COMPETING RISKS WITHIN SHOCK MODELS

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Abstract. We consider a competing risks model, in which system failures are due to one out of two mutually exclusive causes, formulated within the framework of shock models driven by bivariate Poisson process. We obtain the failure densities and the survival functions as well as other related quantities under three different schemes. Namely, system failures are assumed to occur at the first instant in which a random constant threshold is reached by (a) the sum of received shocks, (b) the minimum of shocks, (c) the maximum of shocks.

1 Introduction The classical competing risks model deals with failure times subject to multiple causes of failure. This model is of specific interest in various applied fields such as survival analysis and reliability theory. Indeed, it is appropriate for instance for describing failures of organisms or devices in the presence of different types of risks. Usually this model involves an observable pair (T, δ), where T is the time of failure and δ describes cause or type of failure. General properties of this model can be found in the literature (see Bedford and Cooke [3] and Crowder [6]). For a list of references and recent results on competing risks model see Di Crescenzo and Longobardi [9], and other contributions in [1].

In this paper we introduce a formulation of the competing risks model within the framework of stochastic shock models. These have been introduced and studied with the aim of describing systems subject to shocks occurring randomly such as events in counting processes. About 40 years ago A.W. Marshall and I. Olkin began to investigate probabilistic shock models. In particular, in [25] and [26] they obtained certain bivariate exponential distributions from fatal and non-fatal shock models. Later, by means of the total positivity theory, Esary *et al.* [11] introduced a classical preservation problem in shock models, i.e. to show that certain properties of the discrete distribution of the number of shocks that leads a system to failure are reflected in corresponding properties of the continuous distribution of the system lifetime. This investigation line has been followed by numerous authors including, for instance, El-Neweihi *et al.* [10]. We also recall the contributions by Gottlieb [16] and by Ghosh and Ebrahimi [15], concerning the increasing failure rate property of the life distribution in shock models. Furthermore, preservations results for partial stochastic orderings and for classes of failure distributions in various types of shock models have been obtained by Klefsjö [21], Fagiuoli and Pellerey [12], [13], [14], Kebir [19], Kochar [22], Pellerey [29], Singh and Jain [34]. Moreover, cumulative shock models in univariate and multivariate cases have been considered by Gut [18], Kijima and Nakagawa [20], Pérez-Ocón and Gámiz-Pérez [31], [32], Shaked and Shanthikumar [33].

Recently, new general classes of shock models of interest in reliability theory and survival analysis have been proposed, in which failure is due to the competing causes of degradation and trauma (see Lehmann [23], where the survival function of the failure time has been computed). This research direction is motivated by the circumstance that failure mechanisms

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can often be ascribed to an underlying degradation process and stochastically changing covariates. The present paper falls within such a line of investigation, aiming to include the presence of competing risks into the classical shock models scheme.

In our setting we consider systems characterized by two types of shocks and assume that failures are due to a single cause, with shocks arriving according to a pair of independent counting processes $N_1(t)$ and $N_2(t)$. We study in detail the quantities of interest under three different failure schemes. In each scheme, failures are assumed to occur at the first instant in which one of the following relations holds:

(a) $N_1(t) + N_2(t) = M$,
(b) $\min\{N_1(t), N_2(t)\} = M$,
(c) $\max\{N_1(t), N_2(t)\} = M$,

where M is a random counting number. We examine various cases arising from suitable choices of the distribution of M.

In Section 2 we describe the general setting of the model by introducing the relevant hazard rates and obtain failure densities, survival function, failure probability and the failure time moments conditional on cause or type of failure δ. In Section 3 we study this model for each of the above schemes under the assumption that $N_1(t)$ and $N_2(t)$ are homogeneous Poisson processes. When failures are due to sum of shocks, failure time T and type of failure δ are shown to be independent, and an approximation of the survival function in the presence of an arbitrary (large) number h of types of shocks is given. Finally, in Section 4 we suggest how to extend this model to the case of two non-exclusive types of shocks.

2 A model with two kinds of shocks Recently, various investigations have been oriented towards multidimensional shock models. See, for instance, Ohi and Nishida [27], [28], and Balu and Sabnis [2] for various results on bivariate shock models, including properties of the joint survival probability and preservation of reliability structures, and Wong [35] and Pellerey [30] for preservation of stochastic orders in multivariate shock models with underlying counting processes. A multidimensional shock model driven by counting process is considered in Li and Xu [24], where each shock may simultaneously destroy a subset of the components of a system consisting of several components. We shall now introduce a new kind of shock model characterized by two types of shocks, the extension to a higher number of types of shocks being straightforward. This has been inspired by the shock model with aftereffects treated in Marshall and Olkin [26] and in Ghurye and Marshall [17].

Let T be an absolutely continuous non-negative random variable describing the random failure time of a system or of a living organism. We set $\delta = i$ if the failure occurs due to a shock of type i, for $i = 1, 2$. Let $N(t)$ denote the total number of shocks occurring in $[0, t]$, with $t \geq 0$. We have

$$N(t) = N_1(t) + N_2(t), \qquad t \geq 0,$$

where $N_i(t)$ is the counting process representing the number of shocks of type i occurring in $[0, t]$, $i = 1, 2$.

The state-space $\mathbb{N}_0^2$ of $(N_1(t), N_2(t))$ is partitioned into non-empty subsets S_k, $k = 0, 1, 2, \ldots$, that will be called *failure sets*, S_0 including the numbers of non-fatal shocks. In other terms, if $(N_1(t), N_2(t)) \in S_0$ then the shocks that arrived up to time t do not cause a failure. Let M be the integer-valued random variable that represents the index of which failure set S_k, $k = 1, 2, \ldots$, containes the numbers of fatal shocks. This means that if $M = k$ then the system fails at the first instant $t > 0$ in which $(N_1(t), N_2(t)) \in S_k$. The probability distribution and the survival probability of M will be respectively denoted by

$$p_k = P(M = k), \qquad k = 1, 2, \ldots, \tag{1}$$

and

$$\overline{P}_k = P(M > k), \qquad k = 0, 1, 2, \dots. \tag{2}$$

For $k = 1, 2, \dots$ we now define

$$\begin{aligned} \tilde{S}_k^{(1)} &= \{(x_1, x_2) \in \mathbb{N}_0^2 - S_k : (x_1 + 1, x_2) \in S_k\}, \\ \tilde{S}_k^{(2)} &= \{(x_1, x_2) \in \mathbb{N}_0^2 - S_k : (x_1, x_2 + 1) \in S_k\}. \end{aligned} \tag{3}$$

These will be called *risky sets*, because $\tilde{S}_k^{(i)}$, $i = 1, 2$, containes all states of $\mathbb{N}_0^2$ that lead to a failure at time t at the next occurrence of a shock of type i, given that $M = k$. Note that definitions (3) assume that failures are due to a single cause. Moreover, note that in general $\tilde{S}_k^{(i)}$ is different from S_{k-1}.

Denoting by $f_T(t)$, $t \geq 0$, the probability density function of the failure time T, we have

$$f_T(t) = f_1(t) + f_2(t), \qquad t \geq 0, \tag{4}$$

where $f_i(t)$ is the sub-density defined by

$$f_i(t) = \frac{\mathrm{d}}{\mathrm{d}t} P\{T \leq t,\, \delta = i\}, \qquad t \geq 0, \quad i = 1, 2.$$

Recalling that $\delta = i$ if the failure occurs due to a shock of type i, we have

$$P(\delta = i) = \int_0^\infty f_i(t)\,\mathrm{d}t, \qquad i = 1, 2. \tag{5}$$

In order to express the sub-densities $f_i(t)$, $i = 1, 2$, in terms of the joint probability distribution of $(N_1(t), N_2(t))$ we now introduce the hazard rates

$$\begin{aligned} r_1(x_1, x_2; t) &= \lim_{\tau \to 0^+} \frac{1}{\tau} P\{N_1(t + \tau) = x_1 + 1, N_2(t + \tau) = x_2 | N_1(t) = x_1, N_2(t) = x_2\}, \\ r_2(x_1, x_2; t) &= \lim_{\tau \to 0^+} \frac{1}{\tau} P\{N_1(t + \tau) = x_1, N_2(t + \tau) = x_2 + 1 | N_1(t) = x_1, N_2(t) = x_2\}, \end{aligned} \tag{6}$$

with $(x_1, x_2) \in \mathbb{N}_0^2$ and $t \geq 0$. Given that x_1 shocks of type 1 and x_2 shocks of type 2 occurred in $[0, t]$, $r_i(x_1, x_2; t)$ gives the intensity of the occurrence of a shock of type i immediately after t, with $i = 1, 2$. We note that $r_1(x_1, x_2; t) + r_2(x_1, x_2; t)$ is the hazard rate of a shock of type 1 or 2.

From the above assumptions it follows that the system fails around time t due to a shock of type i if the two-dimensional counting process $(N_1(t), N_2(t))$ takes values in $\tilde{S}_k^{(i)}$ at time t *and* a shock of type i occurs immediately after. Hence, conditioning on M and recalling (1), (3) and (6), for $t \geq 0$ and $i = 1, 2$, failure densities can be expressed as

$$f_i(t) = \sum_{k=1}^{+\infty} p_k \sum_{(x_1, x_2) \in \tilde{S}_k^{(i)}} P\{N_1(t) = x_1, N_2(t) = x_2\}\, r_i(x_1, x_2; t). \tag{7}$$

A relation similar to (7) holds for the survival function of T, denoted by

$$\overline{F}_T(t) = P\{T > t\}, \qquad t \geq 0.$$

Indeed, conditioning on $(N_1(t), N_2(t))$ and recalling (2), we obtain

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} \overline{P}_k \sum_{(x_1,x_2)\in S_k} P\{N_1(t) = x_1, N_2(t) = x_2\}, \qquad t \geq 0, \tag{8}$$

where $\overline{P}_0 = 1$. Other quantities of interest are the moments of the failure time conditional on the cause of failure:

$$E(T^s \mid \delta = i) = \frac{1}{P(\delta = i)} \int_0^{+\infty} t^s \, f_i(t) \, \mathrm{d}t, \qquad i = 1, 2. \tag{9}$$

This can be evaluated making use of (5) and (7).

3 Shocks driven by a Poisson process In this Section we assume that the two kinds of shocks affect the system according to independent homogeneous Poisson processes $N_1(t)$ and $N_2(t)$. Therefore, for $x_1, x_2 = 0, 1, 2, \ldots$ and $t \geq 0$ one has:

$$P\{N_1(t) = x_1, N_2(t) = x_2\} = \frac{e^{-\lambda_1 t}(\lambda_1 t)^{x_1}}{x_1!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{x_2}}{x_2!}, \tag{10}$$

with $\lambda_1 > 0$ and $\lambda_2 > 0$. Under assumption (10), from (6) we now have

$$r_i(x_1, x_2; t) = \lambda_i, \qquad (x_1, x_2) \in \mathbb{N}_0^2, \quad t \geq 0, \quad i = 1, 2.$$

Hence, from (7) and (8) we obtain:

$$f_i(t) = \sum_{k=1}^{+\infty} p_k \sum_{(x_1,x_2)\in \tilde{S}_k^{(i)}} \frac{e^{-\lambda_1 t}(\lambda_1 t)^{x_1}}{x_1!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{x_2}}{x_2!} \lambda_i, \qquad t \geq 0, \quad i = 1, 2 \tag{11}$$

and

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} \overline{P}_k \sum_{(x_1,x_2)\in S_k} \frac{e^{-\lambda_1 t}(\lambda_1 t)^{x_1}}{x_1!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{x_2}}{x_2!} \qquad t \geq 0. \tag{12}$$

Recalling (5) and (11), it is not difficult to see that

$$P(\delta = i) = \pi_i \sum_{k=1}^{+\infty} p_k \sum_{(x_1,x_2)\in \tilde{S}_k^{(i)}} \frac{(x_1 + x_2)!}{x_1! \, x_2!} \, \pi_1^{x_1} \, \pi_2^{x_2}, \qquad i = 1, 2, \tag{13}$$

where we have set $\pi_i = \lambda_i/(\lambda_1 + \lambda_2)$, $i = 1, 2$. Similarly, from (9) we obtain the conditional moments

$$E(T^s \mid \delta = i) = \frac{1}{(\lambda_1 + \lambda_2)^s} \frac{\displaystyle\sum_{k=1}^{+\infty} p_k \sum_{(x_1,x_2)\in \tilde{S}_k^{(i)}} \frac{(x_1 + x_2 + s)!}{x_1! \, x_2!} \, \pi_1^{x_1} \, \pi_2^{x_2}}{\displaystyle\sum_{k=1}^{+\infty} p_k \sum_{(x_1,x_2)\in \tilde{S}_k^{(i)}} \frac{(x_1 + x_2)!}{x_1! \, x_2!} \, \pi_1^{x_1} \, \pi_2^{x_2}}, \qquad i = 1, 2. \tag{14}$$

The above expressions show that the structures of the failure sets and of the risky sets are essential to specify the nature of the shock model. Hereafter we consider three special cases of interest, in which these sets are chosen according to typical models of reliability theory, as sketched in Figure 1 for S_k.

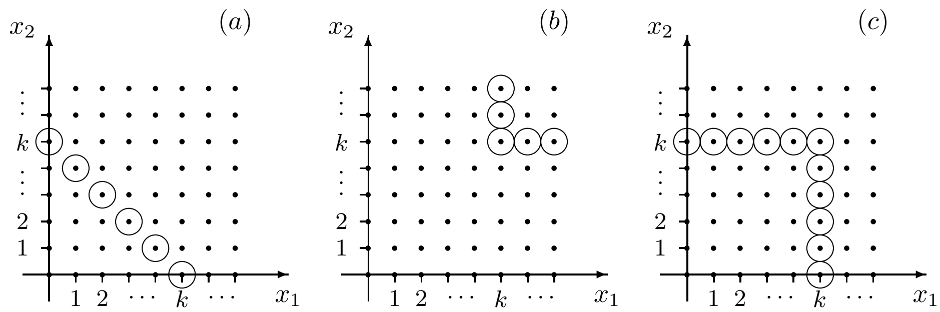

Figure 1: Failure sets S_k when failures are due to (a) sum of shocks, (b) minimum of shocks, and (c) maximum of shocks.

3.1 Failures due to sum of shocks Assume that the system fails when the sum of shocks of type 1 and of type 2 reaches a random threshold that takes values in $\{1, 2, \ldots\}$. In this case we have

$$S_k = \{(x_1, x_2) \in \mathbb{N}_0^2 : x_1 + x_2 = k\}, \qquad k = 0, 1, 2, \ldots.$$

and

$$\tilde{S}_k^{(1)} = \tilde{S}_k^{(2)} = S_{k-1}, \qquad k = 1, 2, \ldots.$$

Under these assumptions, from (11) and by use of Newton's binomial theorem for $i = 1, 2$ we obtain

$$f_i(t) = \lambda_i e^{-(\lambda_1+\lambda_2)t} \sum_{k=1}^{+\infty} p_k \frac{[(\lambda_1 + \lambda_2)t]^{k-1}}{(k-1)!}. \tag{15}$$

Similarly, making use of (12) we have

$$\overline{F}_T(t) = e^{-(\lambda_1+\lambda_2)t} \sum_{k=0}^{+\infty} \overline{P}_k \frac{[(\lambda_1 + \lambda_2)t]^{k}}{k!}. \tag{16}$$

Moreover, from (13) the probability that the failure ultimately occurs due to a shock of type i follows:

$$P(\delta = i) = \frac{\lambda_i}{\lambda_1 + \lambda_2}, \qquad i = 1, 2. \tag{17}$$

Finally, from (14) it is not hard to prove that the conditional moments for $s = 1, 2, \ldots$ in this case are given by

$$E(T^s \,|\, \delta = i) = \frac{1}{(\lambda_1 + \lambda_2)^s}\, E[(M + s - 1)_s], \qquad i = 1, 2, \tag{18}$$

where $(m)_s$ denotes the descending factorial $m(m-1)(m-2)\cdots(m-s+1)$. Note that the right-hand-side of (18) does not depend on δ. This is not surprising, since the time of failure T and the cause of failure δ are independent if the system failures are due to sum of shocks, as will soon be proved. It is interesting to recall that the relevance of independence between T and δ has been pointed out by various authors (see, for instance, Carriere and Kochar [4]).

	$p_k; k \ge 1$	$f_i(t)$	$\overline{F}_T(t)$
(i)	$p(1-p)^{k-1}$	$p\lambda_i e^{-p\tau}$	$e^{-p\tau}$
(ii)	$\dfrac{\eta^{k-1}e^{-\eta}}{(k-1)!}$	$\lambda_i e^{-\eta-\tau} I_0\Big(2\sqrt{\eta\tau}\Big)$	$e^{-\eta-\tau}\displaystyle\sum_{n=0}^{+\infty}\Big(\frac{\eta}{\tau}\Big)^{n/2} I_n\Big(2\sqrt{\eta\tau}\Big)$
(iii)	$\dfrac{1}{k(k+1)}$	$\dfrac{\lambda_i}{\tau^2}\Big\{1-e^{-\tau}(1+\tau)\Big\}$	$\dfrac{1-e^{-\tau}}{\tau}$
(iv)	$kp^2(1-p)^{k-1}$	$\lambda_i p^2\Big\{1+(1-p)\tau\Big\}e^{-p\tau}$	$e^{-p\tau}\Big\{1+p(1-p)\tau\Big\}$

Table 1: Subdensities and survival function for the model with failures due to sum of shocks, with $\tau = (\lambda_1+\lambda_2)t$.

Proposition 3.1 *T and δ are independent under the model assumptions of Section 3.1.*

Proof. From (4), (15) and (17), expressing p_k as $\overline{P}_{k-1} - \overline{P}_k$ there holds:

$$\begin{aligned}
P(\delta = i) f_T(t) &= \lambda_i e^{-(\lambda_1+\lambda_2)t}\left\{\sum_{k=0}^{+\infty}\sum_{j=k+1}^{+\infty} p_j \frac{[(\lambda_1+\lambda_2)t]^k}{k!} - \sum_{k=1}^{+\infty}\sum_{j=k+1}^{+\infty} p_j \frac{[(\lambda_1+\lambda_2)t]^{k-1}}{(k-1)!}\right\} \\
&= \lambda_i e^{-(\lambda_1+\lambda_2)t}\left\{\sum_{j=1}^{+\infty} p_j \sum_{k=0}^{j-1}\frac{[(\lambda_1+\lambda_2)t]^k}{k!} - \sum_{j=2}^{+\infty} p_j \sum_{k=1}^{j-1}\frac{[(\lambda_1+\lambda_2)t]^{k-1}}{(k-1)!}\right\} \\
&= \lambda_i e^{-(\lambda_1+\lambda_2)t}\sum_{j=1}^{+\infty} p_j \frac{[(\lambda_1+\lambda_2)t]^{j-1}}{(j-1)!} = f_i(t), \qquad i = 1,2.
\end{aligned}$$

The proof is thus completed, for T and δ are independent if and only if $f_i(t) = P(\delta = i) f_T(t)$ for all $t \ge 0$ and $i = 1,2$. ∎

For the model in which failures are due to sum of shocks, the subdensity $f_i(t)$ and the survival function $\overline{F}_T(t)$ are shown in Table 1 when the distribution of M is (i) geometric, (ii) Poisson over $\{1,2,\ldots\}$, (iii) $p_k = \frac{1}{k(k+1)}$, $k = 1,2,\ldots$, and (iv) a suitable negative binomial. $I_n(x)$ denotes the modified Bessel function of the first kind.

We notice that $f_i(t)$ and $\overline{F}_T(t)$ can be expressed similarly to Eqs. (15) and (16) if $N_1(t)$ and $N_2(t)$ are independent non-homogeneous Poisson processes. However, in this case $P(\delta = i)$ would be equal to the right-hand-side of (17) only if the ratio of the time-varying intensity functions of $N_1(t)$ and $N_2(t)$ is a constant. Hence, Proposition 3.1 still holds if $N_i(t)$ is a Poisson process characterized by an intensity function of the form $\lambda_i\, u(t)$, with $i = 1,2$ and $t \ge 0$.

The model considered in this paper can be easily extended to the case of an arbitrary number h of types of shocks, by introducing a multidimensional counting process $(N_1(t),\ldots,N_h(t))$, where $N_i(t)$ describes the number of shocks of i-th type occurred in $[0,t]$, $i = 1,2,\ldots,h$. In this case, under the assumption that failures are due to sum of shocks, the survival function of T can be expressed as

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} p_k\, P\{N_1(t)+\ldots+N_h(t) < k\}, \qquad t \ge 0. \tag{19}$$

The form of the right-hand-side of (19) suggests that a central limit theorem might be implemented. For instance, assuming that the components of $(N_1(t),\ldots,N_h(t))$ are independent and identically distributed with finite mean and variance, Eq. (19) at a fixed time

$t_0 > 0$ for large h gives

$$\overline{F}_T(t_0) \approx \sum_{k=0}^{+\infty} p_k \, \Phi\left(\frac{k - h\,\mu_0}{\sqrt{h}\,\sigma_0}\right),$$

where $\mu_0 = E[N_i(t_0)]$ and $\sigma_0^2 = Var[N_i(t_0)]$, with $\Phi(\cdot)$ denoting the standard normal distribution function.

3.2 Failures due to minimum of shocks In this Section we consider another failure scheme for the shock model characterized by two kinds of shocks. Namely, in this case the system fails when the minimum of the shocks of type 1 and of type 2 reaches a random threshold that takes values in $\{1, 2, \ldots\}$. Such a model may for instance be appropriate to describe the failure of systems made out of units connected in parallel.

In this model the failure sets and the risky sets are respectively given by

$$S_k = \{(x_1, x_2) \in \mathbb{N}_0^2 : \min(x_1, x_2) = k\}, \qquad k = 1, 2, \ldots, \tag{20}$$

and

$$\tilde{S}_k^{(1)} = \{(k-1, x_2) \in \mathbb{N}_0^2 : x_2 = k, k+1, \ldots\},$$
$$\tilde{S}_k^{(2)} = \{(x_1, k-1) \in \mathbb{N}_0^2 : x_1 = k, k+1, \ldots\}.$$

From (11) and (20) the following subdensities now follow:

$$\begin{aligned} f_1(t) &= \lambda_1 e^{-(\lambda_1+\lambda_2)t} \sum_{k=1}^{+\infty} p_k \frac{(\lambda_1 t)^{k-1}}{(k-1)!} \, \overline{E}_k(\lambda_2 t), \qquad t \geq 0, \\ f_2(t) &= \lambda_2 e^{-(\lambda_1+\lambda_2)t} \sum_{k=1}^{+\infty} p_k \frac{(\lambda_2 t)^{k-1}}{(k-1)!} \, \overline{E}_k(\lambda_1 t), \qquad t \geq 0, \end{aligned} \tag{21}$$

where for $k = 1, 2, \ldots$ we have set

$$\overline{E}_k(x) = \sum_{j=k}^{+\infty} \frac{x^j}{j!}, \qquad x \in \mathbb{R}.$$

The survival function of T can now be easily obtained from (12) and (20):

$$\overline{F}_T(t) = e^{-(\lambda_1+\lambda_2)t} \sum_{k=0}^{+\infty} \overline{P}_k \left\{ \frac{(\lambda_2 t)^k}{k!} \, \overline{E}_k(\lambda_1 t) + \frac{(\lambda_1 t)^k}{k!} \, \overline{E}_{k+1}(\lambda_2 t) \right\}, \qquad t \geq 0. \tag{22}$$

Tables 2 and 3 show respectively $f_i(t)$ and $\overline{F}_T(t)$ for the present model, for three choices of the distribution of M. Note that the function

$$M_{n,k}(x) = \sum_{r=0}^{+\infty} \frac{x^{3r+n+k}}{r!(r+n)!(r+k)!},$$

appearing in case (ii), is a modified two-index Bessel function (also known as a modified Humbert function, see for instance Dattoli *et al.* [7]). It appears also in the probability distributions of certain two-dimensional random motions (see [5] and [8]). The survival function in case (ii) of Table 3 has been obtained by recalling the relation (see, for instance, Eq. (5.3) of [5])

$$\sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \alpha^n \beta^m \, M_{n,m}(x) = \exp\left\{ x \left(\alpha + \beta + \frac{1}{\alpha\beta} \right) \right\}.$$

	$f_i(t)$
(i)	$p\lambda_i e^{-\tau}\sum_{n=0}^{+\infty}\left[\frac{\lambda_{3-i}}{(1-p)\lambda_i}\right]^{\frac{n+1}{2}} I_{n+1}(\alpha t)$
(ii)	$\lambda_i e^{-\eta-\tau}\sum_{n=0}^{+\infty}\left(\frac{\beta^2}{\alpha\lambda_i t}\right)^{n+1} M_{0,n+1}(\beta)$
(iii)	$\frac{e^{-\tau}}{\lambda_i t^2}\left\{\sqrt{\frac{\lambda_i}{\lambda_{3-i}}}\, I_1\left(2t\sqrt{\lambda_1\lambda_2}\right)-\lambda_i t+\sum_{r=0}^{+\infty}\left[\sqrt{\frac{\lambda_{3-i}}{\lambda_i}} I_r\left(2t\sqrt{\lambda_1\lambda_2}\right)-\frac{(\lambda_{3-i}t)^r}{r!}-\frac{\lambda_i t(\lambda_{3-i}t)^{r+1}}{(r+1)!}\right]\right\}$

Table 2: Subdensities for the model with failures due to minimum of shocks, for cases (i)-(iii) of Table 1, with $\tau=(\lambda_1+\lambda_2)t$, $\alpha=2\sqrt{(1-p)\lambda_1\lambda_2}$ and $\beta=(\eta\lambda_1\lambda_2 t^2)^{1/3}$.

	$\overline{F}_T(t)$
(i)	$e^{-\tau}\sum_{n=0}^{+\infty}\left\{\left[\frac{\lambda_1}{(1-p)\lambda_2}\right]^{\frac{n}{2}} I_n(\alpha t)+\left[\frac{\lambda_2}{(1-p)\lambda_1}\right]^{\frac{n+1}{2}} I_{n+1}(\alpha t)\right\}$
(ii)	$e^{-\eta-\tau}\left(\exp\left\{\beta\left[\left(\frac{\eta^2}{\lambda_1\lambda_2 t^2}\right)^{\frac13}+\left(\frac{\lambda_1^2 t^2}{\eta}\right)^{\frac13}+\left(\frac{\lambda_2}{\lambda_1\eta^2}\right)^{\frac13}\right]\right\}+\exp\left\{\beta\left[\left(\frac{\eta^2}{\lambda_1\lambda_2 t^2}\right)^{\frac13}+\left(\frac{\lambda_2^2 t}{\lambda_1\eta}\right)^{\frac13}+\right.\right.\right.$ $\left.\left.\left.+\left(\frac{\lambda_1^2 t}{\lambda_2\eta}\right)^{\frac13}\right]\right\}-\left(\frac{\lambda_2^2 t}{\lambda_1\eta}\right)^{\frac13}\sum_{j=0}^{+\infty} M_{j,0}(\beta)\left(\frac{\eta^2}{\lambda_1\lambda_2 t^2}\right)^{\frac{j}{3}}\right)$
(iii)	$\frac{e^{-\tau}}{t}\left\{\sum_{j=0}^{+\infty} I_j\left(2t\sqrt{\lambda_1\lambda_2}\right)\frac{\lambda_1^{j+1}+\lambda_2^{j+1}}{(\lambda_1\lambda_2)^{\frac{j}{2}+1}}+I_1\left(2t\sqrt{\lambda_1\lambda_2}\right)\frac{1}{\sqrt{\lambda_1\lambda_2}}-\frac{e^{\lambda_1 t}}{\lambda_2}-\frac{e^{\lambda_2 t}}{\lambda_1}\right\}$

Table 3: Survival function for the model with failures due to minimum of shocks, for the same cases of Table 2.

3.3 Failures due to maximum of shocks Let us now consider a failure scheme in which the system fails when the maximum of the shocks of type 1 and of type 2 reaches a random threshold taking values in $\{1,2,\ldots\}$. For instance, this model is suitable to describe the failure of systems composed by units serially interconnected.

In this case we assume that the failure sets are given by

$$S_k=\{(x_1,x_2)\in\mathbb{N}_0^2:\max(x_1,x_2)=k\},\qquad k=1,2,\ldots, \tag{23}$$

so that, for $k=1,2,\ldots$, the risky sets are

$$\tilde{S}_k^{(1)}=\{(k-1,x_2)\in\mathbb{N}_0^2: x_2=0,1,\ldots k-1\},$$

$$\tilde{S}_k^{(2)}=\{(x_1,k-1)\in\mathbb{N}_0^2: x_1=0,1,\ldots k-1\}.$$

Note that, even if definition (23) includes state (k,k) in S_k, the assumption that failures are due to a single cause excludes in this model the possibility that states of the type (k,k) are failure states.

From (11) it follows

$$\begin{aligned} f_1(t)&=\lambda_1 e^{-(\lambda_1+\lambda_2)t}\sum_{k=1}^{+\infty}p_k\frac{(\lambda_1 t)^{k-1}}{(k-1)!}E_k(\lambda_2 t), \qquad t\ge 0,\\ f_2(t)&=\lambda_2 e^{-(\lambda_1+\lambda_2)t}\sum_{k=1}^{+\infty}p_k\frac{(\lambda_2 t)^{k-1}}{(k-1)!}E_k(\lambda_1 t), \qquad t\ge 0, \end{aligned} \tag{24}$$

where $E_k(x)$, $x \in \mathbb{R}$, is the auxiliary function such that $E_0(x) = 0$ and

$$E_k(x) = \sum_{j=0}^{k-1} \frac{x^j}{j!} = e^x - \overline{E}_k(x), \qquad k = 1, 2, \dots . \tag{25}$$

It is not hard to see that the subdensities $f_i(t)$ for the two models in which failures are due to minimum and maximum of shocks are closely related. This is due to the similar nature of failure sets S_k defined in (20) and (23). Indeed, making use of identity (25), the sum of subdensities (21) and (24) is seen to be

$$\lambda_i e^{-\lambda_i t} \sum_{k=1}^{+\infty} p_k \frac{(\lambda_i t)^{k-1}}{(k-1)!}, \qquad t \geq 0, \quad i = 1, 2.$$

Moreover, from (12) and (23) the survival function of T is obtained:

$$\overline{F}_T(t) = e^{-(\lambda_1+\lambda_2)t} \sum_{k=0}^{+\infty} \overline{P}_k \left\{ \frac{(\lambda_2 t)^k}{k!} E_k(\lambda_1 t) + \frac{(\lambda_1 t)^k}{k!} E_{k+1}(\lambda_2 t) \right\}, \qquad t \geq 0. \tag{26}$$

As for subdensities $f_i(t)$, the survival functions of T for the last two considered models are related. Indeed, making again use of (25) the sum of functions (22) and (26) is

$$\sum_{i=1}^{2} e^{-\lambda_i t} \sum_{k=0}^{+\infty} \overline{P}_k \frac{(\lambda_i t)^k}{k!}, \qquad t \geq 0.$$

4 Concluding remarks We have introduced a formulation of shock models driven by bivariate Poisson process that includes a competing risks set-up, in which failures are due to one out of two mutually exclusive causes. We have obtained specific expressions for the failure densities and the survival function under three different failure schemes, in which the variable that causes the failure is (a) the sum of shocks, (b) the minimum of shocks, (c) the maximum of shocks.

Future developments of this model will be oriented to the construction of other structures for failure sets S_k, and to the study of a case in which the two fonts of shocks are not mutually exclusive, thus including the possibility that failures are due to both types of shocks. The underlying counting process will not have independent components; for instance the following bivariate Poisson process could be employed (see Marshall and Olkin [26] and references therein):

$$P\{N_1(t) = x_1, N_2(t) = x_2\} = e^{-(\lambda_1+\lambda_2+\lambda_3)t} \sum_{k=0}^{\min(x_1,x_2)} \frac{\lambda_1^{x_1-k}\, \lambda_2^{x_2-k}\, \lambda_3^k\, t^{x_1+x_2-k}}{(x_1-k)!\,(x_2-k)!\,k!}, \qquad t \geq 0.$$

Furthermore, the structure of the failure sets and of the risky sets will be modified accordingly, by taking also into account the necessity that a third hazard rate must be added to those defined in Eqs. (6).

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