & \cdots & p_{2n-5} \\ p_{n-4} & p_{n-3} & \cdots & p_{2n-7} \\ \vdots & \vdots & & \vdots \\ p_{-n+4} & p_{-n+5} & \cdots & p_1 \end{vmatrix} \Bigg/ \begin{vmatrix} p_{n-1} & p_n & \cdots & p_{2n-3} \\ p_{n-3} & p_{n-2} & \cdots & p_{2n-5} \\ \vdots & \vdots & & \vdots \\ p_{-n+3} & p_{-n+4} & \cdots & p_1 \end{vmatrix} \tag{4.15}$$

where the numerator and denominator are determinants of size $n-2$ and $n-1$, respectively. Thus we proved Theorem 1.4. ■

5 The best function

The concrete form of the best function $U(t)$ which appeared in Theorem 1.2, is given by the following lemma.

Lemma 5.1 *The best function*

$$U(t) \;=\; \int_{-\infty}^{\infty} G(t-s)\,G(-s)\,ds \qquad (-\infty < t < \infty) \tag{5.1}$$

is expressed in the following three ways.

$$\text{(1)}\quad U(t) \;=\; (-1)^{n+1}\sum_{j=0}^{n-1} \frac{1}{\displaystyle\prod_{k=0,\ k\neq j}^{n-1}\left(a_j^2-a_k^2\right)}\, H_j(t) \qquad (-\infty < t < \infty) \tag{5.2}$$

$$\text{(2)}\quad U(t) \;=\; (-1)^{n+1}\left|\begin{array}{c} a_j^{2i} \\ \hline \cdots \quad H_j(t) \quad \cdots \end{array}\right| \Big/ \left|\begin{array}{c} a_j^{2i} \end{array}\right| \qquad (-\infty < t < \infty) \tag{5.3}$$

$$\text{(3)}\quad U(t) \;=\; (\,H_0 * \cdots * H_{n-1}\,)(t) \qquad (-\infty < t < \infty) \tag{5.4}$$

$H_j(t)$ is defined as follows.

$$H_j(t) \;=\; \frac{1}{2a_j}\, e^{-a_j|t|} \qquad (-\infty < t < \infty, \quad 0 \le j \le n-1) \tag{5.5}$$

Proof of Lemma 5.1 We first have

$$\widehat{U}(\omega) \;=\; \widehat{G}(\omega)\,\widehat{G}(-\omega) \;=\; \frac{1}{P(z)\,P(-z)} \;=\;$$
$$\frac{(-1)^n}{\displaystyle\prod_{k=0}^{n-1}(z^2-a_k^2)} \;=\; \frac{(-1)^n}{R(z^2)} \;=\; (-1)^n\sum_{j=0}^{n-1}\frac{1}{R'(a_j^2)}\,\frac{1}{z^2-a_j^2} \;=\;$$
$$(-1)^n\left|\begin{array}{c} a_j^{2i} \\ \hline \cdots \quad (z^2-a_j^2)^{-1} \quad \cdots \end{array}\right| \Big/ \left|\begin{array}{c} a_j^{2i} \end{array}\right| \tag{5.6}$$

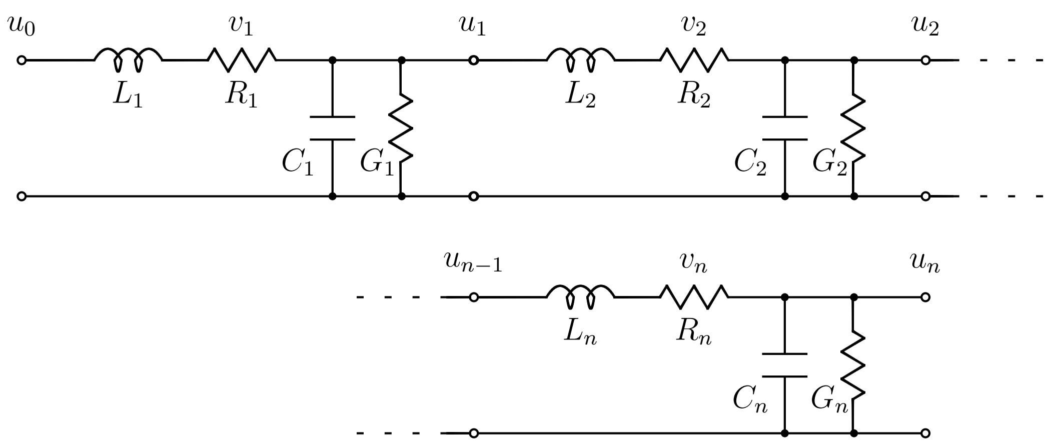
owing to the relation $G(-t) \;\xrightarrow{\;\widehat{}\;}\; \widehat{G}(-\omega)$. From the fact

$$H_j(t) \;=\; \frac{1}{2a_j}\, e^{-a_j|t|} \;=\; \frac{1}{2a_j}\Big(Y(t)\,e^{-a_j t} \,+\, Y(-t)\,e^{a_j t}\Big) \qquad \xrightarrow{\;\widehat{}\;}$$
$$\frac{1}{2a_j}\Big((z+a_j)^{-1} \,+\, (-z+a_j)^{-1}\Big) \;=\; \left(-z^2+a_j^2\right)^{-1} \;=\; \widehat{H}_j(\omega) \tag{5.7}$$

we can show (1) and (2). (3) is derived from $\widehat{U}(\omega) \;=\; \displaystyle\prod_{k=0}^{n-1}\widehat{H}_k(\omega)$. This completes the proof of Lemma 5.1. ∎

6 Heaviside cable and Thomson cable

This section presents an interesting application of the results obtained in the previous sections to engineering field. We consider the following n-cascaded LRCG units.



L_i, R_i, C_i, G_i are inductance, resistance, capacitance and conductance respectively. They are nonnegative constants and not all of them are zero. $u_{i-1} = u_{i-1}(t)$ and $u_i = u_i(t)$ are input and output voltage, respectively. $v_i = v_i(t)$ is current. Output end is open, $v_{n+1}(t) = 0$. Input voltage $u_0(t)$ is a given function of t. We investigate the relation between output voltage $u(t) = u_n(t)$ and input voltage $u_0(t)$.

We treat two cases,

$$\text{HC} \quad (\text{Heaviside cable}) \quad : \quad L_i,\ R_i,\ C_i,\ G_i \,>\, 0 \quad (1 \le i \le n)$$

and

$$\text{TC} \quad (\text{Thomson cable}) \quad : \quad L_i \,=\, G_i \,=\, 0, \quad R_i,\ C_i \,>\, 0 \quad (1 \le i \le n)$$

Heaviside cable is a discrete model of transmission line treated by Oliver Heaviside (See references [2], [3], for example).

In this section, we adopt the following abbreviation.

$$D \,=\, d/dt \tag{6.1}$$

From the Kirchhoff law, we have the following set of differential equations.

$$\begin{cases} (L_i D \,+\, R_i) v_i \,=\, u_{i-1} \,-\, u_i & (6.2) \\ (C_i D \,+\, G_i) u_i \,=\, v_i \,-\, v_{i+1} \qquad (1 \le i \le n,\ -\infty < t < \infty) & (6.3) \end{cases}$$

We introduce vectors

$$\boldsymbol{u} \,=\, {}^t(\, u_1, \cdots, u_n \,), \quad \boldsymbol{v} \,=\, {}^t(\, v_1, \cdots, v_n \,)$$

and $n \times n$ matrices

$$\boldsymbol{L} = \begin{pmatrix} L_i\,\delta_{ij} \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} R_i\,\delta_{ij} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} C_i\,\delta_{ij} \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} G_i\,\delta_{ij} \end{pmatrix},$$

$$\boldsymbol{N} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

These set of differential equations can be expressed in vector form. We treat the following boundary value problems.

BVP

$$\begin{cases} (\,\boldsymbol{L}D + \boldsymbol{R}\,)\,\boldsymbol{v} + (\,\boldsymbol{I} - {}^t\boldsymbol{N}\,)\,\boldsymbol{u} = u_0(t)\,{}^t(\,1, 0, \cdots, 0\,) & (6.4) \\ (\,\boldsymbol{C}D + \boldsymbol{G}\,)\,\boldsymbol{u} - (\,\boldsymbol{I} - \boldsymbol{N}\,)\,\boldsymbol{v} = \boldsymbol{0} \qquad (-\infty < t < \infty) & (6.5) \\ \boldsymbol{v},\ D\,\boldsymbol{v},\ \boldsymbol{u},\ D\,\boldsymbol{u} \in L^2(-\infty, \infty) & (6.6) \end{cases}$$

In the case of Thomson cable, we do not require $D\,\boldsymbol{v} \in L^2(-\infty,\infty)$. The above equation is rewritten as

$$(\,\boldsymbol{L}D + \boldsymbol{R}\,)\,(\,\boldsymbol{I} - \boldsymbol{N}\,)^{-1}(\,\boldsymbol{C}D + \boldsymbol{G}\,)\,\boldsymbol{u} + (\,\boldsymbol{I} - {}^t\boldsymbol{N}\,)\,\boldsymbol{u} = u_0(t)\,{}^t(1, 0, \cdots, 0) \\ (-\infty < t < \infty) \tag{6.7}$$

by eliminating $\boldsymbol{v}$. From now on, we put

$$u_0(t) = \begin{cases} \left(\displaystyle\prod_{i=1}^{n} (L_i C_i)\right) f(t) & \text{(HC)} \\ \left(\displaystyle\prod_{i=1}^{n} (R_i C_i)\right) f(t) & \text{(TC)} \end{cases} \tag{6.8}$$

By Fourier transform, we have the following matrix equation.

BVP$\widehat{\ }$

$$\begin{cases} (\,\boldsymbol{L}z + \boldsymbol{R}\,)\,\widehat{\boldsymbol{v}} + (\,\boldsymbol{I} - {}^t\boldsymbol{N}\,)\,\widehat{\boldsymbol{u}} = \widehat{u}_0(\omega)\,{}^t(\,1, 0, \cdots, 0\,) & (6.9) \\ (\,\boldsymbol{C}z + \boldsymbol{G}\,)\,\widehat{\boldsymbol{u}} - (\,\boldsymbol{I} - \boldsymbol{N}\,)\,\widehat{\boldsymbol{v}} = \boldsymbol{0} \qquad (-\infty < \omega < \infty) & (6.10) \end{cases}$$

The above equation is rewritten as follows.

BVP$\widehat{\ }$

$$\left(\begin{array}{cccc|cccc} L_1 z + R_1 & & & & 1 & & & \\ & & & & -1 & 1 & & \\ & \ddots & & & & \ddots & \ddots & \\ & & & L_n z + R_n & & & -1 & 1 \\ \hline -1 & 1 & & & C_1 z + G_1 & & & \\ & -1 & \ddots & & & & & \\ & & \ddots & 1 & & & \ddots & \\ & & & -1 & & & & C_n z + G_n \end{array}\right) \left(\begin{array}{c} \widehat{v}_1 \\ \vdots \\ \widehat{v}_n \\ \hline \widehat{u}_1 \\ \vdots \\ \widehat{u}_n \end{array}\right) = \widehat{u}_0(\omega) \left(\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) \tag{6.11}$$

We still use the abbreviation $z = \sqrt{-1}\omega$. It is easy to see that

$$\left|\begin{array}{cccc|cccc|c} L_1z+R_1 & & & & 1 & & & & 1 \\ & & & & -1 & \ddots & & & 0 \\ & & \ddots & & & \ddots & & 1 & \vdots \\ & & & L_nz+R_n & & & & -1 & 0 \\ \hline -1 & 1 & & & C_1z+G_1 & & & & 0 \\ & -1 & \ddots & & & \ddots & & & 0 \\ & & \ddots & 1 & & & & C_{n-1}z+G_{n-1} & \vdots \\ & & & -1 & & & & 0 & 0 \end{array}\right| = 1$$

holds. The characteristic polynomial is given by

$$P(z) = \begin{cases} \left|\begin{array}{cccc|cccc} L_1z+R_1 & & & & 1 & & & \\ & & & & -1 & 1 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & L_nz+R_n & & & -1 & 1 \\ \hline -1 & 1 & & & C_1z+G_1 & & & \\ & -1 & \ddots & & & & & \\ & & \ddots & 1 & & & \ddots & \\ & & & -1 & & & & C_nz+G_n \end{array}\right| \Big/ \prod_{i=1}^{n}(L_iC_i) \\ \text{(HC)} \\ \left|\begin{array}{cccc|cccc} R_1 & & & & 1 & & & \\ & & & & -1 & 1 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & R_n & & & -1 & 1 \\ \hline -1 & 1 & & & C_1z & & & \\ & -1 & \ddots & & & & & \\ & & \ddots & 1 & & & \ddots & \\ & & & -1 & & & & C_nz \end{array}\right| \Big/ \prod_{i=1}^{n}(R_iC_i) \\ \text{(TC)} \end{cases}$$

Solving a linear equation (6.11) with respect to $\widehat{u}_n(\omega)$, we have

$$\widehat{u}_n(\omega) = P(z)^{-1}\widehat{f}(\omega) \qquad (-\infty < \omega < \infty) \tag{6.12}$$

This means that

$$P(d/dt)\,u_n = f(t) \qquad (-\infty < t < \infty) \tag{6.13}$$

Now we obtained a concrete example of higher order ordinary differential equation with Hurwitz characteristic polynomial. In fact, we have the following lemma.

Lemma 6.1 *The characteristic polynomial $P(z)$ is a Hurwitz polynomial in both cases.*

Proof of Lemma 6.1 Let

$$P(z) = \begin{cases} \displaystyle\prod_{j=0}^{2n-1} (z + a_j) & \text{(HC)} \\ \displaystyle\prod_{j=0}^{n-1} (z + a_j) & \text{(TC)} \end{cases}$$

be the factorization of the characteristic polynomial.

For $z = -a_j$ we have a $2n$ dimensional vector ${}^t(\boldsymbol{v}, \boldsymbol{u}) \neq {}^t(\boldsymbol{0}, \boldsymbol{0})$ such that

$$\begin{cases} (\boldsymbol{L}z + \boldsymbol{R})\boldsymbol{v} + (\boldsymbol{I} - {}^t\boldsymbol{N})\boldsymbol{u} = \boldsymbol{0} & (6.14) \\ (\boldsymbol{C}z + \boldsymbol{G})\boldsymbol{u} - (\boldsymbol{I} - \boldsymbol{N})\boldsymbol{v} = \boldsymbol{0} & (6.15) \end{cases}$$

holds. We have the following relation.

$$\begin{cases} (\boldsymbol{v}^* \boldsymbol{L}\boldsymbol{v})\, z + \boldsymbol{v}^* \boldsymbol{R}\boldsymbol{v} = -\boldsymbol{v}^*(\boldsymbol{I} - {}^t\boldsymbol{N})\boldsymbol{u} = -\overline{\boldsymbol{u}^*(\boldsymbol{I} - \boldsymbol{N})\boldsymbol{v}} \\ (\boldsymbol{u}^* \boldsymbol{C}\boldsymbol{u})\, z + \boldsymbol{u}^* \boldsymbol{G}\boldsymbol{u} = \boldsymbol{u}^*(\boldsymbol{I} - \boldsymbol{N})\boldsymbol{v} \end{cases}$$

Considering that the relation

$$\Big(\boldsymbol{v}^* \boldsymbol{L}\boldsymbol{v} + \boldsymbol{u}^* \boldsymbol{C}\boldsymbol{u}\Big)\Big(\operatorname{Re} z\Big) = -\Big(\boldsymbol{v}^* \boldsymbol{R}\boldsymbol{v} + \boldsymbol{u}^* \boldsymbol{G}\boldsymbol{u}\Big)$$

and inequalities

$$\boldsymbol{v}^* \boldsymbol{L}\boldsymbol{v} + \boldsymbol{u}^* \boldsymbol{C}\boldsymbol{u} > 0, \qquad \boldsymbol{v}^* \boldsymbol{R}\boldsymbol{v} + \boldsymbol{u}^* \boldsymbol{G}\boldsymbol{u} > 0$$

hold, we have $\operatorname{Re} z < 0$. This completes the proof of Lemma 6.1. ■

Remark 6.1 *If Heaviside cable has no resistance and conductance*

$$R_i = G_i = 0 \qquad (1 \le i \le n) \tag{6.16}$$

then all the characteristic roots are pure imaginary, that is $\operatorname{Re} a_j = 0$ $(1 \le j \le n)$.

Considering the physical background, we believe the following conjecture.

Conjecture 6.1 *In the case of Heaviside cable, if we have*

$$L_i,\ C_i > 0, \quad R_i,\ G_i \ge 0 \qquad (1 \le i \le n) \tag{6.17}$$

and

$$\sum_{i=1}^{n} (R_i + G_i) > 0 \tag{6.18}$$

that is to say at least one of the R_i *or* G_i *is positive, then* $P(z)$ *is Hurwitz polynomial.*

The above conjecture is confirmed to be valid in the case $n = 1, 2$. We believe that this conjecture holds for $n \geq 3$ but it requires much effort and is still an open problem. If $n = 1$

$$L_1 C_1\, P(z) \;=\; (L_1 z + R_1)\,(C_1 z + G_1) \;+\; 1$$

has two negative roots if $R_1 > 0$ or $G_1 > 0$.

We next consider the case $n = 2$. We can show that the above conjecture is true by rational calculation using computer software "Mathematica". The characteristic polynomial is calculated as follows.

$$L_1 L_2 C_1 C_2\, P(z) \;=\; \left|\begin{array}{cc|cc} zL_1 + R_1 & & 1 & \\ & zL_2 + R_2 & -1 & 1 \\ \hline -1 & 1 & zC_1 + G_1 & \\ & -1 & & zC_2 + G_2 \end{array}\right| =$$

$$p_0 z^4 + p_1 z^3 + p_2 z^2 + p_3 z + p_4 \tag{6.19}$$

Since

$$p_4 \;=\; R_1 R_2 G_1 G_2 \;+\; R_1 G_1 \;+\; (R_1 + R_2)\, G_2 \;+\; 1 \;\geq\; 1$$

holds, we have $a_j \neq 0$ $(0 \leq j \leq 3)$. Next we assume that the equation $P(z) = 0$ has a pure imaginary characteristic root $z = a_j = \sqrt{-1}\, y,\ y \in \mathbb{R}\backslash\{0\}$. Then the equation $P(z) = 0$ is rewritten equivalently as follows.

$$y^4 - p_2 y^2 + p_4 \;-\; \sqrt{-1} y \left(p_1 y^2 - p_3 \right) \;=\; 0$$

Hence $Y = y^2$ satisfies the following equation.

$$p_0 Y^2 \;-\; p_2 Y \;+\; p_4 \;=\; 0, \qquad p_1 Y \;-\; p_3 \;=\; 0 \tag{6.20}$$

According to the rigorous calculation of the resultant using computer software "Mathematica", we can show that the inequality

$$\begin{vmatrix} p_0 & -p_2 & p_4 \\ p_1 & -p_3 & 0 \\ 0 & p_1 & -p_3 \end{vmatrix} \;=\; p_0 p_3^2 + p_1^2 p_4 - p_1 p_2 p_3 \;\geq$$

$$\left(L_2^2 R_1^2 + L_1^2 R_2^2 \right) C_1 C_2^3 \;+\; L_1^3 L_2 \left(C_2^2 G_1^2 + C_1^2 G_2^2 \right) \;>\; 0 \tag{6.21}$$

holds. This is a contradiction.

7 Example 1

In this and next section, we apply the results concerning the best constant of the Sobolev-type inequality to the Thomson cable, where corresponding characteristic polynomial $P(z)$ is proved to be a Hurwitz polynomial.

We consider the following special case of the Thomson cable.

$$R_i \;=\; C_i \;=\; 1 \qquad (1 \leq i \leq n) \tag{7.1}$$

The characteristic polynomial is given by

$$
P(n;z) =
$$

$$
\left|\begin{array}{cccc|cccc}
1 & & & & 1 & & & \\
 & 1 & & & -1 & 1 & & \\
 & & \ddots & & & \ddots & \ddots & \\
 & & & 1 & & & -1 & 1 \\
\hline
-1 & 1 & & & z & & & \\
 & -1 & \ddots & & & z & & \\
 & & \ddots & 1 & & & \ddots & \\
 & & & -1 & & & & z
\end{array}\right| = \det\Big(z\boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{N})\,(\boldsymbol{I} - {}^t\boldsymbol{N}) \Big) =
$$

$$
\left|\begin{array}{cccc}
z+2 & -1 & & \\
-1 & \ddots & \ddots & \\
 & \ddots & z+2 & -1 \\
 & & -1 & z+1
\end{array}\right| = \Big(U_{n+1}(x) - U_n(x) \Big)\Big|_{x=(z+2)/2} \tag{7.2}
$$

where $U_n(x)$ defined by $U_n(\cos(\theta)) = \sin(n\theta)\big/\sin(\theta)$ is a second kind Chebyshev polynomial. The coefficients p_{nj} of Taylor expansion and the characteristic roots a_j of

$$
P(n;z) = \sum_{j=0}^{n} p_{nj}\, z^{n-j} = \prod_{j=0}^{n-1} (z + a_j) \tag{7.3}
$$

is determined by

$$
\begin{cases}
p_{n0} = p_{nn} = 1 \qquad (n = 0, 1, 2, \cdots) \\
p_{n1} = 2n - 1 \qquad (n = 1, 2, 3, \cdots) \\
p_{n+1\,j+1} - 2p_{nj} + p_{n-1\,j-1} = p_{n\,j+1} \qquad (n = 1, 2, 3, \cdots,\ 1 \le j \le n-1)
\end{cases} \tag{7.4}
$$

and

$$
a_j = 2(1 - \cos(\theta_j)) = 4\sin^2(\theta_j/2), \quad \theta_j = \frac{2j+1}{2n+1}\,\pi \qquad (0 \le j \le n-1) \tag{7.5}
$$

Due to the inequalities $0 < \theta_0/2 < \cdots < \theta_{n-1}/2 < \pi/2$, it is easy to see that $0 < a_0 < a_1 < \cdots < a_{n-1} < 4$ holds. In fact, all the characteristic roots are negative. We here list the polynomial $P(n;z)$ and the best constants $C(n)$.

$$
\begin{aligned}
&P(0;z) = 1, \quad P(1;z) = z + 1, \quad P(2;z) = z^2 + 3z + 1, \\
&P(3;z) = z^3 + 5z^2 + 6z + 1, \quad P(4;z) = z^4 + 7z^3 + 15z^2 + 10z + 1, \\
&P(5;z) = z^5 + 9z^4 + 28z^3 + 35z^2 + 15z + 1 \\
&P(6;z) = z^6 + 11z^5 + 45z^4 + 84z^3 + 70z^2 + 21z + 1 \\
&P(7;z) = z^7 + 13z^6 + 66z^5 + 165z^4 + 210z^3 + 126z^2 + 28z + 1 \\
&P(8;z) = z^8 + 15z^7 + 91z^6 + 286z^5 + 495z^4 + 462z^3 + 210z^2 + 36z + 1, \quad \cdots
\end{aligned}
$$

$$
\begin{aligned}
&C(1) = \frac{1}{2}, \quad C(2) = \frac{1}{6}, \quad C(3) = \frac{5}{58}, \quad C(4) = \frac{95}{1802}, \quad C(5) = \frac{6389}{179786} \\
&C(6) = \frac{294361}{11517430}, \quad C(7) = \frac{873689}{45374850}, \quad C(8) = \frac{2936193722139}{195451751169362}, \quad \cdots
\end{aligned}
$$

This example shows the usefulness of the expression of the best constants by rational function of p_j.

8 Example 2

In the second special case

$$R_1 = 1/2, \qquad R_i = 1 \quad (2 \le i \le n), \qquad C_i = 1 \quad (1 \le i \le n) \tag{8.1}$$

we have

$$P(n;z) = \left|\begin{array}{cccc|cccc} 1/2 & & & & 1 & & & \\ & 1 & & & -1 & 1 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 1 & & & -1 & 1 \\ \hline -1 & 1 & & & z & & & \\ & -1 & \ddots & & & z & & \\ & & \ddots & 1 & & & \ddots & \\ & & & -1 & & & & z \end{array}\right| =$$

$$\det\left(z\boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{N}) \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} (\boldsymbol{I} - {}^t\boldsymbol{N}) \right) =$$

$$\begin{vmatrix} z+3 & -1 & & & \\ -1 & z+2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & z+2 & -1 \\ & & & -1 & z+1 \end{vmatrix} = 2\,T_n(x)\Big|_{x=(z+2)/2} \tag{8.2}$$

where $T_n(x)$ defined by $T_n(\cos(\theta)) = \cos(n\theta)$ is a first kind Chebyshev polynomial. The coefficients p_{nj} of Taylor expansion and the characteristic roots a_j of

$$P(n;z) = \sum_{j=0}^{n} p_{nj}\, z^{n-j} = \prod_{j=0}^{n-1} (z + a_j) \tag{8.3}$$

is determined by

$$\begin{cases} p_{nn} = 2 \quad (n = 0,1,2,\cdots), \qquad p_{n0} = 1 \quad (n = 1,2,3,\cdots) \\ p_{n1} = 2n \qquad (n = 1,2,3,\cdots) \\ p_{n+1\,j+1} - 2p_{nj} + p_{n-1\,j-1} = p_{n\,j+1} \qquad (n = 1,2,3,\cdots,\ 1 \le j \le n-1) \end{cases} \tag{8.4}$$

and expressed as follows.

$$a_j = 2(1 - \cos(\theta_j)) = 4\sin^2(\theta_j/2), \quad \theta_j = \frac{2j+1}{2n}\pi \qquad (0 \le j \le n-1) \tag{8.5}$$

We also have $0 < a_0 < a_1 < \cdots < a_{n-1} < 4$. We here list the polynomial $P(n;z)$ and the best constants $C(n)$.

$$\begin{aligned} &P(0;z) = 2, \quad P(1;z) = z + 2, \quad P(2;z) = z^2 + 4z + 2, \\ &P(3;z) = z^3 + 6z^2 + 9z + 2, \quad P(4;z) = z^4 + 8z^3 + 20z^2 + 16z + 2, \\ &P(5;z) = z^5 + 10z^4 + 35z^3 + 50z^2 + 25z + 2, \\ &P(6;z) = z^6 + 12z^5 + 54z^4 + 112z^3 + 105z^2 + 36z + 2, \\ &P(7;z) = z^7 + 14z^6 + 77z^5 + 210z^4 + 294z^3 + 196z^2 + 49z + 2, \\ &P(8;z) = z^8 + 16z^7 + 104z^6 + 352z^5 + 660z^4 + 672z^3 + 336z^2 + 64z + 2 \quad \cdots \end{aligned}$$

$$C(1) = \frac{1}{4}, \quad C(2) = \frac{1}{16}, \quad C(3) = \frac{3}{104}, \quad C(4) = \frac{9}{544}, \quad C(5) = \frac{1565}{146248}$$
$$C(6) = \frac{59001}{7889840}, \quad C(7) = \frac{120599227}{21863100232}, \quad C(8) = \frac{402124297}{94961127488}, \quad \cdots$$

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