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WHITE NOISE APPROACH TO TIME OR SPACE-TIME DEPENDENT RANDOM FUNCTIONS DESCRIBING BIOLOGICAL PHENOMENA

Takeyuki Hida

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Abstract. We are interested in some biological phenomena which are evolutional random complex systems. Simplest cases are expressed in the form of stochastic processes $X(t)$ that are functionals of white noise. In more general cases those phenomena in question are viewed as random fields $X(C)$ depending on a manifold C running through a space-time Euclidean space R^n. We may assume that C is an $(n-1)$-dimensional smooth ovaloid so that variational calculus can be applied smoothly.

Our approach starts with the step of **reduction** of the complex random systems. This means that we try to find a system of idealized elemental random variables (abb. i.e.r.v.'s) that has the same information as the given random system, in addition, in a causal manner. Once the system is expressed as functionals of the i.e.r.v.'s, we are ready to analyze them by appealing to the white noise analysis which has been extensively developed in recent years. That is the step of the **analysis**. Having established the analysis, namely after the systems are well investigated, we can finally come to the step of **applications**.

The system of i.e.r.v.'s. is the so-called the **innovation** of the random evolutional system, for either stochastic processes or random fields.

The innovation, if it exists, appears in the stochastic differential equations for stochastic processes $X(t)$ and in the variational equations for random fields $X(C)$. We now assume that i.e.r.v. is a (Gaussian) white noise. The variational equation is a generalization, in a sense, of a stochastic differential equation. In addition to the usual technique for ordinary stochastic differential equations, we need new method of calculus for which white noise analysis can be a powerful tool. Generalized white noise functionals and creation and annihilation operators are efficiently used. Applications to image processing give us interesting questions, in particular, in medical images. More examples illustrate our idea.

1 Introduction We are interested in random complex systems which are developing as space-time parameter goes by. Many biological phenomena that we are interested in are viewed as such systems.

In order to investigate random complex shystems, it is proposed to start with the step of **Reduction**. It means that we should form a system of *independent and atomic* random variables (they may be infinitesimal) that has the same information as the given system. In addition, *causality* should hold in the sense that will be explained later. We call such a system *idealized elemental random variables* abb. i.e.r.v.'s.

For instance, if the given system is a stochastic process $X(t)$ or random field $X(C)$ parameterized by a manifold C, then the i.e.r.v. is the *innovation*. For many important cases,

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the innovation is taken to be a *white noise*, the time derivative $\dot{B}(t)$ of a Brownian motion $B(t)$ or Poisson noise $\dot{P}(t)$, that of a Poisson process $P(t)$, or their linear combination.

Having been obtained the innovation, the given random system should be expressed as a function of those independent random varaibles, also in a causal manner. In reality, we shall introduce a class of *generalized functionals* of white noise and those of Poisson noise. Significance is that to each $\dot{B}(t)$ and to $\dot{P}(t)$ the definite *identity* can be given, although they are infinitesimal random vaiables. Then, they are taken to be the variables of functions, indeed functionals, which are mathematical description of the given random complex shystem. These can be done rigorously in line with white noise analysis.

We should note that in order to carry on the analysis of white noise functionals, in fact generalized functionals, the *infinite dimensional rotation group* plays dominant roles. Also, if we come to the Poisson noise functionals, the *infinite symmetric group* will do similar roles. In additon, two groups, putting together, can describe a *duality* between two noises; Gaussian and Poissonian. The Lie algebra associated with the group involves partial differential operators, which are basic tools of the analysis. These topis are important, however we have to skip them in this report, to our regret.

Although further results cannot be included in this report, the main direction of those approaches can be seen in what follows.

2 Generalized random variables $\{\dot{B}(t)\}$ We are in a position of Reduction, so that we now consider a system of idealized elemental random variables as the innovation of a stochastic process or a random field. Important and natural cases where we often meet is, intuitively speaking, a time derivative of an additive process with stationary increments. Thus we may think of the derivative of a Lévy process which is an independent sum of Brownian motion and a compound Poisson process.

We shall mainly be concerned with Gaussian part, i.e. Brownian motion up to constant factor. The time derivative of a Brownian notion $B(t)$ is called a *white noise* and is denoted by $\dot{B}(t)$. If necessary, we may say *Gaussian* white noise to discriminate from other noises. Usefulness of $\dot{B}(t)$ has been recognized in many ways, although it has had only a formal significance for many years. In fact, such an understanding was efficient in a sense to have ideas. Then, it became to be viewed as a stationary generalized Gaussian process with mean 0 and covariance function is a delta function. The generalized stochastic process $\dot{B}(t)$ has a correct meaning being evaluated at a test function ξ in such a way that

$$\dot{B}(\xi) = -\int \xi'(t)B(t)dt,$$

which is Gaussian in distribution $N(0, \|\xi\|^2)$. The process has the characteristic functional $C(\xi)$ given by

$$C(\xi) = \exp[-\frac{1}{2}\|\xi\|^2],\ \xi \in E,$$

where E is the test function space involving very smooth functions. The $C(\xi)$ determines the probability distribution μ of $\dot{B}$ on the space E^* of generalized functions over R^1.

Thus, a rigorous definition of white noise was given some time ago. Unfortunately, we are given only a random variable after the *smearing* of $\dot{B}(t)$: we miss the time variable t. On the other hand, the idealized random variable $\dot{B}(t)$ is used as an input signal at instant t when an input-output model is considered in applications like the cases where identification of some biological organs or communication channels are discussed. Actually, there has been required a sharp time expression.

We are, therefore, requested to give each $\dot{B}(t)$ a definite position in a certain collection of random variables, more precisely in a set of generalized random variables. This has been done by introducing a reasonably larger space $H_1^{(-1)}$ which is an extension of the ordinary space H_1 spanned by linear functions of a Brownian motion. An identification, as it were, has been given to each $\dot{B}(t)$ as a member of $H_1^{(-1)}$. Once the collection of $\{\dot{B}(t), t \in R^1\}$ is given, we may proceed to form (nonlinear) functions of those variables. The class of those functions, indeed functionals of the $\dot{B}(t), t \in R^1$, can not be constructed in a simple manner, because the variables are random and even idealized generalized variables. Precise discussion on these facts can be seen in [**?**] in particular, in Chapt. 2., however we can explain the main idea by taking an example below. Such an observation may be helpful when applications like in biology or in communication theory will be discussed.

3 Generalized functionals. The theory From theoretical side we provide mathematical tools.

1) The collection $\{\dot{B}(t), t \in R\}$, is a system of *idealized elemental random variables* (*i.e.r.v.*).

2) Linear functionals.

As has been briefly mentioned above, we start with the Hilbert space H_1 involving linear functions of $B(t)$ with finite variance. It is well-known that H_1 is a collection of stochastic integrals of the form

$$\dot{B}(f) = \int f(u) dB(u),$$

which is Gaussian in distribtion: $N(0, \|f\|^2)$, where $\|\cdot\|$ is the $L^2(R^1)$-norm. Further, we konw that the mapping, in the above notations,

$$f \ \to \dot{B}(f)$$

defines an isomorphism:

$$H_1 \ \cong \ L^2(R^1).$$

Both spaces should have extensions keeping the isomorphism. It is noted that $\dot{B}(t)$ may be viewed as $\dot{B}(\delta_t)$, where δ_t is the delta function supported by $\{t\}$. Hence, we are led to introduce a space $K^{(-1)}(R^1)$, the Sobolev space of order -1 over the space R^1 in which the delta function is involved. With this idea we can define a space $H_1^{(-1)}$ such that

$$H_1^{(-1)} \ \cong \ K^{(-1)}(R^1),$$

and that $\dot{B}(t) \in H_1^{(-1)}$.

The space $H_1^{(-1)}$ is not only the space of linear functionals of white noise, but it is the space where all the variables of nonlinear white noise functionals are involved. Namely, it is the basic space of our analysis. Now we can state

Theorem 1. The collection $\{\dot{B}(t), t \in R^1\}$ is not a base, but it is *total* in the sense that all finite linear combinations of the memebers span the entire space $H_1^{(-1)}$.

Proof. The system $\{\frac{e^{it\lambda}}{\sqrt{1+|\lambda|^2}}\}$ is total in $L^2(R^1)$. This implies the assertion.

So far as the space $H_1^{(-1)}$ is concerned, it is not suitable to consider the dimension, but the notion of a total set is more suitable.

Then, nonlinear functionals of white noise are going to be introduced. Let us begin with the most simplest case.

3) Quadratic polinomials in $\dot{B}(t)$ and quadratic functionals of $\dot{B}(t)$ are introduced. In the classical stochastic calculus (e.g. for stochastic differential equations) it is permitted to have the equality

$$(dB(t))^2 = dt.$$

If we come to microscopic world, the differece must be taken into account, since it is still random although it is infinitesimal. We therefore apply *renormalization*. Namely, first we apply orthogonalization to the trivial space of constants and to the space $H_1^{(-1)}$. In fact, we note the approximating equality $E[(dB(t))^2] = dt$, then we have

$$(dB(t))^2 - dt,$$

Renormalization (actually magnification in this case) as much as $\frac{1}{(dt)^2}$ to have

$$: \dot{B}(t)^2 := \frac{(dB(t))^2}{(dt)^2} - \frac{1}{dt}.$$

Note that each term in the right-hand side is neither ordinary nor generalized functions of white noise. What we claim is that we first approximate both terms by ordinary functions, namely $(\frac{\Delta B}{\Delta})^2$ and by $\frac{1}{\Delta}$ then let them go to their limits. The limit should be taken not in the ordinary topology, but in the topology that is to be introduced to the new space $H_2^{(-2)}$. In this case, only the difference of two terms can approach to the limit, but not independently, of course. Again, let us note that the difference is not in the ordinary sense, but it does have meanings only in the approximation stage.

Having been suggested by the case 2) and by the naive interpretations on generalized Brownian (indeed, white noise) functionals in [3], we start from the isomorphism

$$H_2 \ \cong \ \hat{L}^2(R^2),$$

where the space H_2 is spanned by quadratic functionals of white nopise which are orthogonal to the spaces H_1 and to H_0 that is involving only constants. As in the case 2), both sides are extended to wider spaces with weaker topology such that

$$H_2^{(-2)} \ \cong \ \hat{K}^{(-3/2)}(R^2),$$

where $\hat{K}^{(-3/2)}(R^2)$ is the symmetric Sobolev space of degree -3/2 over R^2 and where the constant $\sqrt{2}$ is ignored in the above isomorphism.

Remark. For an information, a definition of Sobolev spaces is reviewed. A Sobolev space $K^m(R^n)$ of degree m over R^n is defined as

$$K^m(R^n) = \{f; \hat{f}(1 + |\lambda|^2)^{m/2} \in L^2(R^n)\},$$

where m can be any number, but in our case m is taken to be positive or negative half-integer. In our case, $(n+1)/2$ is the *magic number* associated to R^n. The notation $\hat{}$ used there means "symmetric".

It is easily checked that the symmetrization of $\delta_t \otimes \delta_s$ which is a member of $\hat{K}^{(-3/2)}(R^2)$ and is associated to $: \dot{B}(t)\dot{B}(s) :$ the renormalized product of $\dot{B}(t)$ and $\dot{B}(s)$, t and s may or may not be equal.

Interesting example, according to Volterra and Lévy, is a *normal* functionals expressed in ther form

$$\int f(t) : \dot{B}(t)^2 : dt.$$

Functionals of this form play very important roles in the harmonic analysis of white noise functionals, and also in applications.

4) More general functionals.

Now generalizations of $H_1^{(1)}$ and of $H_1^{(-2)}$ to the space $H_n^{(-n)}$ is almost straightforward. We can take the symmetric Soblolev space of order $-(n+1)/2$ over R^n is taken to define the space $H_n^{(-n)}$:

$$H_n^{(-n)} \;\cong\; \hat{K}^{(-(n+1)/2)}(R^n),$$

where the constant $\sqrt{n!}$ is ignored.

Finally, we define the space $(L^2)^+$ of **test white noise functionals**:

$$(L^2)^+ = \oplus_0^\infty c_n H_n^{(-n)},$$

where $H_0^{(0)} = \{1\}$ and where the c_n is an increasing sequence of positive numbers. The choice of the c_n depends on the place where the analysis in question is done.

The dual space $(L^2)^-$ of $(L^2)^+$ is expressed as a sum

$$(L^2)^- = \oplus_0^\infty c_n^{-1} H_n^(- n),$$

and it is called the space of **generalized white noise functionals**.

Theorem 2 The collection of the Wick products (the Hermite polynomials in $\dot{B}(t)$'s) $\{: \Pi_{j=1}^n \dot{B}(t_j) :; t_j \in Ra, n \geq 0\}$ is total in $(L^2)^-$, where Ra is the set of rational number. The space $(L^2)^-$ is therefore separable.

Proof. As in the proof of Theorem 1, the proof can be reduced to the totality of the system $\{\Pi_1^n e^{it_j\lambda_j}\}$ in $L^2(R^n)$.

5) For the innovation approach to *random fields*, white noise with multi-dimensional parameter plays essential roles. It is often denoted by $W(t)$ and appears in an explicit form, where t may run through higher dimensional Euclidean space.

Note. We shall not state explicitly, but similar theory can be proceeded in the case where the noise is Poisson.

4 Calculus on $(L^2)^-$ Having introduced a space of generalized white noise functionals, we are ready to discuss analysis of those functionals.

Since $\dot{B}(t)$ is one of the variables, we can define a partial *differential operator* ∂_t:

$$\partial_t = \frac{\partial}{\partial \dot{B}(t)}.$$

Its domain is rich including $(L^2)^+$.

The operator ∂_t is also called an annihilation operator. We can further define its adjoint operator, denoted by ∂_t^*, by using the Gel'fand triple introduced in the last section. It is called a *creation operator*. The domain is wide enough in $(L^2)^-$.

Theorem 3 The creation operator defines a stochastic integral in such a way that for $\varphi \in (L^2)^-$

$$\int \partial_t^* \varphi dt,$$

where the integrand is *not necessarily non-anticipating*.

Actual computations of the commutation relations of operators are listed in the following theorem.

Theorem 4

1) The following commutation relations hold

$$[\partial_t, \partial_s] = [\partial_t^*, \partial_s^*] = 0.$$

$$[\partial_s, \partial_t^*] = \delta(t-s)1.$$

2) Multiplication by $\dot{B}(t)$ is expressed by

$$\partial_t + \partial_t^*.$$

3) The generator $r_{t,s}$ of the rotation acting on $(\dot{B}(t), \dot{B}(s))$-plane is expressed in the form

$$r_{t,s} = \partial_t^* \partial_s - \partial_s^* \partial_t$$

and the commutation relations are as follows

$$[r_{s,t}, r_{t,u}] = \delta(0) r_{s,u}, \ \ s \neq u,$$

$$[r_{s,t}, \partial_u] = \delta(u-t)\partial_s - \delta(u-s)\partial_t,$$

$$[r_{s,t}, \partial_u^*] = \delta(u-t)\partial_s^* - \delta(u-s)\partial_t^*,$$

For further calculus, like harmonic analysis, can be proceeded by using operators arising from the collections of ∂_t and of the ∂_t^*. We shall omit those operators since there will be, at present, no connections with the biocomputations.

The analysis of Poisson noise functionals can be discussed in a similar manner to the Gaussian case, however it is noted that there are many dissimilarities between them, and they are quite significant. Even we can see *dualities* between two noises. But at present stage, we do not find good connections in applications, so they are omitted here.

5 Generalization to random fields Multi-dimensional parameter case can be dealt with in a similar manner with some modifications.

1) The R^d-parameter stochastic processes (random fields).

The most important process is again a Brownian motion, called Lévy's Brownian motion $B(a), a \in R^d$. We may tacitly understand that a is the space-time variable, but space and time will not necessarily be discriminated.

A Brownian motion $B(a)$ is defined by

1) It is a Gaussian system,

2) $B(b) - B(a)$ is Gaussian $N(0, |b-a|)$,

3) $B(o) = 0$, o being the origin of R^d.

To fix the idea, we now consider the case $d = 2$.

Set $a = (t, \theta)$, and let $\{\varphi_n(\theta), n = 0, 1, 2, \cdots\}$ be a complete orthonormal base of $L^2(S^1)$. Then, $X(a) = X(r, \theta)$ has Fourier series expansion:

$$X(t,\theta) = \sum_1^\infty X_n(t)\varphi_n(\theta),$$

where

$$X_n(t) = \int_0^{2\pi} X(t,\theta)\varphi_n(\theta)d\theta.$$

We know how to obtain the innovations which are family $\{\dot{B}_n(t)\}$ of white noises. There are, then, the canonical representations of $X_n(t)$ in terms of the white noises $\dot{B}_n(t), n \geq 1$:

$$X_n(t) = \int_0^t F_n(t,u)\dot{B}(u)du.$$

By taking all of the $\dot{B}_n$, we can proceed the analysis. Therefore, if a random fields are functionals of Brownian motion or white noises, we can discuss analysis of them in line with innovation approach, namely within the framework of white noise analysis that has been established so far. And, of course a similar and somewhat complicated analysis follows. The idea is, however, just the same.

In short, a functional of $B(a), a \in R^2, |a| \leq t$, can be expressed as a functional of $X_n(s), |s| \leq t, n \geq 1$. In this sense, the *causality* holds. Namely, the information obtained by $X(a), |a| \leq t$, is exactly equal to that obtained by the innovation $B_n(s), s \leq t, n \geq 1$. This can be proved by using the fact that the representation of $X_n(t)$ is canonical in terms of the $\dot{B}_n(t)$ for every n.

6 Applications There are various directions of application. We shall restrict our attention to biological fields. We shall list some of fruitful directions of applications below.

i) Partial differential operator $\partial_t = \frac{\partial}{\partial \dot{B}(t)}$ and its adjoint operator ∂_t^* have been defined, and they are used efficiently in the calculus. If the input signal is expressed in terms of white noise, the instanteneous change of the input at instant t may be expressed by ∂_t.

ii) With the help of the creation operators ∂_t^*, the stochastic integral can be defined. The integral is more general and even simpler than the classical stochastic integral.

iii) Stochastic differential equations can be dealt with in a similar manner to the non random case in the world of white noise.

iv) Apply to physics and engineering ; like representation of the kinetic energy of a Brownian movement (e.g. **Feynman's path integral**), generalized white noise functionals are involved and we can proceed calculus.

v) Applications to random fields. See [11].

Some examples are now in order.

1. Method of identification of a black box that admits a white noise input.

A good example can be seen in the Naka's approach to a method of identification of function of retina (of a catfish). See [12] and [13]. In the experimental laboratory, the input is taken to be a white noise, so that the output $X(t)$ is a functional of white noise. The $X(t)$ is the observed data so that it is to be expressed in the form

$$X(t) = f(\dot{B}(s), s \le t; t).$$

It can be a causal representation, so $X(t)$ is a function only of $\dot{B}(s), s \le t$. We may assume that the system is stationary in t; that is, the passage of time, say $t \to t + h$, implies the shift of the input $\dot{B}(t) \to \dot{B}(t + h)$; as a result we have the shift from $X(t) \to X(t + h)$. Therefore, we may write

$$X(t) = f(\dot{B}(s), s \le t).$$

The unitary group $\{U_t, t \in R^1\}$ acts in the form

$$U(t)f(\dot{B}(s), s \in R^1) = f(\dot{B}(s + t), s \in R^1).$$

First we have a quick review of the classical method of the identification, which is the determination of f.

We know the spectrum of the $U(t)$ is of countably Lebesgue type, more precisely, for the linear (Gaussian) component of $X(t)$ has single Lebesgue spectrum, and for the nonlinear component (i.e. homogeneous chaos) of any degree $n \ge 2$ has Lebesgue spectrum with countably many multiplicity. It is known that $X(t)$ is a sum of the form

$$X(t) = \sum_{n=1}^{\infty} X_n(t),$$

where $X_n(t)$ is in H_n, i.e. a homogeneous chaos of degree n. For $n = 1$ it is a Gaussian process, and it is actually a cyclic subspace, namely there exists a single random variable, say X_1 and $U_s X_1 = X(s), s \in R^1$ spans the space H_1. To determine the structure of $X_1(t)$ we need to know the canonical representation expressed in the form

$$X_1(t) = U_t X_1 = \int_{-\infty}^{t} F_1(t - u)\dot{B}(u)du,$$

where $\dot{B}(u)$ is the innovation. The identification problem now turns out to the determination of the kernel F_1. It is a Volterra kernel that vanishes on the negative half line.

Wiener's idea (see [18]) is that we provide a channel the kernel (Volterra kernel) of which is known, say $G(s-u)$ such that it admits white noise input to have $Y(t)$. Then, we have the correlation $r(h)$ between $X_1(t)$ and $Y(t+h)$. By assumption we have a representation of $Y(s)$ as follows:

$$Y(s) = \int_{-\infty}^{s} G(s-u)\dot{B}(u)du.$$

Assume that the input to the two channels is the same $\dot{B}$. Let s vary and compute the correlation to have

$$r(h) = E(X(t)Y(t+h)) = \int F(t-u)G(t+h-u)du.$$

Let $\hat{F}(\lambda)$ and $\hat{G}(\lambda)$ be the Fourier transforms of F and G, respectively. Then, we have

$$r(h) = (2\pi)\int e^{ih\lambda}\hat{F}(\lambda)\hat{G}(\lambda)d\lambda.$$

We may assume that the known channel is constructed so that $\hat{G}(\lambda)$ never vanishes. Using the fact that the system $\{e^{ih\lambda}, h \in R^1\}$ is total in $L^2(R^1)$, we can conclude that F is uniquely determined by the information $r(h)$ just obtained.

Note The known channel that corresponds to the kernel G can actually constructed as an LCR circuit. The picture of this circuit can be seen in [18].

We then come to the determination of $X_n(t)$ for $n \geq 2$. Again, use the unitary group U_t. It is known that $X_n(t)$ lives in a space H_n of homogeneous chaos of degree n (n-ple Wiener integrals), where the U_t has the Lebesgue spectrum with countably many multiplicity. This means that H_n has direct sum decomposition into countably many cyclic subspaces which are mutually orthogonal and are isomorphic to H_1. Hence, in order to identify the component $X_n(t)$, we have to prepare infinitely many known LCR circuits and we compute correlations. Then, we must repeat the above procedure infinitely many times at least theoretical level to identify the system.

In the concrete example like Naka's approch (see [13])to the identification of the function of letina, he can choose finitely many significant cyclic subspaces chosen according to the biologist's experience and had enough information for the identification.

We can, in what follows, propose a method of using the partial differential operators $\partial_t, t \in R^1$. The steps are now in order.

(1) The expressions of $X(t)$ and $X_n(t)$ are the same as before.

(2) In terms of the Wick products $:\dot{B}(t_1)\dot{B}(t_2)\cdots\dot{B}(t_n):$, the $X_n(t)$ can be written as

$$X_n(t) = \int^t \int^t \cdots \int^t F(t-u_1, t-u_2, \cdots, t-u_n) : \dot{B}(u_1)\dot{B}(u_2)\cdots\dot{B}(u_n) : du^n,$$

which is written as

$$X_n(t) = n! \int^t \int^{u_1} \cdots \int^{u_{n-1}} F(t-u_1, t-u_2, \cdots, t-u_n) : \dot{B}(u_1)\dot{B}(u_2)\cdots\dot{B}(u_n) : du^n.$$

Remind that ∂_t is an annihilation operator. Then, we have

Theorem 5 The kernel F_n is obtained by

$$E(\partial_{t_1}\partial_{t_2}\cdots\partial_{t_n}X(t)) = n!F(t-t_1, t-t_2, \cdots, t-t_n),$$

where $t \geq t_1 \geq t_2 \geq \cdots \geq t_n$.

Proof. By the n-th order partial differential operator the components $X_k(t), k < n$ disappear. As for the components $X_j(t), j > n$, vanishes by the expectation after compound differential operator is applied. Thus remains only F_n to conclude the assertion of the theorem.

By this theorem the kernels can be obtained theoretically.

There remain several questions. First, one may ask if it is possible to change the input a little instantaneously to realize the derivative. Continuously many points $(t_1, t_2, \cdots, t_n)$ can not be chosen, but we wonder how many n-tupples should be chosen in oreder to have reasonable approximation of the kernel F_n. May we assume that F_n's are smooth enlough ? And so forth.

2. Stochastic differential equations for stochastic processes.

Example 1. Bilinear fluctuation.

$$\frac{d}{dt}X(t) = aX(t) + \partial_t^*(bX(t) + b').$$

The solution $X(t)$ is given by the kernels. The n-th kernel is

$$F_n(u_1, \cdots, u_n) = \frac{1}{n!}b^{n-1}b'e^{-a\min_j u_j}\chi_{(-\infty,0]^n}(u_1, \cdots, u_n).$$

Example 2. Modified Langevin equation, which can be solved easily.

$$\frac{d}{dt} = -\lambda X(t) + a : \dot{B}(t)^2 :$$

Example 3. Consider the equation

$$\partial_t\varphi(\dot{B}) = a\partial_t^*\varphi(\dot{B}),\ t \geq 0,$$

with $a > 0$ and

$$< 1, \varphi >= 1.$$

Then, the solution is a Gauss kernel

$$\varphi_c(\dot{B}) = N\exp[c\int : \dot{B}(s)^2 : ds],$$

where $c = \frac{a}{2a+1}$ and N is the renormalizing constant.

Incidentally, the solution is an eigenfunctional of the Lévy Laplacian.

3. Stochastic variational equations for random fields.

Directions to random fields and applications to biology would be very fruitful area and attractive for us.

Historically speaking, we are influenced by the monographs [16], [17] and [9] from mathematical side and have come to variational calculus. Then, we have gradually come to interesting problems arising from natural phenomena expressed as random fields which are evolutional. We have discussed the random fields such as $X(a), a \in R^d$, and as $X(C)$ parameterized by a smooth manifold, in fact ovaloid, C that runs through a multi-dimensional Euclidean space (see papers in [1] and [5]).

At present, it can be said that we have always discussed in line with reduction; namely, by using the innovations of those fields. The innovation theory is somewhat complicated in these cases for random fields, but can be discussed in the same spirit as in the case of a stochastic process. We will, therefore, not go into details regarding the innovation, but we should like to mention two facts; one is the identification problem of a black box where the parameter is multi-dimensional. Lévy's Brownian motion $B(a)$ is most important and we can introduce white noise that is derived from the $B(s)$. Functionals of a Brownian motion can be dealt with after they are expressed as functionals of white noise.

Such a whitening procedure is much significamt in the case where functionals of Poisson noise are discussed.

The other fact is that the innovation of $X(C)$ is quite powerful for the analysis when we discuss the fields determined by stochastic variational equations. There classical theory of (non-random) variational calculus is applicable efficiently. To shift the theory to the random fields we have to determine a generalization of an additive process with R^1-parameter. As was seen in the step of whitening of Lévy's Brownian motion, the innovation can not be given by simple differential, but by complicated computations.

While we were trying to find a suitable method of analysis of random fields $X(C)$, we have come across a example that is related to generic images in geometry (see e.g. [11]). They are represented by image plus fluctuation, where the fluctuation is taken to be a genralization of the Lévy noise with higher dimensional parameter. We need profound considerations to clarify the Lévy-Itô decomposition of a Lévy process in higher dimensional parameter case.

We have recently recognized an interesting approach to a medical image processing in [2] where the idea by Mumford (cf [11]) is seen. Having been stimulated by those literatures, we have made some contribution with the hope that the results would make contribution to BIOCOMP.

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Takeyuki Hida
Meijo University
Nagoya, Japan
E-mail: thida@ccmfs.meijo-u.ac.jp