# ASYMPTOTICS AND EVALUATIONS OF FPT DENSITIES THROUGH VARYING BOUNDARIES FOR GAUSS-MARKOV PROCESSES

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Abstract. For Gauss-Markov processes the asymptotic behaviors of the first passage time probability density functions through certain time-varying boundaries are determined. Computational results for Wiener, Ornstein-Uhlenbeck and Brownian bridge processes show that for certain large boundaries and for large times excellent asymptotic approximations hold for such densities.

**1 Introduction and mathematical background** First-passage time (FPT) probability density functions (pdf's) through generally time-dependent boundaries play a relevant role in a variety of problems in biology, physics and engineering (see, for instance, [16] and related references). Since closed form results for diffusion and Gauss-Markov processes are scarce, efforts have been made to devise numerical algorithms and simulation techniques for their evaluation (cf. [1]–[4],[8], [9], [11], [15], [17], [19]) and for the analysis of their asymptotic behaviors as boundaries or time grow large (cf. [5], [6], [7], [12]–[14], [18]).

The present contribution, focusing on Gauss-Markov processes, is the natural extension of previous investigations on the asymptotic behavior of FPT pdf's in the presence of single asymptotically constant boundaries or of single asymptotically periodic boundaries carried out for one-dimensional diffusion processes admitting steady state densities ([7], [13], [14], [18]) and for a class of stationary Gaussian processes ([5], [6]). Some preliminary investigations for Gauss-Markov processes [12] have shown that for the estimated FPT pdf's through certain large boundaries for large times excellent asymptotic approximations hold. The validity of such unexpected computational results is confirmed in this paper by theoretical considerations.

We shall briefly recall some basic notation of Gauss-Markov processes that will be used throughout this paper.

Let $\{Z(t), t \in T\}$, where $T$ is a continuous parameter set, be a real continuous Gauss-Markov process. It can be represented as:

$$Z(t) = m(t) + h_2(t)\, W\big[r(t)\big], \tag{1.1}$$

where

*(i)* $\{W(t), t \geq 0\}$ denotes the standard Wiener process such that $P\{W(0) = 0\} = 1$, $E[W(t)] = 0$ and $E[W(s)\, W(t)] = \min(s, t)$;

*(ii)* $m(t) := E[Z(t)]$ is continuous in $T$;

*(iii)* the covariance $c(s, t) := E\{[Z(s) - m(s)]\, [Z(t) - m(t)]\}$ is continuous in $T \times T$, with $c(s, t) = h_1(s)\, h_2(t)$ $(s < t)$;

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*(iv)* $\{Z(t)\}$ is nonsingular except possibly at the end points of $T$, i.e. if $T = [a, b]$, $\{Z(t)\}$ has a nonsingular normal distribution except possibly at $t = a$ or $t = b$, where $Z(t)$ could be equal to $m(t)$ with probability one.

*(v)* $r(t) = h_1(t)/h_2(t)$ is a monotonically increasing function and $h_1(t)\,h_2(t) > 0$ because of the assumed nonsingularity of the process on the interior of $T$.

The transition pdf $f_Z(x,t|y,\tau)$ of a Gauss-Markov process is a normal density characterized by mean and variance:

$$
\begin{aligned}
&E[Z(t) \mid Z(\tau)=y] = m(t) + \frac{h_2(t)}{h_2(\tau)}\,\big[y - m(\tau)\big],\\
&Var[Z(t) \mid Z(\tau)=y] = h_2(t)\Big[h_1(t) - \frac{h_2(t)}{h_2(\tau)} h_1(\tau)\Big],
\end{aligned}
\tag{1.2}
$$

where $t, \tau \in T, \tau < t$. It satisfies the Fokker-Planck equation and the associated initial condition

$$
\begin{aligned}
&\frac{\partial f(x,t|y,\tau)}{\partial t} = -\,\frac{\partial}{\partial x}\,[A_1(x,t)\,f(x,t|y,\tau)] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,[A_2(t)\,f(x,t|y,\tau)],\\
&\lim_{\tau\uparrow t} f(x,t|y,\tau) = \delta(x-y),
\end{aligned}
\tag{1.3}
$$

with $A_1(x,t)$ and $A_2(t)$ given by

$$
A_1(x,t) = m'(t) + [x - m(t)]\,\frac{h_2'(t)}{h_2(t)}\,, \qquad A_2(t) = h_2^2(t)\,r'(t),
\tag{1.4}
$$

the prime denoting derivative with respect to the argument.

Let $\{X(t), t \geq 0\}$ be the non-stationary Ornstein-Uhlenbeck (OU) process with zero mean and covariance function $E[X(s)X(t)] = \sigma^2\left(e^{\beta\,s} - e^{-\beta\,s}\right)e^{-\beta\,t}/(2\,\beta)$ with $s < t$ and $\beta > 0$. Then (cf., for instance, [10]),

$$
X(t) = e^{-\beta\,t}\,W\Big[\frac{\sigma^2}{2\,\beta}\left(e^{2\,\beta\,t} - 1\right)\Big] \qquad (t \geq 0).
\tag{1.5}
$$

The infinitesimal moments (1.4) of the non-stationary OU process $X(t)$ are given by

$$
A_1(x) = -\beta\,x, \qquad A_2 = \sigma^2 \qquad (x \in \mathbb{R},\ \beta > 0,\ \sigma > 0).
$$

Due to (1.1) and (1.5), any Gauss-Markov process can be represented in terms of a non-stationary OU process as follows:

$$
Z(t) = m(t) + k(t)\;X\big[\varphi(t)\big] \qquad (t \in T),
\tag{1.6}
$$

where we have set:

$$
k(t) = h_2(t)\,\sqrt{1 + \frac{2\,\beta}{\sigma^2}\,r(t)}, \qquad \varphi(t) = \frac{1}{2\,\beta}\,\ln\Big(1 + \frac{2\,\beta}{\sigma^2}\,r(t)\Big).
\tag{1.7}
$$

Note that, by virtue of *(v)*, from (1.7) $\varphi(t) : T \to [0, +\infty)$ is a continuous and monotonically increasing function and $k(t)$ is a continuous non-vanishing function in the interior of $T$.

We shall now focus our attention on the random variable

$$\mathcal{T}_z = \inf_{t \geq \tau} \{t : Z(t) > S(t)\}, \qquad Z(\tau) = z < S(\tau), \quad \tau, t \in T, \tag{1.8}$$

that represents the FPT of $Z(t)$ from $Z(\tau) = z$ to the supposed continuous boundary $S(t)$. Making use of (1.1), the FPT pdf $g_Z$ of $Z(t)$ can be obtained from the FPT pdf $g_W$ of the Wiener process $W(t)$ as:

$$g_Z[S(t), t \mid z, \tau] := \frac{\partial}{\partial t} P\big(\mathcal{T}_z < t\big) = \frac{dr(t)}{dt}\, g_W \left\{ \frac{S(t) - m(t)}{h_2(t)}, r(t) \,\Big|\, \frac{z - m(\tau)}{h_2(\tau)}, r(\tau) \right\} \tag{1.9}$$

for $z < S(\tau)$. By virtue of (1.6), $g_Z$ can be also expressed in terms of the FPT pdf $g_X$ of the non-stationary OU process $X(t)$ as follows:

$$g_Z[S(t), t \mid z, \tau] = \frac{d\varphi(t)}{dt}\, g_X \Big[ \frac{S(t) - m(t)}{k(t)}, \varphi(t) \,\Big|\, \frac{z - m(\tau)}{k(\tau)}, \varphi(\tau) \Big], \qquad z < S(\tau). \tag{1.10}$$

As proved in [4], if $S(t), m(t), h_1(t), h_2(t)$ are $C^1(T)$ functions, the FPT pdf of the Gauss-Markov process $Z(t)$ satisfies the following nonsingular second-kind Volterra integral equation

$$g_Z[S(t), t \mid z, \tau] = -2\, \Psi_Z[S(t), t \mid z, \tau] + 2 \int_\tau^t g_Z[S(\zeta), \zeta \mid z, \tau]\, \Psi_Z[S(t), t \mid S(\zeta), \zeta]\, d\zeta \qquad [z < S(\tau)], \tag{1.11}$$

where

$$\Psi_Z[S(t), t \mid z, \vartheta] = \left\{ \frac{S'(t) - m'(t)}{2} - \frac{S(t) - m(t)}{2}\, \frac{h_1'(t)\, h_2(\vartheta) - h_2'(t)\, h_1(\vartheta)}{h_1(t)\, h_2(\vartheta) - h_2(t)\, h_1(\vartheta)} - \frac{z - m(\vartheta)}{2}\, \frac{h_2'(t)\, h_1(t) - h_2(t)\, h_1'(t)}{h_1(t)\, h_2(\vartheta) - h_2(t)\, h_1(\vartheta)} \right\} f_Z[S(t), t \mid z, \vartheta]. \tag{1.12}$$

In Section 2 we shall focus on the asymptotics of the FPT pdf for the non-stationary OU process $X(\vartheta)$ through a continuous and bounded boundary $\eta(\vartheta)$ that is asymptotically constant or asymptotically periodic. In Section 3, making use of the transformation (1.6), we shall show that the asymptotic results of the FPT pdf for the non-stationary OU process through the boundary $\eta(\vartheta)$ can be used in order to obtain quantitative information on the FPT pdf of a Gauss-Markov process $Z(t)$ through the transformed boundary $S(t) = m(t) + k(t)\, \eta[\varphi(t)]$. In Section 4, by making use of the non-singular integral equation (1.11), a numerical algorithm with variable step-size will be proposed. Finally, extensive computations will be performed for special Gauss-Markov processes with the aim to pinpoint the goodness of the asymptotic behaviors of FPT densities.

**2 Non-stationary OU process** In this section we study the asymptotic behavior of the FPT pdf $g_X[\eta(\vartheta), \vartheta \mid y, \vartheta_0]$ for the non-stationary OU process $X(\vartheta)$ through a continuous and bounded boundary $\eta(\vartheta)$. From (1.11) and (1.12) it follows that $g_X$ is solution of the Volterra integral equation

$$g_X[\eta(\vartheta), \vartheta \mid y, \vartheta_0] = -2\, \Psi_X[\eta(\vartheta), \vartheta \mid y, \vartheta_0] + 2 \int_{\vartheta_0}^{\vartheta} g_X[\eta(\zeta), \zeta \mid y, \vartheta_0]\, \Psi_X[\eta(\vartheta), \vartheta \mid \eta(\zeta), \zeta]\, d\zeta \qquad [y < \eta(\vartheta_0)], \tag{2.1}$$

where

$$\Psi_X[\eta(\vartheta),\vartheta|y,\vartheta_0] = \frac{1}{2}\Big\{\frac{d\eta(\vartheta)}{d\vartheta} - \beta\,\frac{\eta(\vartheta)\,[1+e^{-2\beta(\vartheta-\vartheta_0)}] - 2\,y\,e^{-\beta(\vartheta-\vartheta_0)}}{1-e^{-2\beta(\vartheta-\vartheta_0)}}\Big\}\, f_X[\eta(\vartheta),\vartheta|y,\vartheta_0] \qquad (\vartheta>\vartheta_0\ge 0), \tag{2.2}$$

the transition pdf of $X(\vartheta)$ being given by

$$f_X(x,\vartheta \mid y,\vartheta_0) = \sqrt{\frac{\beta}{\pi\,\sigma^2\,\big(1-e^{-2\beta(\vartheta-\vartheta_0)}\big)}}\ \exp\Big\{-\frac{\beta\,\big[x-y\,e^{-\beta(\vartheta-\vartheta_0)}\big]^2}{\sigma^2\,\big(1-e^{-2\beta(\vartheta-\vartheta_0)}\big)}\Big\} \qquad (x,y\in\mathbb{R},\vartheta>\vartheta_0\ge 0). \tag{2.3}$$

Denote by

$$G_X[\eta(\vartheta),\vartheta \mid y,\vartheta_0] := \int_{\vartheta_0}^{\vartheta} g_X[\eta(u),u \mid y,\vartheta_0]\,du, \qquad y<\eta(\vartheta_0), \tag{2.4}$$

the FPT distribution function of $X(\vartheta)$. Since the first passage through a constant boundary for a non-stationary OU process is a sure event, it follows that also the first passage of $X(\vartheta)$ through a continuous and bounded boundary $\eta(\vartheta)$ is a sure event, i.e.

$$\int_{\vartheta_0}^{+\infty} g_X[\eta(u),u \mid y,\vartheta_0]\,du = 1, \qquad y<\eta(\vartheta_0). \tag{2.5}$$

Resorting to the results obtained in [7] for a class of one-dimensional diffusion processes, we now focus our analysis on the non-stationary OU process by considering separately two cases: *(i)* $\eta(\vartheta)$ is an asymptotically constant boundary and *(ii)* $\eta(\vartheta)$ is an asymptotically periodic boundary.

**2.1 Asymptotically constant boundary** We consider the FPT problem for the non-stationary OU process through the asymptotically constant boundary

$$\eta(\vartheta) = S + \varrho(\vartheta), \qquad [S\in\mathbb{R},\vartheta\ge 0], \tag{2.6}$$

where $\varrho(\vartheta)\in C^1[0,+\infty)$ is a bounded function independent of $S$ and such that

$$\lim_{\vartheta\to+\infty}\varrho(\vartheta)=0 \qquad \text{and} \qquad \lim_{\vartheta\to+\infty}\dot\varrho(\vartheta)=0. \tag{2.7}$$

The function $\Psi_X[\eta(\vartheta),\vartheta \mid y,\vartheta_0]$, given in (2.2), approaches a constant value as $\vartheta$ increases. Indeed, taking the limit $\vartheta\to+\infty$ in (2.2) and making use of (2.6) and (2.7), for all $\vartheta_0,\vartheta$ $(\vartheta>\vartheta_0\ge 0)$ there holds:

$$R(S) := -2\lim_{\vartheta\to+\infty}\Psi_X[\eta(\vartheta),\vartheta \mid y,\vartheta_0] = \beta\,S\,\sqrt{\frac{\beta}{\pi\,\sigma^2}}\ \exp\Big\{-\frac{\beta\,S^2}{\sigma^2}\Big\}, \qquad y<\eta(\vartheta_0). \tag{2.8}$$

We note that $R(S)>0$ for all $S>0$ and

$$\lim_{S\to+\infty} R(S) = 0. \tag{2.9}$$

**Proposition 2.1** *For the non-stationary OU process $X(\vartheta)$, let $\eta(\vartheta)=S+\varrho(\vartheta)$ be an asymptotically constant boundary, with $S\in\mathbb{R}$ and with $\varrho(\vartheta)\in C^1[0,+\infty)$ a bounded function independent of $S$ such that (2.7) hold. Then,*

$$\lim_{S\to+\infty}\frac{1}{R(S)}\,g_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\,\Big|\,y,\vartheta_0\Big] = e^{-\vartheta}, \qquad y<\eta(\vartheta_0), \tag{2.10}$$

*for $\vartheta>0$ and $\vartheta_0\ge 0$, with $R(S)$ defined in (2.8).*

**Proof.** Changing $\vartheta$ in $\vartheta+\vartheta_0$ in (2.1), for the non-stationary OU process $X(t)$ one has

$$
\begin{aligned}
&g_X[\eta(\vartheta+\vartheta_0),\vartheta+\vartheta_0 \mid y,\vartheta_0] = -2\,\Psi_X[\eta(\vartheta+\vartheta_0),\vartheta+\vartheta_0 \mid y,\vartheta_0]\\
&\quad +2\int_{\vartheta_0}^{\vartheta+\vartheta_0} g_X[\eta(\zeta),\zeta \mid y,\vartheta_0]\,\Psi_X[\eta(\vartheta+\vartheta_0),\vartheta+\vartheta_0 \mid \eta(\zeta),\zeta]\,d\zeta, \qquad y<\eta(\vartheta_0).
\end{aligned}
\tag{2.11}
$$

If $R(S)>0$, changing $\vartheta$ in $\vartheta/R(S)$ and making use of the change of variable $\zeta = u/R(S)+\vartheta_0$ in (2.11), for $\vartheta_0 \geq 0$ and $\vartheta>0$ we obtain:

$$
\begin{aligned}
&\frac{1}{R(S)}\,g_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\Big|y,\vartheta_0\Big] = -\frac{2}{R(S)}\,\Psi_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\Big|y,\vartheta_0\Big]\\
&\quad +\int_0^{\vartheta}\Big\{\frac{1}{R(S)}\,g_X\Big[\eta\Big(\frac{u}{R(S)}+\vartheta_0\Big),\frac{u}{R(S)}+\vartheta_0\Big|y,\vartheta_0\Big]\Big\}\\
&\quad \times\Big\{\frac{2}{R(S)}\,\Psi_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\Big|\eta\Big(\frac{u}{R(S)}+\vartheta_0\Big),\frac{u}{R(S)}+\vartheta_0\Big]\Big\}\,du,
\end{aligned}
\tag{2.12}
$$

with $y<\eta(\vartheta_0)$. By virtue of (2.2), making use of (2.8) and (2.9), one has

$$
\lim_{S\to+\infty}\frac{2}{R(S)}\,\Psi_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\Big|y,\vartheta_0\Big] = -1,
$$

for $\vartheta_0 \geq 0, \vartheta>0$, and

$$
\lim_{S\to+\infty}\frac{2}{R(S)}\,\Psi_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\Big|\eta\Big(\frac{u}{R(S)}+\vartheta_0\Big),\frac{u}{R(S)}+\vartheta_0\Big] = -1
$$

for $\vartheta_0\geq 0,\ 0<u<\vartheta$. Hence, taking the limit as $S\to+\infty$ in (2.12), one finally obtains (2.10). □

**Corollary 2.1** *Under the assumptions of Proposition 2.1, for $S\to+\infty$ and for large times one has:*

$$
g_X\big[\eta(\vartheta),\vartheta \mid y,\vartheta_0\big] \simeq R(S)\,e^{-R(S)\,(\vartheta-\vartheta_0)}, \qquad y<\eta(\vartheta_0),
\tag{2.13}
$$

*with $0\leq\vartheta_0<\vartheta<+\infty$, where $R(S)$ is given in (2.8).*

**Proof.** By virtue of (2.10), for $S\to+\infty$ one obtains:

$$
g_X\Big[\eta\Big(\frac{\vartheta}{R(S)}+\vartheta_0\Big),\frac{\vartheta}{R(S)}+\vartheta_0\;\Big|\;y,\vartheta_0\Big] \simeq R(S)\,e^{-\vartheta},
$$

so that by changing $\vartheta/R(S)+\vartheta_0$ in $\vartheta$ one is immediately led to (2.13). □

Corollary 2.1 expresses the asymptotic exponential trend of the FPT density of the non-stationary OU process as the asymptotically constant boundary moves away from the starting point.

The right-hand side of (2.13) has the following functional form:

$$
\gamma_X(\vartheta \mid \vartheta_0) = \lambda\,e^{-\lambda\,(\vartheta-\vartheta_0)} \qquad (0\leq\vartheta_0<\vartheta<+\infty),
\tag{2.14}
$$

with $\lambda > 0$. We now denote by $\tilde{g}_X\big[\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0\big]$ the estimated FPT pdf and by $\tilde{G}_X[\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0]$ the estimated FPT distribution function at times $\vartheta_k = \vartheta_0 + k\,p$ $(k = 1, 2, \dots, N)$, where $p > 0$ is the time discretization step. We then use the method of least squares to fit the computed FPT distribution function $\tilde{G}_X[\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0]$ with the exponential distribution function:

$$(2.15) \qquad \Gamma_X(\vartheta \mid \vartheta_0) := \int_{\vartheta_0}^{\vartheta} \gamma_X(u \mid \vartheta_0)\, du = 1 - e^{-\lambda(\vartheta - \vartheta_0)} \qquad (0 \le \vartheta_0 < \vartheta < +\infty).$$

To this aim, we evaluate the minimum with respect to $\lambda$ of the function:

$$\sum_{k=1}^{N} \Big\{ \ln\big[1 - \Gamma_X(\vartheta_k \mid \vartheta_0)\big] - \ln\big[1 - \tilde{G}_X\big(\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0\big)\big] \Big\}^2$$
$$= \sum_{k=1}^{N} \Big\{ \lambda\,(\vartheta_k - \vartheta_0) + \ln\big[1 - \tilde{G}_X\big(\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0\big)\big] \Big\}^2,$$

which is equivalent to solving the equation

$$\sum_{k=1}^{N} (\vartheta_k - \vartheta_0) \ln\Big\{1 - \tilde{G}_X\big[\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0\big]\Big\} + \lambda \sum_{k=1}^{N} (\vartheta_k - \vartheta_0)^2 = 0,$$

with respect to $\lambda$. Hence, the least squares estimate of $\lambda$ can be determined as (cf., for instance, [5]):

$$\widehat{\lambda} = -\frac{\sum_{k=1}^{N} \big(\vartheta_k - \vartheta_0\big) \ln\Big\{1 - \tilde{G}_X[\eta(\vartheta_k), \vartheta_k \mid y, \vartheta_0]\Big\}}{\sum_{k=1}^{N} \big(\vartheta_k - \vartheta_0\big)^2}$$
$$(2.16) \qquad = -\frac{6 \sum_{k=1}^{N} k \ln\Big\{1 - \tilde{G}_X[\eta(\vartheta_0 + k\,p), \vartheta_0 + k\,p \mid y, \vartheta_0]\Big\}}{N\,(N+1)\,(2N+1)\,p},$$

with $\vartheta_k = \vartheta_0 + k\,p$ $(k = 1, 2, \dots, N)$, where $p > 0$ is the time discretization step.

Table 1 shows, for some choices of the constant boundary $\eta(\vartheta) = S$, the values of $\widehat{\lambda}$, obtained by means of (2.16) with integration step $10^{-2}$, the values of $R(S)$, obtained via (2.8), and the relative error $e_r(S) = \{R(S) - \widehat{\lambda}\}/R(S)$ for the non-stationary OU process with $\beta = 1$ and $\sigma^2 = 2$ originating in $y = 0$ at time $\vartheta_0 = 0$. We note that for $S \ge 2.7$ the relative error $e_r(S)$ decreases as the boundary increases.

Figure 1 shows the evaluated FPT density $\tilde{g}_X(\vartheta) = \tilde{g}_X(S, \vartheta|0, 0)$ with $S = 2.5$ and the exponential density $\gamma_X(\vartheta \mid 0) = \lambda \exp(-\lambda\,\vartheta)$, with $\lambda$ estimated by means of (2.16), for the non-stationary OU process with $\beta = 1$ and $\sigma^2 = 2$. Use has been made of the numerical algorithm proposed in [4] with integration step $10^{-2}$. The goodness of the exponential approximation increases as the boundary is progressively moved away from the starting point of the process.

**2.2 Asymptotically periodic boundary** We shall now focus on the FPT problem of non-stationary OU process for an asymptotically periodic boundary (2.6), with $S \in \mathbb{R}$ and where $\varrho(\vartheta) \in C^1[0, +\infty)$ is a bounded function independent of $S$ such that

$$(2.17) \qquad \lim_{k \to +\infty} \varrho(\vartheta + k\,Q) = V(\vartheta) \qquad \text{and} \qquad \lim_{k \to +\infty} \dot{\varrho}(\vartheta + k\,Q) = \dot{V}(\vartheta),$$

| $S$ | $\widehat{\lambda}$ | $R(S)$ | $e_r(S)$ | $S$ | $\widehat{\lambda}$ | $R(S)$ | $e_r(S)$ |
|---|---|---|---|---|---|---|---|
| 2.1 | 0.081498 | 0.092366 | 0.11766 | 3.1 | 0.0088390 | 0.010127 | 0.12718 |
| 2.2 | 0.067961 | 0.078044 | 0.12920 | 3.2 | 0.0066986 | 0.0076291 | 0.12197 |
| 2.3 | 0.056229 | 0.065152 | 0.13696 | 3.3 | 0.0050220 | 0.0056845 | 0.11654 |
| 2.4 | 0.046139 | 0.053747 | 0.14155 | 3.4 | 0.0037241 | 0.0041895 | 0.11109 |
| 2.5 | 0.037532 | 0.043821 | 0.14352 | 3.5 | 0.0027314 | 0.0030544 | 0.10575 |
| 2.6 | 0.030239 | 0.035316 | 0.14376 | 3.6 | 0.0019814 | 0.0022028 | 0.10051 |
| 2.7 | 0.024125 | 0.028137 | 0.14259 | 3.7 | 0.0014215 | 0.0015717 | 0.09557 |
| 2.8 | 0.019060 | 0.022163 | 0.14001 | 3.8 | 0.0010086 | 0.0011094 | 0.09086 |
| 2.9 | 0.014907 | 0.017262 | 0.13643 | 3.9 | 0.00070783 | 0.00077476 | 0.08639 |
| 3.0 | 0.011539 | 0.013296 | 0.13215 | 4.0 | 0.00049129 | 0.00053532 | 0.08225 |

Table 1: For the non-stationary OU process $X(\vartheta)$ with $\beta = 1$ and $\sigma^2 = 2$ originating in $y = 0$ at time $\vartheta_0 = 0$, the values of $\widehat{\lambda}$ and $R(S)$ are shown for various choices of the constant boundary $\eta(\vartheta) = S$.

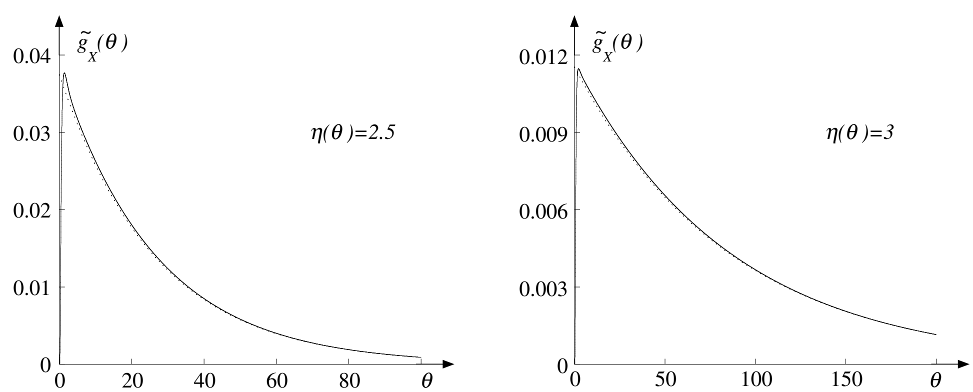



Figure 1: For the non-stationary OU process with $\beta = 1$ and $\sigma^2 = 2$, $\tilde{g}_X(\vartheta) = \tilde{g}_X[\eta(\vartheta), \vartheta|0, 0]$ (solid line) is compared with the exponential density $\gamma_X(\vartheta \mid 0) = \lambda \exp(-\lambda\,\vartheta)$ (dotted line), with $\lambda$ estimated as $\widehat{\lambda} = 0.037532$ for $\eta(\vartheta) = 2.5$ and as $\widehat{\lambda} = 0.011539$ for $\eta(\vartheta) = 3$.

where $V(\vartheta)$ is a periodic function of period $Q > 0$ satisfying

$$\int_0^Q V(u)\, du = 0. \tag{2.18}$$

Changing $\vartheta$ in $\vartheta + k\,Q$ in (2.2) and taking the limit as $k \to +\infty$, by virtue of (2.6), (2.17) and (2.18), one obtains

$$\begin{aligned} R[V(\vartheta)] &:= -2 \lim_{k\to+\infty} \Psi_X[\eta(\vartheta + k\,Q), \vartheta + k\,Q \mid y, \vartheta_0] \\ &= \sqrt{\frac{\beta}{\pi\,\sigma^2}}\,\Big\{\beta\,[S + V(\vartheta)] - \dot{V}(\vartheta)\Big\}\,\exp\Big\{-\frac{\beta\,\big[S + V(\vartheta)\big]^2}{\sigma^2}\Big\} \end{aligned} \tag{2.19}$$

for all $y \in \mathbb{R}$ and for all $\vartheta_0, \vartheta$ ($\vartheta > \vartheta_0 \geq 0$). For all $\vartheta > 0$ the function $R[V(\vartheta)]$ defined in (2.19) is periodic with period $Q$. Furthermore, there exists an $S^* \in \mathbb{R}$ such that $R[V(\vartheta)] > 0$

for all $S > S^*$ and

$$\lim_{S\to+\infty} R[V(\vartheta)] = 0. \tag{2.20}$$

**Lemma 2.1** *For all $S > S^*$ let*

$$\alpha = \alpha(S) := \frac{1}{Q}\int_0^Q R[V(\vartheta)]\,d\vartheta, \tag{2.21}$$

*with $R[V(\vartheta)]$ defined in (2.19). Then, there exists a non-negative monotonically increasing function $\chi(\vartheta)$ which is a solution of*

$$\int_0^{\chi(\vartheta)} R[V(u)]\,du = \alpha\,\vartheta, \qquad \forall\vartheta \geq 0 \tag{2.22}$$

*such that*

$$\chi(0) = 0, \quad \lim_{\vartheta\to+\infty}\chi(\vartheta) = +\infty, \quad \chi(\vartheta + k\,Q) = \chi(\vartheta) + k\,Q \qquad (k = 0, 1, \dots). \tag{2.23}$$

**Proof.** Since $R[V(\vartheta)] > 0$ for all $S > S^*$, from (2.21) it follows $\alpha > 0$, and from (2.22) one has $\chi(0) = 0$ and $\chi(\vartheta) > 0$ for all $\vartheta > 0$. Let $h(\vartheta)$ be any primitive function of $R[V(\vartheta)]$. From (2.22) we have $h[\chi(\vartheta)] = h(0) + \alpha\,\vartheta$. Since $R[V(\vartheta)] > 0$, $\forall\vartheta > 0$, $\chi(\vartheta)$ possesses an inverse, and hence $\chi(\vartheta) = h^{-1}[h(0) + \alpha\,\vartheta]$. Furthermore, since $\alpha > 0$, from (2.22) one has:

$$\frac{d\chi(\vartheta)}{d\vartheta} = \frac{\alpha}{R\big\{V[\chi(\vartheta)]\big\}} > 0, \tag{2.24}$$

for all $S > S^*$. Therefore, $\chi(\vartheta)$ is a monotonically increasing function for all $\vartheta > 0$. Furthermore, since $R[V(\vartheta)]$ is a positive function for $S > S^*$, the second of (2.23) holds. We now remark that from (2.21) and (2.22) one has:

$$\begin{aligned}\int_{\chi(\vartheta)}^{\chi(\vartheta+k\,Q)} R[V(u)]\,du &= \int_0^{\chi(\vartheta+k\,Q)} R[V(u)]\,du - \int_0^{\chi(\vartheta)} R[V(u)]\,du = \alpha\,k\,Q \\ &= k\int_0^Q R[V(u)]\,du = \int_0^{k\,Q} R[V(u)]\,du,\end{aligned} \tag{2.25}$$

where the last equality follows since $R[V(\vartheta)]$ is a periodic function with period $Q$. Relation (2.25) finally implies the last of (2.23). $\square$

**Lemma 2.2** *For all $S > S^*$ and for all $\vartheta > 0$ one has:*

$$\begin{aligned}
&(i) && \chi\Big(\frac{\vartheta}{\alpha}\Big) > 0,\\
&(ii) && \frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big) = \frac{1}{R\Big\{V\Big[\chi\Big(\dfrac{\vartheta}{\alpha}\Big)\Big]\Big\}},\\
&(iii) && \lim_{S\to+\infty}\chi\Big(\frac{\vartheta}{\alpha}\Big) = +\infty,\\
&(iv) && \lim_{S\to+\infty}\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big) - \chi\Big(\frac{\vartheta_0}{\alpha}\Big)\Big] = +\infty \qquad (0 < \vartheta_0 < \vartheta).
\end{aligned}$$

**Proof.** Since $\chi(\vartheta)$ is a non-negative function for all $S > S^*$ and $\alpha > 0$, condition *(i)* follows from Lemma 2.1, whereas from (2.24) one immediately obtains *(ii)*. Making use of (2.20), from (2.21) we have:

$$\lim_{S\to+\infty} \alpha = \frac{1}{Q} \lim_{S\to+\infty} \int_0^Q R[V(\vartheta)]\, d\vartheta = 0, \tag{2.26}$$

that, due to the second of (2.23), implies *(iii)*. Finally, making use of the Lagrange mean-value theorem one has:

$$\chi\Big(\frac{\vartheta}{\alpha}\Big) - \chi\Big(\frac{\vartheta_0}{\alpha}\Big) = \frac{\vartheta-\vartheta_0}{\alpha}\, \dot\chi\Big(\frac{\zeta}{\alpha}\Big) = \frac{\vartheta-\vartheta_0}{R\Big\{V\Big[\chi\Big(\frac{\zeta}{\alpha}\Big)\Big]\Big\}} \qquad (0<\vartheta_0\le\zeta\le\vartheta), \tag{2.27}$$

where the last identity follows from *(ii)*. We note that, due to (2.20), there holds:

$$\lim_{S\to+\infty} R\Big\{V\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\Big\} = \lim_{\vartheta\to+\infty} R[V(\vartheta)] = 0,$$

so that *(iv)* follows after taking the limit as $S\to+\infty$ in (2.27). □

**Proposition 2.2** *For the non-stationary OU process $X(\vartheta)$, let $\eta(\vartheta) = S + \varrho(\vartheta)$ be an asymptotically periodic boundary, with $S\in\mathbb{R}$ and with $\varrho(\vartheta)\in C^1[0,+\infty)$ a bounded function independent of $S$ such that (2.17) and (2.18) hold. Then, for $\vartheta>0$ and $\vartheta_0\ge 0$ one has*

$$\lim_{S\to+\infty}\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\; g_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\,\Big|\, y,\vartheta_0\Big] = e^{-\vartheta}, \qquad y<\eta(\vartheta_0), \tag{2.28}$$

*with $\alpha$ defined in (2.21) and $\chi(\vartheta)$ in (2.22).*

**Proof.** If $S > S^*$, changing $\vartheta$ in $\chi(\vartheta/\alpha)$ and $\zeta = \chi(u/\alpha)+\vartheta_0$ in (2.11) for $\vartheta_0\ge 0$ and $\vartheta>0$ we obtain:

$$\begin{aligned}
&\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\; g_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\,\Big|\, y,\vartheta_0\Big]\\
&\quad = -2\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\; \Psi_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\,\Big|\, y,\vartheta_0\Big]\\
&\qquad +2\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\int_0^\vartheta \Big\{\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\; g_X\Big[\eta\Big(\chi\Big(\frac{u}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{u}{\alpha}\Big)+\vartheta_0\,\Big|\, y,\vartheta_0\Big]\Big\}\\
&\qquad\quad \times\Psi_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\,\Big|\, \eta\Big(\chi\Big(\frac{u}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{u}{\alpha}\Big)+\vartheta_0\Big]\, du,
\end{aligned} \tag{2.29}$$

with $y<\eta(\vartheta_0)$. By virtue of (2.2), making use of (2.19), (2.20), Lemma 2.1 and Lemma 2.2, one has

$$\lim_{S\to+\infty} 2\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\; \Psi_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\,\Big|\, y,\vartheta_0\Big] = -1,$$

for $\vartheta_0\ge 0, \vartheta>0$, and

$$\lim_{S\to+\infty} 2\Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\Psi_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\,\Big|\, \eta\Big(\chi\Big(\frac{u}{\alpha}\Big)+\vartheta_0\Big), \chi\Big(\frac{u}{\alpha}\Big)+\vartheta_0\Big] = -1$$

for $\vartheta_0\ge 0,\ 0<u<\vartheta$, so that taking the limit as $S\to+\infty$ in (2.29), one finally obtains (2.28). □

**Corollary 2.2** *Under the assumptions of Proposition 2.2, for large times as $S \to +\infty$ one has:*

$$g_X\big[\eta(\vartheta), \vartheta \mid y, \vartheta_0\big] \simeq R[V(\vartheta - \vartheta_0)] \exp\Big\{-\int_{\vartheta_0}^{\vartheta} R[V(u-\vartheta_0)]\, du\Big\}, \qquad y < \eta(\vartheta_0), \tag{2.30}$$

*with $0 \le \vartheta_0 < \vartheta < +\infty$ and $R[V(\vartheta)]$ given in (2.19). Furthermore, (2.30) can be also written as:*

$$g_X\big[\eta(\vartheta), \vartheta \mid y, \vartheta_0\big] \simeq \xi_X(\vartheta - \vartheta_0)\, e^{-\alpha\,(\vartheta - \vartheta_0)}, \qquad y < \eta(\vartheta_0), \tag{2.31}$$

*where $\xi_X(\vartheta)$ is a periodic function of period $Q$ given by*

$$\xi_X(\vartheta) = R[V(\vartheta)] \exp\Big\{\alpha\,\vartheta - \int_0^{\vartheta} R[V(u)]\, du\Big\}, \tag{2.32}$$

*with $\alpha$ defined in (2.21).*

**Proof.** By virtue of (2.28), recalling (2.22) and *(ii)* of Lemma 2.2, as $S \to +\infty$ one obtains:

$$\begin{aligned} g_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0 \,\Big|\, y, \vartheta_0\Big] &\simeq \Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]^{-1} e^{-\vartheta} \\ &= R\Big\{V\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\Big\} \exp\Big\{-\int_0^{\chi(\vartheta/\alpha)} R[V(u)]\, du\Big\}, \qquad y < \eta(\vartheta_0), \end{aligned} \tag{2.33}$$

with $0 \le \vartheta_0 < \vartheta < +\infty$. Hence, changing $\chi(\vartheta/\alpha) + \vartheta_0$ into $\vartheta$ in (2.33), one is led to (2.30). Furthermore, since $R[V(\vartheta)]$ is a periodic function of period $Q$, due to (2.21) and (2.22), one has:

$$\begin{aligned} \xi_X(\vartheta + k\,Q) &= R[V(\vartheta + k\,Q)] \exp\Big\{\alpha\,(\vartheta + k\,Q) - \int_0^{\vartheta + k\,Q} R[V(u)]\, du\Big\} \\ &= R[V(\vartheta)] \exp\Big\{\alpha\,\vartheta + k\int_0^{Q} R[V(u)]\, du - \int_0^{\vartheta + k\,Q} R[V(u)]\, du\Big\} \\ &= R[V(\vartheta)] \exp\Big\{\alpha\,\vartheta + k\int_{k\,Q}^{\vartheta + k\,Q} R[V(u)]\, du\Big\} = \xi_X(\vartheta). \end{aligned}$$

so that $\xi_X(\vartheta)$ is a periodic function of period $Q$. Hence, (2.31) and (2.32) immediately follow from (2.30). ◻

Corollary 2.2 shows that FPT pdf exhibits a non-homogeneous exponential behavior when the asymptotically periodic boundary moves away from the starting point of the non-stationary OU process.

The right-hand side of (2.31) has the following functional form:

$$\gamma_X(\vartheta \mid \vartheta_0) = R^*(\vartheta - \vartheta_0) \exp\Big\{-\int_{\vartheta_0}^{\vartheta} R^*(u - \vartheta_0)\, du\Big\}, \qquad (0 \le \vartheta_0 < \vartheta < +\infty), \tag{2.34}$$

where $R^*(\vartheta)$ is a periodic function of period $Q$. Note that (2.34) can also be re-written as

$$\gamma_X(\vartheta \mid \vartheta_0) = \xi^*_X(\vartheta - \vartheta_0)\, e^{-\alpha^*(\vartheta - \vartheta_0)} \qquad (0 \le \vartheta_0 < \vartheta < +\infty), \tag{2.35}$$

where

$$\alpha^* = \frac{1}{Q} \int_0^Q R^*(u)\, du \tag{2.36}$$

and

$$\xi^*_X(\vartheta) = R^*(\vartheta)\, \exp\left\{\alpha^*\, \vartheta - \int_{\vartheta_0}^{\vartheta} R^*(u-\vartheta_0)\, du\right\}. \tag{2.37}$$

By virtue of (2.34), it follows that the non-homogeneous exponential distribution is

$$\Gamma_X(\vartheta \mid \vartheta_0) := \int_{\vartheta_0}^{\vartheta} \gamma_X(u \mid \vartheta_0)\, du = 1 - \exp\left\{-\int_{\vartheta_0}^{\vartheta} R^*(u-\vartheta_0)\, du\right\}, \tag{2.38}$$

with $0 \le \vartheta_0 < \vartheta < +\infty$. Note that for $\vartheta = \vartheta_0 + k\, Q$ $(k = 1, 2, \dots)$ one has:

$$\begin{aligned}\Gamma_X(\vartheta_0 + k\, Q \mid \vartheta_0) &= 1 - \exp\left\{-\int_{\vartheta_0}^{\vartheta_0 + k\, Q} R^*(u-\vartheta_0)\, du\right\} = 1 - \exp\left\{-\int_0^{k\, Q} R^*(u)\, du\right\}\\ &= 1 - \exp\left\{-k\int_0^{Q} R^*(u)\, du\right\} = 1 - e^{-\alpha^* k\, Q}\end{aligned}$$

with $\alpha^*$ defined in (2.36). Hence, recalling (2.16), the least squares estimate of $\alpha^*$ is:

$$\widehat{\alpha} = -\frac{6\sum_{k=1}^{M} k\, \ln\left\{1 - \tilde{G}_X[\eta(\vartheta_0 + k\, Q), \vartheta_0 + k\, Q \mid\; y, \vartheta_0]\right\}}{M\,(M+1)\,(2\,M+1)\,Q}, \tag{2.39}$$

where the period $Q > 0$ is the time discretization step and where $\tilde{G}_X$ is the computed FPT distribution function. With such a value of $\widehat{\alpha}$, the function $\xi_X(\vartheta - \vartheta_0)$ can be finally evaluated via (2.31) as:

$$\tilde{\xi}_X(\vartheta - \vartheta_0) = e^{\widehat{\alpha}\,(\vartheta - \vartheta_0)}\, \tilde{g}_X\left[\eta(\vartheta), \vartheta \mid y, \vartheta_0\right], \tag{2.40}$$

where $\tilde{g}_X$ is the computed FPT pdf. Massive computations have shown that the function (2.40) exhibits a periodic behavior having the same period $Q$ of $V(\vartheta)$. Hence, the computed FPT pdf $\tilde{g}_X$ is susceptible of a non-homogeneous exponential approximation for asymptotically periodic boundaries, as far as these are not too close to the initial position of the non-stationary OU process. This is clearly indicated in Figure 2 in which the function $\tilde{g}_X(\vartheta) = \tilde{g}_X[\eta(\vartheta), \vartheta|0, 0]$ is plotted for a non-stationary OU process with $\beta = 1$, $\sigma^2 = 2$ and periodic boundary $\eta(\vartheta) = 3 + 0.2\, \sin(2\pi\, \vartheta/5)$. We have used the numerical algorithm proposed in [4] with the integration step taken as $10^{-2}$. The FPT pdf $\tilde{g}_X(\vartheta)$ exhibits damped oscillations having the same period $Q = 5$ as the boundary. The function $\tilde{\xi}_X(\vartheta)$ is finally obtained via (2.40), with the value $\widehat{\alpha} = 0.01229$ estimated via (2.39). This function is actually periodic to a very high degree of accuracy.

In Figure 3 is instead considered the boundary $\eta(\vartheta) = 3 + 0.1\, \sin(2\pi\, \vartheta/0.2)$. On the left the FPT pdf $\tilde{g}_X(\vartheta) = \tilde{g}_X[\eta(\vartheta), \vartheta|0, 0]$ (solid line) is compared with the exponential density $\alpha\, \exp(-\alpha\, \vartheta)$ (dotted line) with $\alpha$ estimated as $\widehat{\alpha} = 0.011950$, whereas on the right the function $\tilde{\xi}_X(\vartheta)$ is plotted.

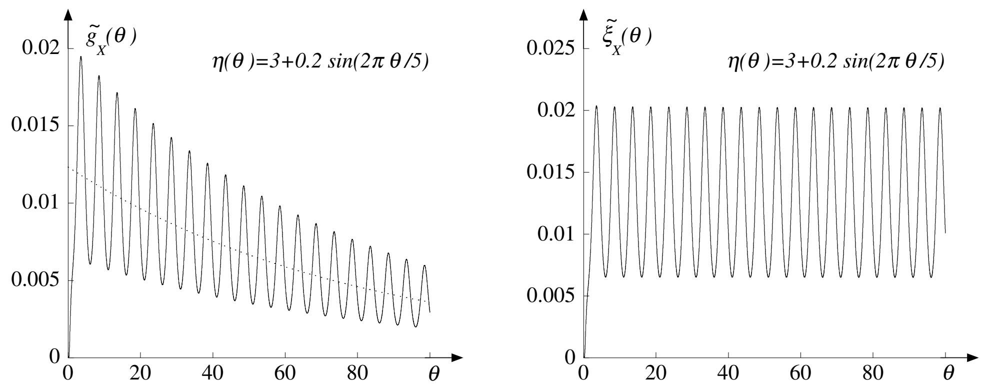



Figure 2: Non-stationary OU process with $\beta = 1$ and $\sigma^2 = 2$ and boundary $\eta(\vartheta) = 3 + 0.2\,\sin(2\pi\,\vartheta/5)$. On the left the FPT pdf $\tilde{g}_X(\vartheta) = \tilde{g}_X[\eta(\vartheta), \vartheta|0, 0]$ (solid line) is compared with the exponential density $\alpha\,\exp(-\alpha\,\vartheta)$ (dotted line), with $\alpha$ estimated as $\widehat{\alpha} = 0.012337$. On the right the function $\tilde{\xi}_X(\vartheta)$ computed via (2.40) is plotted.

**3 Asymptotic behavior of Gauss-Markov processes** In this section we prove that our results on the asymptotic behavior of the FPT pdf for the non-stationary OU process $X(\vartheta)$ through a continuous bounded boundary $\eta(\vartheta)$, asymptotically constant or asymptotically periodic, can be used in order to obtain quantitative information on the FPT pdf of a Gauss-Markov process $Z(t)$ through the transformed boundary

$$S(t) = m(t) + k(t)\,\eta\big[\varphi(t)\big] \qquad (t \in T), \tag{3.1}$$

with $k(t)$ and $\varphi(t)$ defined in (1.7). By virtue of (3.1), from (1.10) one obtains:

$$g_Z[S(t), t \mid z, \tau] = \frac{d\varphi(t)}{dt}\, g_X\Big\{\eta\big[\varphi(t)\big], \varphi(t)\Big|\frac{z - m(\tau)}{k(\tau)}, \varphi(\tau)\Big\}, \qquad z < S(\tau) \tag{3.2}$$

for $\tau < t$, $\tau, t \in T$. Denoting by $G_Z\big[S(t), t \mid z, \tau\big]$ the FPT distribution function of the Gauss-Markov process $Z(t)$, from (3.1) and (3.2) one has

$$\begin{aligned} G_Z\big[S(t), t \mid z, \tau\big] &:= \int_\tau^t g_Z\big[S(u), u \mid z, \tau\big]\, du \\ &= \int_\tau^t \frac{d\varphi(\zeta)}{d\zeta}\, g_X\Big\{\eta\big[\varphi(\zeta)\big], \varphi(\zeta)\Big|\frac{z - m(\tau)}{k(\tau)}, \varphi(\tau)\Big\}\, d\zeta \\ &= \int_{\varphi(\tau)}^{\varphi(t)} g_X\Big\{\eta(u), u\Big|\frac{z - m(\tau)}{k(\tau)}, \varphi(\tau)\Big\}\, du = G_X\Big\{\eta\big[\varphi(t)\big], \varphi(t)\Big|\frac{z - m(\tau)}{k(\tau)}, \varphi(\tau)\Big\}, \end{aligned} \tag{3.3}$$

where $G_X\big[\eta(\vartheta), \vartheta \mid y, \vartheta_0\big]$, defined in (2.4), is the FPT distribution function of the non-stationary OU process $X(\vartheta)$. If $\eta(\vartheta)$ $(\vartheta \geq 0)$ is a continuous and bounded boundary for $X(\vartheta)$ and if $\varphi(t) : T \to [0, +\infty)$ is such that

$$\lim_{t \to \sup T} \varphi(t) = +\infty, \tag{3.4}$$

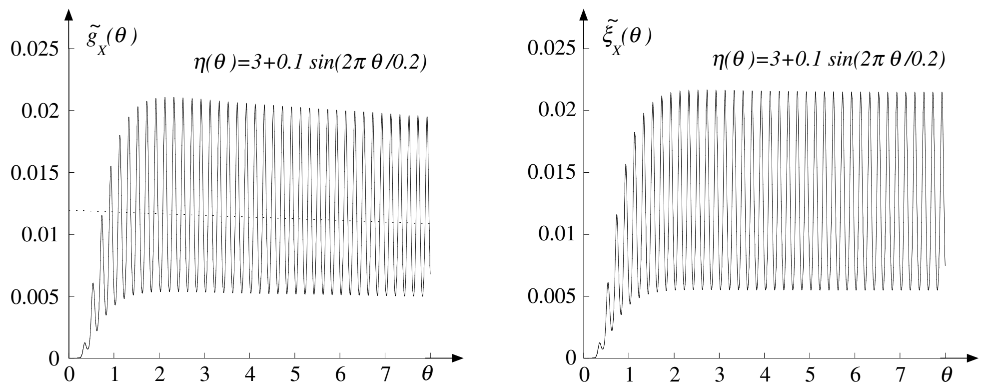



Figure 3: Same as in Figure 2 with $\eta(\vartheta) = 3 + 0.1 \sin(2\pi\,\vartheta/0.2)$ and $\widehat{\alpha} = 0.011950$. Integration step is $5 \cdot 10^{-3}$.

by virtue of (2.5) one has:

$$\int_{\tau}^{\sup T} g_Z\,[S(t), t \mid z, \tau]\, dt = \int_{\varphi(\tau)}^{+\infty} g_X\left\{\eta(u), u \Big| \frac{z - m(\tau)}{k(\tau)}, \varphi(\tau)\right\} du = 1, \tag{3.5}$$

so that the first passage of the Gauss-Markov process $Z(t)$ through the boundary (3.1) is a sure event.

**Theorem 3.1** *Let $\eta(\vartheta) = S + \varrho(\vartheta)$ be an asymptotically constant boundary, with $S \in \mathbb{R}$ and $\varrho(\vartheta) \in C^1[0, +\infty)$ a bounded function independent of $S$ such that (2.7) hold, and let $Z(t)$ be a Gauss-Markov process, with $m(t), h_1(t), h_2(t) \in C^1(T)$ independent of $S$. Denote by $S(t) = m(t) + k(t)\,\eta\big[\varphi(t)\big]$ $(t \in T, S \in \mathbb{R})$ a boundary, with $k(t)$ and $\varphi(t)$ defined in (1.7). Then, under the assumption (3.4), for $\vartheta_0 \geq 0$ and $\vartheta > 0$ one has:*

$$\begin{aligned}\lim_{S\to+\infty} \frac{d}{d\vartheta}\varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big) g_Z \left\{ S\Big[\varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big], \varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big| z, \varphi^{-1}(\vartheta_0)\right\}&\\ = e^{-\vartheta}, \qquad z < S\big[\varphi^{-1}(\vartheta_0)\big].&\end{aligned} \tag{3.6}$$

**Proof.** Setting $\varphi(t) = \vartheta/R(S) + \vartheta_0$ and $\varphi(\tau) = \vartheta_0$ in (3.2), for $\vartheta_0 \geq 0$ and $\vartheta > 0$ one has:

$$\begin{aligned}\Big[\frac{d}{d\vartheta}\varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big]\, g_Z \left\{ S\Big[\varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big], \varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big| z, \varphi^{-1}(\vartheta_0)\right\}&\\ = \frac{1}{R(S)}\, g_X\Big[\eta\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big), \frac{\vartheta}{R(S)} + \vartheta_0 \,\Big|\, y, \vartheta_0\Big], \qquad y < \eta(\vartheta_0),&\end{aligned} \tag{3.7}$$

where we have set:

$$y = \frac{z - m\big[\varphi^{-1}(\vartheta_0)\big]}{k\big[\varphi^{-1}(\vartheta_0)\big]}\,. \tag{3.8}$$

Taking the limit as $S \to +\infty$ in (3.7) and making use of (2.10), Eq. (3.6) immediately follows. ◻

**Corollary 3.1** *Under the assumption of Theorem 3.1, for $S \to +\infty$ there holds:*

$$g_Z\big[S(t), t \mid z, \tau\big] \simeq R(S)\,\frac{d\varphi(t)}{dt} e^{-R(S)\,[\varphi(t)-\varphi(\tau)]}, \qquad z < S(\tau), \tag{3.9}$$

*with $\tau < t$, $\tau, t \in T$ and $R(S)$ given in (2.8).*

**Proof.** By virtue of (3.6), for $S \to +\infty$ one obtains:

$$g_Z\Big\{S\Big[\varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big], \varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big| z, \varphi^{-1}(\vartheta_0)\Big\} \simeq \Big[\frac{d}{d\vartheta}\varphi^{-1}\Big(\frac{\vartheta}{R(S)} + \vartheta_0\Big)\Big]^{-1} e^{-\vartheta}.$$

Hence, setting $t = \varphi^{-1}\big[\vartheta/R(S) + \vartheta_0\big]$ and $\tau = \varphi^{-1}(\vartheta_0)$, one is immediately led to (3.9). □

Corollary 3.1 expresses the asymptotic trend of the FPT density of the Gauss-Markov process $Z(t)$ through the boundary $S(t) = m(t) + k(t)\,\eta\big[\varphi(t)\big]$, where $\eta(\vartheta) = S + \varrho(\vartheta)$ is an asymptotically constant boundary for the non-stationary OU process $X(\vartheta)$.

The right-hand side of (3.9) has the following functional form:

$$\gamma_Z(t \mid \tau) = \lambda\,\frac{d\varphi(t)}{dt} e^{-\lambda\,[\varphi(t)-\varphi(\tau)]} \qquad (\tau, t \in T, \tau < t). \tag{3.10}$$

Recalling (2.16), the least squares estimate of $\lambda$ can be obtained as:

$$\begin{aligned}\widehat{\lambda} &= -\frac{\sum_{k=1}^N \big(\vartheta_k - \vartheta_0\big)\, \ln\Big\{1 - \tilde{G}_X\Big[\eta(\vartheta_k), \vartheta_k\Big|\frac{z - m\big[\varphi^{-1}(\vartheta_0)\big]}{k\big[\varphi^{-1}(\vartheta_0)\big]}, \vartheta_0\Big]\Big\}}{\sum_{k=1}^N \big(\vartheta_k - \vartheta_0\big)^2}\\ &= -\frac{6\sum_{k=1}^N k\, \ln\Big\{1 - \tilde{G}_X\Big[\eta(\vartheta_0 + k\,p), \vartheta_0 + k\,p\Big|\frac{z - m\big[\varphi^{-1}(\vartheta_0)\big]}{k\big[\varphi^{-1}(\vartheta_0)\big]}, \vartheta_0\Big]\Big\}}{N\,(N+1)\,(2\,N+1)\,p}.\end{aligned} \tag{3.11}$$

with $\vartheta_k = \vartheta_0 + k\,p$ $(k = 1, 2, \ldots, N)$, where $p > 0$ is the time discretization step, and where $\tilde{G}_X$ is the computed FPT distribution function of the non-stationary OU process.

**Theorem 3.2** *Let $\eta(\vartheta) = S + \varrho(\vartheta)$ be an asymptotically periodic boundary, with $S \in \mathbb{R}$ and $\varrho(\vartheta) \in C^1[0, +\infty)$ a bounded function independent of $S$ such that (2.17) and (2.18) hold, and let $Z(t)$ be a Gauss-Markov process, with $m(t), h_1(t), h_2(t) \in C^1(T)$ independent of $S$. Denote by $S(t) = m(t) + k(t)\,\eta\big[\varphi(t)\big]$ $(t \in T, S \in \mathbb{R})$ a boundary, with $k(t)$ and $\varphi(t)$ given in (1.7). Then, under the assumption (3.4), for $\vartheta_0 \geq 0$ and $\vartheta > 0$ one has:*

$$\begin{aligned}\lim_{S\to+\infty}&\Big\{\frac{d}{d\vartheta}\varphi^{-1}\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big]\Big\} g_Z\Big\{S\Big[\varphi^{-1}\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big)\Big], \varphi^{-1}\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big)\Big| z, \varphi^{-1}(\vartheta_0)\Big\}\\ &= e^{-\vartheta}, \qquad z < S\big[\varphi^{-1}(\vartheta_0)\big].\end{aligned} \tag{3.12}$$

**Proof.** Setting $\varphi(t) = \chi(\vartheta/\alpha) + \vartheta_0$ and $\varphi(\tau) = \vartheta_0$ in (3.2), for $\vartheta_0 \geq 0$ and $\vartheta > 0$ one obtains:

$$\begin{aligned}&\Big\{\frac{d}{d\vartheta}\varphi^{-1}\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big]\Big\} g_Z\Big\{S\Big[\varphi^{-1}\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big)\Big], \varphi^{-1}\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big)\Big| z, \varphi^{-1}(\vartheta_0)\Big\}\\ &= \Big[\frac{d}{d\vartheta}\chi\Big(\frac{\vartheta}{\alpha}\Big)\Big]\, g_X\Big[\eta\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0\Big), \chi\Big(\frac{\vartheta}{\alpha}\Big) + \vartheta_0 \,\Big|\, y, \vartheta_0\Big], \qquad y < \eta(\vartheta_0)\end{aligned} \tag{3.13}$$

where $y$ is given in (3.8). Taking the limit as $S \to +\infty$ in (3.13) and making use of (2.28), Eq. (3.12) immediately follows. □

**Corollary 3.2** *Under the assumption of Theorem 3.2, for $S \to +\infty$ there holds:*

$$g_Z\big[S(t), t \mid z, \tau\big] \simeq R\Big\{V[\varphi(t)-\varphi(\tau)]\Big\}\,\frac{d\varphi(t)}{dt}\exp\Big\{-\int_{\varphi(\tau)}^{\varphi(t)} R\Big\{V\big[u-\varphi(\tau)\big]\Big\}\,du\Big\},\qquad z < S(\tau), \tag{3.14}$$

*with $\tau < t,\ \tau, t \in T$ and $R[V(\vartheta)]$ given in (2.19). Furthermore, (3.14) can be also written as:*

$$g_Z\big[S(t), t \mid z, \tau\big] \simeq \frac{d\varphi(t)}{dt}\,\xi_X\big[\varphi(t)-\varphi(\tau)\big]\,e^{-\alpha\,[\varphi(t)-\varphi(\tau)]},\qquad z < S(\tau), \tag{3.15}$$

*with $\alpha$ given in (2.21) and where $\xi_X(\vartheta)$, defined in (2.32), is a periodic function of period $Q$.*

**Proof.** By virtue of (3.12), recalling (2.33), for $S \to +\infty$ one has:

$$\begin{aligned} g_Z&\Big\{S\Big[\varphi^{-1}\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big)\Big], \varphi^{-1}\Big(\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big)\Big| z, \varphi^{-1}(\vartheta_0)\Big\} \simeq \Big\{\frac{d}{d\vartheta}\varphi^{-1}\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big]\Big\}^{-1} e^{-\vartheta}\\ &= \Big\{\frac{d}{d\chi(\vartheta/\alpha)}\varphi^{-1}\Big[\chi\Big(\frac{\vartheta}{\alpha}\Big)+\vartheta_0\Big]\Big\}^{-1}\Big[\frac{d\chi(\vartheta/\alpha)}{d\vartheta}\Big]^{-1}\exp\Big\{-\int_0^{\chi(\vartheta/\alpha)} R[V(u)]\,du\Big\}\\ &\qquad z < S\big[\varphi^{-1}(\vartheta_0)\big]. \end{aligned} \tag{3.16}$$

with $0 \le \vartheta_0 < \vartheta < +\infty$. Hence, setting $t = \varphi^{-1}\big\{\chi\big(\vartheta/\alpha\big)+\vartheta_0\big\}$ and $\tau = \varphi^{-1}(\vartheta_0)$ in (3.16), one obtains (3.14). Finally, by virtue of Corollary 2.2, (3.15) immediately follows from (3.14). □

Corollary 3.2 expresses the asymptotic trend of the FPT density of the Gauss-Markov process $Z(t)$ through the boundary $S(t) = m(t) + k(t)\,\eta\big[\varphi(t)\big]$, where $\eta(\vartheta)$ is an asymptotically periodic boundary for the non-stationary OU process $X(\vartheta)$.

The right-hand side of (3.14) has the following functional form:

$$\gamma_Z(t \mid \tau) = R^*\big[\varphi(t)-\varphi(\tau)\big]\,\frac{d\varphi(t)}{dt}\,\exp\Big\{-\int_{\varphi(\tau)}^{\varphi(t)} R^*[u-\varphi(\tau)]\,du\Big\}, \tag{3.17}$$

with $\tau < t,\ \tau, t \in T$ and where $R^*(\vartheta)$ is a periodic function of period $Q$. We note that (3.17) can also be written as

$$\gamma_Z(t \mid \tau) = \xi^*_X\big[\varphi(t)-\varphi(\tau)\big]\,\frac{d\varphi(t)}{dt}\,e^{-\alpha^*\,[\varphi(t)-\varphi(\tau)]}\qquad (\tau < t,\ \tau, t \in T), \tag{3.18}$$

where $\alpha^*$ is defined in (2.36) and $\xi^*_X(\vartheta)$ is given in (2.37). Recalling (2.39), the least squares estimate of $\alpha^*$ is:

$$\widehat{\alpha} = -\frac{6\sum_{k=1}^{M} k\,\ln\Big\{1-\tilde{G}_X\Big[\eta(\vartheta_0+k\,Q), \vartheta_0+k\,Q\Big|\frac{z-m\big[\varphi^{-1}(\vartheta_0)\big]}{k\big[\varphi^{-1}(\vartheta_0)\big]}, \vartheta_0\Big]\Big\}}{M\,(M+1)\,(2\,M+1)\,Q}, \tag{3.19}$$

where the period $Q > 0$ is the time discretization step and where $\tilde{G}_X$ is the computed FPT distribution function of the non-stationary OU process. With such a value of $\widehat{\alpha}$, the function $\xi_X[\varphi(t)-\varphi(\tau)]$ can be finally evaluated via (3.15) as:

$$\tilde{\xi}_X[\varphi(t)-\varphi(\tau)] = \Big[\frac{d\varphi(t)}{dt}\Big]^{-1} e^{\widehat{\alpha}\,[\varphi(t)-\varphi(\tau)]}\,\tilde{g}_Z\big[S(t), t \mid z, \tau\big] \tag{3.20}$$

where $\tilde{g}_Z$ is the computed FPT pdf.

**4 Estimations of FPT pdf's for Gauss-Markov processes** A computationally simple, speedy and accurate method based on the repeated Simpson rule has been proposed in [4] to construct FPT pdf of Gauss-Markov processes $Z(t)$ through time-dependent boundaries. Following [4] and denoting by $\tilde g_Z[S(t_k),t_k \mid z,\tau]$ the numerical evaluation of the FPT pdf $g_Z[S(t_k),t_k \mid z,\tau]$ at times $t_k=\tau+k\,p$ $(k=1,2,\dots,N)$, where $p>0$ is the time discretization step, one has:

$$
\begin{aligned}
&\tilde g_Z\big[S(t_1),t_1 \mid z,\tau\big] = -2\,\Psi_Z\big[S(t_1),t_1 \mid z,\tau\big]\\
&\tilde g_Z\big[S(t_k),t_k \mid z,\tau\big] = -2\Psi_Z\big[S(t_k),t_k \mid z,\tau\big]\\
&\qquad +2\,p\sum_{j=1}^{k-1} w_{k,j}\,\tilde g_Z\big[S(t_j),t_j \mid z,\tau\big]\,\Psi_Z\big[S(t_k),t_k \mid S(t_j),t_j\big] \qquad (k=2,3,\dots),
\end{aligned}
\tag{4.1}
$$

with $\tau<t_k$ $(t_k,\tau\in T)$ and where the weights $w_{k,j}$ are specified as follows:

$$
\begin{aligned}
&w_{2n,2j-1}=\frac{4}{3}\ \ (j=1,2,\dots,n;\ n=1,2,\dots)\\
&w_{2n,2j}=\frac{2}{3}\ \ (j=1,2,\dots,n-1;\ n=2,3,\dots)\\
&w_{2n+1,2j-1}=\frac{4}{3}\ \ (j=1,2,\dots,n-1;\ n=2,3,\dots)\\
&w_{2n+1,2j}=\frac{2}{3}\ \ (j=1,2,\dots,n-2;\ n=3,4,\dots)\\
&w_{2n+1,2(n-1)}=\frac{17}{24}\ \ (n=2,3,\dots)\\
&w_{2n+1,2n-1}=w_{2n+1,2n}=\frac{9}{8}\ \ (n=1,2,\dots).
\end{aligned}
\tag{4.2}
$$

An alternative numerical procedure to obtain the FPT pdf $g_Z[S(t),t \mid z,\tau]$ for the Gauss-Markov process $Z(t)$ through the boundary $S(t)=m(t)+k(t)\,\eta\big[\varphi(t)\big]$ is based on (3.2). Indeed, $g_Z$ can be computed as:

$$
\tilde g_Z[S(t_k),t_k \mid z,\tau]=\frac{d\varphi(t)}{dt}\Big|_{t=t_k}\tilde g_X\Big\{\eta[\varphi(t_k)],\varphi(t_k)\Big|\frac{z-m(\tau)}{k(\tau)},\varphi(\tau)\Big\}, \qquad z<S(\tau)
\tag{4.3}
$$

for $\tau<t_k=t_0+k\,p$ $(\tau,t_k\in T)$, where $\tilde g_X$ denotes the numerical evaluation of the FPT pdf $g_X$ for the non-stationary OU process $X(t)$.

In order to compute (4.3), we have designed a numerical algorithm with variable step-size to evaluate the FPT pdf $g_X\big\{\eta\big[\varphi(t)\big],\varphi(t) \mid y,\varphi(\tau)\big\}$ for the non-stationary OU process $X(t)$ through the boundary $\eta\big[\varphi(t)\big]$. Let $\varphi(t):T\to[0,+\infty)$ be a continuous and monotonically increasing function. Setting $\vartheta=\varphi(t)$ and $\vartheta_0=\varphi(\tau)$ in (2.1) one has:

$$
\begin{aligned}
&g_X\big\{\eta[\varphi(t)],\varphi(t) \mid y,\varphi(\tau)\big\} = -2\,\Psi_X\big\{\eta[\varphi(t)],\varphi(t) \mid y,\varphi(\tau)\big\}\\
&\qquad +2\int_{\varphi(\tau)}^{\varphi(t)} g_X\big\{\eta(\zeta),\zeta \mid y,\varphi(\tau)\big\}\,\Psi_X\big\{\eta[\varphi(t)],\varphi(t) \mid \eta(\zeta),\zeta\big\}\,d\zeta, \qquad y<\eta[\varphi(\tau)],
\end{aligned}
$$

where $\Psi_X[\eta(t),t|y,\tau]$ is given in (2.2), or equivalently:

$$
\begin{aligned}
&g_X\big\{\eta[\varphi(t)],\varphi(t) \mid y,\varphi(\tau)\big\} = -2\,\Psi_X\big\{\eta[\varphi(t)],\varphi(t) \mid y,\varphi(\tau)\big\}\\
&\qquad +2\int_{\tau}^{t} g_X\big\{\eta[\varphi(u)],\varphi(u) \mid y,\varphi(\tau)\big\}\,\frac{d\varphi(u)}{du}\,\Psi_X\big\{\eta[\varphi(t)],\varphi(t) \mid \eta[\varphi(u)],\varphi(u)\big\}\,du,
\end{aligned}
\tag{4.4}
$$

with $y < \eta[\varphi(\tau)]$. Hence, for $t_k = \tau + k\,p \quad (k = 1, 2, \dots)$, where $p > 0$ is the time discretization step, one is led to the following iterative algorithm:

$$
\begin{aligned}
&\tilde{g}_X\big\{\eta[\varphi(t_1)], \varphi(t_1) \mid y, \varphi(\tau)\big\} = -2\,\Psi_X\big\{\eta[\varphi(t_1)], \varphi(t_1) \mid y, \varphi(\tau)\big\},\\
&\tilde{g}_X\big\{\eta[\varphi(t_k)], \varphi(t_k) \mid y, \varphi(\tau)\big\} = -2\,\Psi_X\big\{\eta[\varphi(t_k)], \varphi(t_k) \mid y, \varphi(\tau)\big\}\\
&\qquad +2\,p \sum_{j=1}^{k-1} w_{k,j} \frac{d\varphi(u)}{du}\Big|_{u=t_j} \tilde{g}_X\big\{\eta[\varphi(t_j)], \varphi(t_j) \mid y, \varphi(\tau)\big\}\\
&\qquad \times \Psi_X\big\{\eta[\varphi(t_k)], \varphi(t_k) \mid \eta[\varphi(t_j)], \varphi(t_j)\big\} \quad (k = 2, 3, \dots),
\end{aligned}
\tag{4.5}
$$

where $\Psi_X$ is given in (2.2) and where the weights $w_{k,j}$ are defined in (4.2).

The goodness of asymptotic approximations (3.9) and (3.14) has been confirmed by the numerical computations. Indeed, making use of (4.3) and (4.5), we shall show that the asymptotic results on the FPT pdf for the non-stationary OU process $X(t)$ through a constant or a periodic boundary $\eta(\vartheta)$ can be implemented in order to obtain information on the FPT pdf for particular Gauss-Markov processes in the presence of special time-varying boundaries $S(t) = m(t) + k(t)\,\eta[\varphi(t)]$.

**4.1 Wiener process** Let $Z(t)$ be the Wiener process in the time domain $T = [0, +\infty)$, such that

$$
Z(t) = \mu\,t + W(\omega^2\,t) \qquad (\mu \in \mathbb{R}, \omega > 0).
\tag{4.6}
$$

Recalling (1.1) one has:

$$
m(t) = \mu\,t, \qquad h_1(t) = \omega^2\,t, \qquad h_2(t) = 1, \qquad r(t) = \omega^2\,t.
$$

Then, $Z(t)$ can be represented in terms of a non-stationary OU process $X(t)$ by means of (1.6) with

$$
k(t) = \sqrt{1 + \frac{2\,\beta}{\sigma^2}\,\omega^2\,t}, \qquad \varphi(t) = \frac{1}{2\,\beta}\,\ln\Big(1 + \frac{2\,\beta}{\sigma^2}\,\omega^2\,t\Big) \qquad (t \geq 0).
\tag{4.7}
$$

*Case a)* Setting $\beta = 1$ and $\sigma^2 = 2$ in (4.7) and $\eta(\vartheta) = S$ in (3.1) one obtains:

$$
S(t) = \mu\,t + S\,\sqrt{1 + \omega^2\,t} \qquad (t \geq 0),
\tag{4.8}
$$

so that, recalling (3.9), the FPT pdf of the Wiener process (4.6) through the boundary (4.8) for large $S$ and large times exhibits the following asymptotic behavior:

$$
g_Z\Big[\mu\,t + S\,\sqrt{1 + \omega^2\,t},\, t \mid z, \tau\Big] \simeq \frac{R(S)\,\omega^2}{2}\,\frac{\big(1 + \omega^2\,\tau\big)^{R(S)/2}}{\big(1 + \omega^2\,t\big)^{1 + R(S)/2}} \quad (0 \leq \tau < t < +\infty),
\tag{4.9}
$$

with $R(S)$ given in (2.8). The right-hand side of (4.9) has the following functional form:

$$
\gamma_Z(t \mid \tau) = \frac{\lambda\,\omega^2}{2}\,\frac{\big(1 + \omega^2\,\tau\big)^{\lambda/2}}{\big(1 + \omega^2\,t\big)^{1 + \lambda/2}},
\tag{4.10}
$$

with $0 \leq \tau < t < +\infty$.

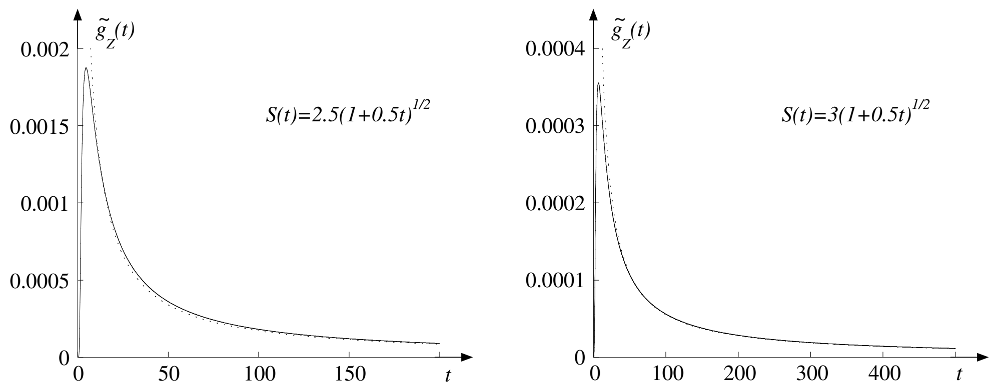



Figure 4: For the Wiener process with $\mu = 0$ and $\omega^2 = 0.5$, $\tilde{g}_Z(t) = \tilde{g}_Z(S\sqrt{1+0.5\,t}, t \mid 0, 0)$ (solid line) is compared with the asymptotic density $\gamma_Z(t \mid 0)$ (dotted line) given in (4.10) for the same choices of $\lambda$ and $S$ as in Fig.1.

For the Wiener process with $\mu = 0$ and $\omega^2 = 0.5$, Fig. 4 shows the evaluated FPT density $\tilde{g}_Z(t) = \tilde{g}_Z(S\sqrt{1+0.5\,t}, t \mid 0, 0)$ with $S = 2.5$ and $S = 3$ and the asymptotic density $\gamma_Z(t \mid 0)$, given in (4.10), with $\lambda$ estimated by means of (3.11). The integration step in (4.3) and (4.5) has been taken as 0.1.

*Case b)* Setting $\beta = 1$ and $\sigma^2 = 2$ in (4.7) and $\eta(\vartheta) = S + B\,\cos(2\pi\vartheta/Q) + C\,\sin(2\pi\vartheta/Q)$ in (3.1), for $t \geq 0$ one has:

$$S(t) = \mu\,t + \sqrt{1+\omega^2\,t}\,\Big\{S + B\,\cos\Big[\frac{\pi}{Q}\,\ln\big(1+\omega^2\,t\big)\Big] + C\,\sin\Big[\frac{\pi}{Q}\,\ln\big(1+\omega^2\,t\big)\Big]\Big\}, \tag{4.11}$$

so that, recalling (3.15), the FPT pdf of the Wiener process through the boundary (4.11) for large $S$ and large times exhibits the following asymptotic behavior:

$$g_Z\big[S(t), t \mid z, \tau\big] \simeq \frac{\omega^2}{2}\,\frac{\big(1+\omega^2\,\tau\big)^{\alpha/2}}{\big(1+\omega^2\,t\big)^{1+\alpha/2}}\,\xi_X\left[\frac{1}{2}\,\ln\Big(\frac{1+\omega^2\,t}{1+\omega^2\,\tau}\Big)\right], \tag{4.12}$$

for $0 \leq \tau < t < +\infty$, with $\alpha$ given in (2.21) and where $\xi_X(\vartheta)$, defined in (2.32), is a periodic function of period $Q$.

For the Wiener process with $\mu = 0$ and $\omega^2 = 0.5$, Fig. 5 shows the evaluated FPT density $\tilde{g}_Z(t) = \tilde{g}_Z[S(t), t \mid 0, 0)$ with $S(t) = \sqrt{1+0.5\,t}\,\{3 + 0.1\,\sin[\pi\,\ln(1+0.5\,t)/0.2]\}$ and the estimated function $\tilde{\xi}_X(\vartheta)$ with $\widehat{\alpha} = 0.011950$. Integration step in (4.3) and (4.5) is $10^{-2}$.

**4.2 Stationary OU process** Let $Z(t)$ be the stationary OU process in the time domain $T = \mathbb{R}$, such that (cf., for instance, [10]):

$$Z(t) = e^{-\beta\,t}\,W\Big(\frac{\sigma^2}{2\,\beta}\,e^{2\,\beta\,t}\Big) \qquad (t \in \mathbb{R}). \tag{4.13}$$

Recalling (1.1) one has:

$$m(t) = 0, \qquad h_1(t) = \frac{\sigma^2}{2\,\beta}\,e^{\beta\,t}, \qquad h_2(t) = e^{-\beta\,t}, \qquad r(t) = \frac{\sigma^2}{2\,\beta}\,e^{2\,\beta\,t}.$$

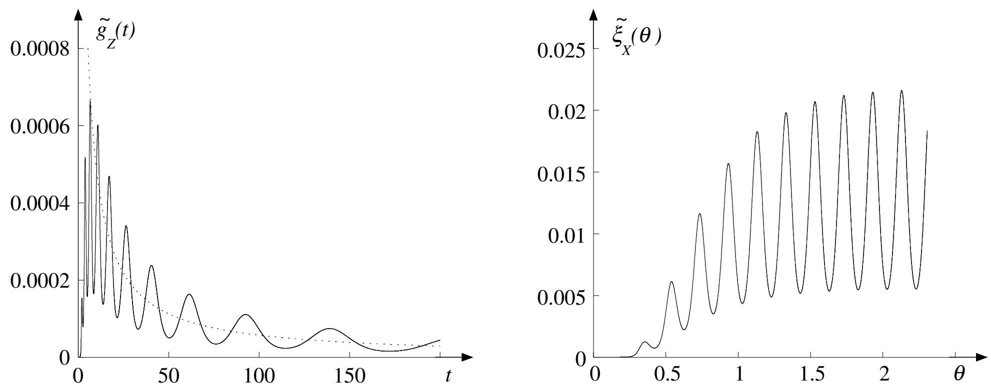



Figure 5: Wiener process with $\mu = 0, \omega^2 = 0.5$ and boundary $S(t) = \sqrt{1+0.5\,t}\,\{3{+}0.1\,\sin[\pi\,\ln(1{+}0.5\,t)/0.2]\}$. On the left the FPT pdf $\tilde{g}_Z(t) = \tilde{g}_Z[S(t), t \mid 0, 0]$ (solid line) is compared with the density $0.5\alpha\,(1{+}0.5\,t)^{-1-\alpha/2}/2$ (dotted line), with $\alpha$ estimated by means of (3.19) as $\widehat{\alpha} = 0.011950$. On the right the function $\tilde{\xi}_X(\vartheta)$, computed via (3.20), is plotted for $\vartheta = \ln(1+0.5\,t)/2$.

The process $Z(t)$ can be represented in terms of a non-stationary OU process $X(t)$ by means of (1.6) with

$$k(t) = \sqrt{1+e^{-2\beta t}}, \qquad \varphi(t) = \frac{1}{2\beta}\ln\Bigl(1+e^{2\beta t}\Bigr) \qquad (t \in \mathbb{R}). \tag{4.14}$$

*Case a)* Setting $\eta(\vartheta) = S$ in (3.1) one obtains:

$$S(t) = S\sqrt{1+e^{-2\beta t}} \qquad (t \in \mathbb{R}), \tag{4.15}$$

so that, recalling (3.9), the FPT pdf of the stationary OU process (4.13) through the boundary (4.15) for large $S$ and large times exhibits the following asymptotic behavior:

$$g_Z\Bigl[S\sqrt{1+e^{-2\beta t}}, t \mid z, \tau\Bigr] \simeq \frac{R(S)\,e^{2\beta t}\,\bigl(1+e^{2\beta\tau}\bigr)^{R(S)/(2\beta)}}{\bigl(1+e^{2\beta t}\bigr)^{1+R(S)/(2\beta)}}, \tag{4.16}$$

for $-\infty < \tau < t < +\infty$, with $R(S)$ given in (2.8). The right-hand side of (4.16) has the following functional form:

$$\gamma_Z(t \mid \tau) = \frac{\lambda\,e^{2\beta t}\,\bigl(1+e^{2\beta\tau}\bigr)^{\lambda/(2\beta)}}{\bigl(1+e^{2\beta t}\bigr)^{1+\lambda/(2\beta)}} \qquad (-\infty < \tau < t < +\infty). \tag{4.17}$$

Fig. 6 shows $\tilde{g}_Z(t) = \tilde{g}_Z(S\sqrt{1+e^{-2t}}, t \mid 0, 0)$ for the stationary OU process ($\beta = 1$, $\sigma^2 = 2$) with $S = 2.5$ and $S = 3$ and the asymptotic density $\gamma_Z(t \mid 0)$ given in (4.17) with $\lambda$ estimated by means of (3.11). The integration step in (4.3) and (4.5) has been taken as $10^{-2}$.

*Case b)* Setting $\eta(\vartheta) = S + B\,\cos(2\pi\vartheta/Q) + C\,\sin(2\pi\vartheta/Q)$ in (3.1), for $t \in \mathbb{R}$ one has:

$$S(t) = \sqrt{1+e^{-2\beta t}}\,\Bigl\{S + B\,\cos\Bigl[\frac{\pi}{\beta Q}\,\ln(1+e^{2\beta t})\Bigr]\Bigr\} + C\,\sin\Bigl[\frac{\pi}{\beta Q}\,\ln(1+e^{2\beta t})\Bigr]\Bigr\}, \tag{4.18}$$

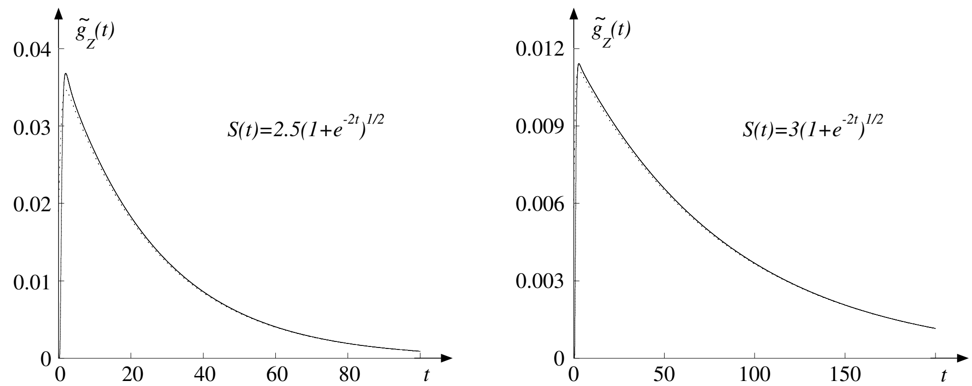



Figure 6: The FPT pdf $\tilde{g}_Z(t) = \tilde{g}_Z(S\sqrt{1+e^{-2t}}, t \mid 0, 0)$ (solid line) for the stationary OU process ($\beta = 1$, $\sigma^2 = 2$) is compared with the asymptotic density $\gamma_Z(t \mid 0)$ (dotted line) given in (4.17) for the same choices of $\lambda$ and $S$ as in Fig.1.

so that, recalling (3.15), the FPT pdf of the stationary OU process through the boundary (4.18) exhibits for large $S$ and large times the following asymptotic behavior:

$$g_Z\big[S(t), t \mid z, \tau\big] \simeq \frac{e^{2\beta t}\left(1+e^{2\beta\tau}\right)^{\alpha/(2\beta)}}{\left(1+e^{2\beta t}\right)^{1+\alpha/(2\beta)}}\,\xi_X\left[\frac{1}{2\beta}\ln\Big(\frac{1+e^{2\beta t}}{1+e^{2\beta\tau}}\Big)\right], \tag{4.19}$$

for $-\infty < \tau < t < +\infty$, with $\alpha$ given in (2.21) and where $\xi_X(\vartheta)$, defined in (2.32), is a periodic function of period $Q$.

For the stationary OU process with $\beta = 1$ and $\sigma^2 = 2$, Fig. 7 shows the evaluated FPT density $\tilde{g}_Z(t) = \tilde{g}_Z[S(t), t \mid 0, 0)$ with $S(t) = \sqrt{1+e^{-2t}}\,\{3 + 0.2\,\sin[\pi\,\ln(1+e^{2t})/5]\}$ and the estimated function $\tilde{\xi}_X(\vartheta)$ with $\widehat{\alpha} = 0.012337$. Integration step in (4.3) and (4.5) is $10^{-2}$.

**4.3 Brownian bridge** We now consider the Brownian bridge with time domain $T = [0, 1)$, defined as:

$$Z(t) = (1-t)\,W\Big(\frac{t}{1-t}\Big). \tag{4.20}$$

In this case, by virtue of (1.1), one has:

$$m(t) = 0, \qquad h_1(t) = t, \qquad h_2(t) = 1-t, \qquad r(t) = \frac{t}{1-t}\cdot$$

The Brownian bridge can be represented in terms of a non-stationary OU process $X(t)$ via (1.6) with

$$k(t) = (1-t)\sqrt{1+\frac{2\beta}{\sigma^2}\,\frac{t}{1-t}}, \qquad \varphi(t) = \frac{1}{2\beta}\,\ln\Big(1+\frac{2\beta}{\sigma^2}\,\frac{t}{1-t}\Big) \qquad (0 \le t < 1). \tag{4.21}$$

*Case a)* Setting $\beta = 1$ and $\sigma^2 = 2$ in (4.21) and $\eta(\vartheta) = S$ in (3.1) one obtains:

$$S(t) = S\sqrt{1-t} \qquad (0 \le t < 1), \tag{4.22}$$

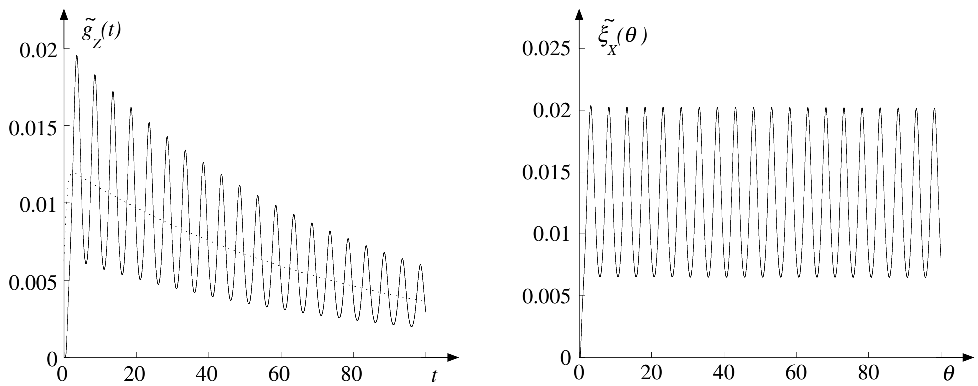



Figure 7: Stationary OU process with $\beta = 1$ and $\sigma^2 = 2$ and boundary $S(t) = \sqrt{1 + e^{-2t}}\,\{3 + 0.2\,\sin[\pi\,\ln(1+e^{2t})/5]\}$. On the left the FPT pdf $\tilde{g}_Z(t) = \tilde{g}_Z[S(t), t \mid 0, 0]$ (solid line) is compared with the density $\alpha\, e^{2t}\, 2^{\alpha/2}(1 + e^{2t})^{-1-\alpha/2}$ (dotted line), with $\alpha$ estimated as $\widehat{\alpha} = 0.012337$. On the right the function $\tilde{\xi}_X(\vartheta)$, computed via (3.20), is plotted for $\vartheta = [\ln(1 + e^{2t}) - \ln 2]/2$.

so that, recalling (3.9), the FPT pdf of the Brownian bridge (4.20) through the boundary (4.22) for large $S$ and large times exhibits the following asymptotic behavior:

$$g_Z\Big[S\sqrt{1-t}, t \mid z, \tau\Big] \simeq \frac{R(S)}{2}\,\frac{(1-t)^{-1+R(S)/2}}{(1-\tau)^{R(S)/2}} \qquad (0 \le \tau < t < 1), \tag{4.23}$$

with $R(S)$ given in (2.8). The right-hand side of (4.23) has the following functional form:

$$\gamma_Z(t \mid \tau) = \frac{\lambda}{2}\,\frac{(1-t)^{-1+\lambda/2}}{(1-\tau)^{\lambda/2}} \qquad (0 \le \tau < t < 1). \tag{4.24}$$

Fig. 8 shows the evaluated FPT density $\tilde{g}_Z(t) = \tilde{g}_Z(S\sqrt{1-t}, t \mid 0, 0)$ for the Brownian bridge with $S = 2.5$ and $S = 3$ and the density $\gamma_Z(t \mid 0)$ given in (4.24) with $\lambda$ estimated by means of (3.11). The integration step in (4.3) and (4.5) has been taken as $10^{-3}$.
*Case b)* Setting $\beta = 1$ and $\sigma^2 = 2$ in (4.21) and $\eta(\vartheta) = S + B\,\cos(2\pi\vartheta/Q) + C\,\sin(2\pi\vartheta/Q)$ in (3.1), for $t \in [0, 1)$ one has:

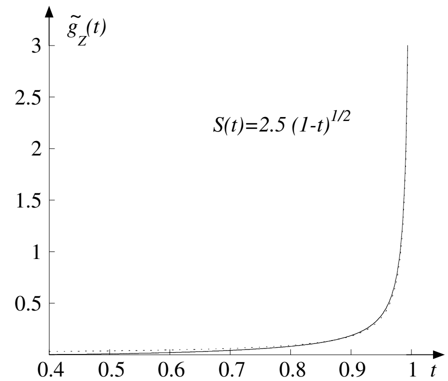
$$S(t) = \sqrt{1-t}\,\Big\{S + B\,\cos\Big[\frac{\pi}{Q}\,\ln(1-t)\Big] + C\,\sin\Big[\frac{\pi}{Q}\,\ln(1-t)\Big]\Big\} \qquad (0 \le t < 1), \tag{4.25}$$

so that, recalling (3.15), the FPT pdf of the Brownian bridge through the boundary (4.25) for large $S$ and large times exhibits the following asymptotic behavior:

$$g_Z\big[S(t), t \mid z, \tau\big] \simeq \frac{1}{2}\,\frac{(1-t)^{-1+\alpha/2}}{(1-\tau)^{\alpha/2}}\,\xi_X\left[\frac{1}{2}\,\ln\Big(\frac{1-\tau}{1-t}\Big)\right] \quad (0 \le \tau < t < 1), \tag{4.26}$$

with $\alpha$ given in (2.21) and where $\xi_X(\vartheta)$, defined in (2.32), is a periodic function of period $Q$.

For the Brownian bridge, Fig. 5 shows the evaluated FPT density $\tilde{g}_Z(t) = \tilde{g}_Z[S(t), t \mid 0, 0)$ with $S(t) = \sqrt{1-t}\,\{3 + 0.2\,\sin[\pi\,\ln(1-t)/5]\}$ and the estimated function $\tilde{\xi}_X(\vartheta)$ with $\widehat{\alpha} = 0.011950$. Integration step in (4.3) and (4.5) is $10^{-4}$.



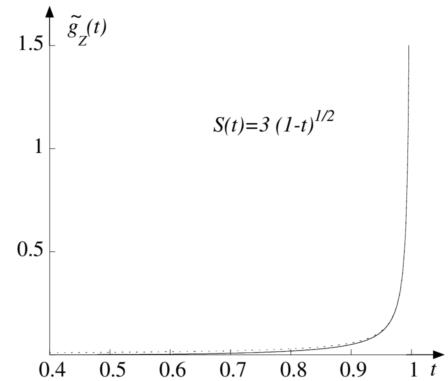




Figure 8: For the Brownian bridge $\tilde{g}_Z(t) = \tilde{g}_Z(S\sqrt{1-t}, t \mid 0, 0)$ (solid line) is compared with the asymptotic density $\gamma_Z(t \mid 0)$ (dotted line) given in (4.24) for the same choices of $\lambda$ and $S$ as in Fig.1.

**4.4 A transformation of the non-stationary OU process** Let $Z(t)$ be the following Gauss-Markov process with time domain $T = [0, 1)$, defined as:

$$Z(t) = (1-t)\, X\Big(\frac{t}{1-t}\Big). \tag{4.27}$$

Recalling (1.6) one has

$$k(t) = 1-t, \qquad \varphi(t) = \frac{t}{1-t} \qquad (0 \le t < 1), \tag{4.28}$$

so that $Z(t)$ is characterized by

$$m(t) = 0, \qquad h_1(t) = \frac{\sigma^2}{2\beta}(1-t)\Big[\exp\Big\{\frac{\beta t}{1-t}\Big\} - \exp\Big\{-\frac{\beta t}{1-t}\Big\}\Big],$$
$$h_2(t) = (1-t)\exp\Big\{-\frac{\beta t}{1-t}\Big\}, \qquad r(t) = \frac{\sigma^2}{2\beta}\Big[\exp\Big\{-\frac{2\beta t}{1-t}\Big\} - 1\Big].$$

The infinitesimal moments of $Z(t)$ are $A_1(x,t) = -x\,(1+\beta-t)/(1-t)^2$ and $A_2(t) = \sigma^2$ $(0 \le t < 1)$, so that as $\beta \to 0$ and $\sigma^2 = 1$, $Z(t)$ becomes the Brownian bridge (4.20).
*Case a)* Setting $\beta = 1$ and $\sigma^2 = 2$ in (4.28) and $\eta(\vartheta) = S$ in (3.1) one obtains:

$$S(t) = S\,(1-t) \qquad (0 \le t < 1), \tag{4.29}$$

so that, recalling (3.9), the FPT pdf of the process (4.27) through the boundary (4.29) exhibits for large $S$ and large times the following asymptotic behavior:

$$g_Z\Big[S\,(1-t), t \mid z, \tau\Big] \simeq \frac{R(S)}{(1-t)^2}\exp\Big\{-R(S)\Big[\frac{t}{1-t} - \frac{\tau}{1-\tau}\Big]\Big\} \quad (0 \le \tau < t < 1), \tag{4.30}$$

with $R(S)$ given in (2.8). The right-hand side of (4.30) has the following functional form:

$$\gamma_Z(t \mid \tau) = \frac{\lambda}{(1-t)^2}\exp\Big\{-\lambda\Big[\frac{t}{1-t} - \frac{\tau}{1-\tau}\Big]\Big\} \qquad (0 \le \tau < t < 1). \tag{4.31}$$

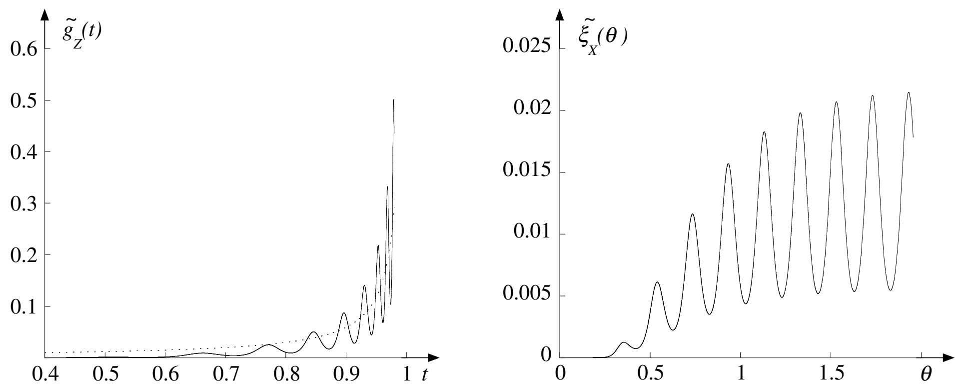



Figure 9: Brownian bridge and boundary $S(t)=\sqrt{1-t}\,\{3+0.1\,\sin[\pi\,\ln(1-t)/0.2]\}$. On the left the FPT pdf $\tilde{g}_Z(t)=\tilde{g}_Z[S(t),t\mid 0,0]$ (solid line) is compared with the density $\alpha\,(1-t)^{-1+\alpha/2}/2$ (dotted line), with $\alpha$ estimated as $\widehat{\alpha}=0.011950$. On the right the function $\tilde{\xi}_X(\vartheta)$, computed via (3.20), is plotted for $\vartheta=-\ln(1-t)/2$.

Fig. 10 shows the evaluated FPT density $\tilde{g}_Z(t)=\tilde{g}_Z[S\,(1-t),t\mid 0,0]$ for the process (4.27) with $S=2.5$ and $S=3$ and the asymptotic density $\gamma_Z(t\mid 0)$ given in (4.31) for $\beta=1$ and $\sigma^2=2$. The integration step in (4.3) and (4.5) has been taken as $10^{-4}$.
*Case b)* Setting $\beta=1$ and $\sigma^2=2$ in (4.28) and $\eta(\vartheta)=S+B\,\cos(2\pi\vartheta/Q)+C\,\sin(2\pi\vartheta/Q)$ in (3.1), for $t\in[0,1)$ one has:

$$S(t)=(1-t)\Big\{S+B\,\cos\Big[\frac{2\,\pi}{Q}\,\frac{t}{1-t}\Big]+C\,\sin\Big[\frac{2\,\pi}{Q}\,\frac{t}{1-t}\Big]\Big\}\qquad(0\le t<1),\tag{4.32}$$

so that, recalling (3.15), the FPT pdf of the process (4.27) through the boundary (4.32) exhibits for large $S$ and large times the following asymptotic behavior:

$$g_Z\big[S(t),t\mid z,\tau\big]\simeq\frac{1}{(1-t)^2}\,\exp\Big\{-\alpha\Big[\frac{t}{1-t}-\frac{\tau}{1-\tau}\Big]\Big\}\,\xi_X\left[\frac{t}{1-t}-\frac{\tau}{1-\tau}\right],\tag{4.33}$$

for $0\le\tau<t<1$, with $\alpha$ given in (2.21) and where $\xi_X(\vartheta)$, defined in (2.32), is a periodic function of period $Q$.

For the process (4.27), Fig. 11 shows the evaluated FPT density $\tilde{g}_Z(t)=\tilde{g}_Z[S(t),t\mid 0,0)$ with $S(t)=(1-t)\{3+0.2\,\sin[2\,\pi\,t/(5-5\,t)]\}$ and the estimated function $\tilde{\xi}_X(\vartheta)$ with $\widehat{\alpha}=0.012337$. Integration step in (4.3) and (4.5) is $5\cdot 10^{-5}$.

**4.5 Concluding Remarks** The aim of this paper has been to obtain quantitative information on the asymptotic behaviors of the FPT pdf's of Gauss-Markov processes through certain time-varying boundaries. This task has been achieved by proving that the asymptotic forms (2.13) and (2.30) of the FPT pdf's for the non-stationary OU process $X(\vartheta)$ through an asymptotically constant and an asymptotically periodic boundary $\eta(\vartheta)$, respectively, can be used in order to disclose the asymptotic behaviors (3.9) and (3.14) of the FPT pdf of a Gauss-Markov process $Z(t)=m(t)+k(t)\;X\big[\varphi(t)\big]$ through the transformed boundary $S(t)=m(t)+k(t)\,\eta[\varphi(t)]$, with $k(t)$ and $\varphi(t)$ specified in (1.7). In particular,

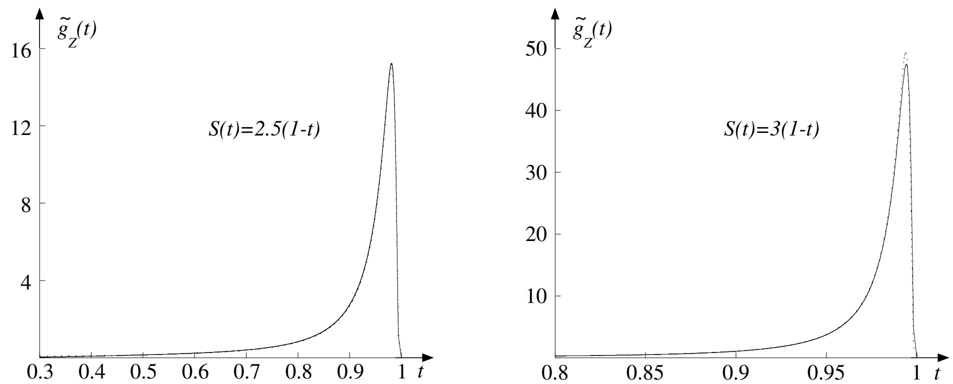



Figure 10: For the process $Z(t)$ given in (4.27) with $\beta = 1$ and $\sigma^2 = 2$, $\tilde{g}_Z(t) = \tilde{g}_Z[S(1-t), t \mid 0, 0]$ (solid line) is compared with the asymptotic density $\gamma_Z(t \mid 0)$ (dotted line) given in (4.31) for the same choices of $\lambda$ and $S$ as in Fig.1.

starting from an asymptotic constant boundary $\eta(\vartheta) = S$ for the non-stationary OU process with $\beta = 1$ and $\sigma^2 = 2$, we have considered the boundaries (4.8), (4.15), (4.22) and (4.29) for Wiener, stationary OU, Brownian bridge and transformed OU processes, respectively, and shown that for large boundaries and for large times the asymptotic results (4.9), (4.16), (4.23) and (4.30) hold. Furthermore, starting from an asymptotic periodic boundary $\eta(\vartheta) = S + B\cos(2\pi\vartheta/Q) + C\sin(2\pi\vartheta/Q)$ for the non-stationary OU process with $\beta = 1$ and $\sigma^2 = 2$, we have considered the boundaries (4.11), (4.18), (4.25) and (4.32) for Wiener, stationary OU, Brownian bridge and transformed OU processes, respectively. We have then proved that for large boundaries and for large times their FPT pdf's exhibit the asymptotic behaviors (4.12), (4.19), (4.26) and (4.33). The goodness of the asymptotic approximations has been confirmed by numerical computations based on the variable step-size algorithm (4.3) and (4.5).

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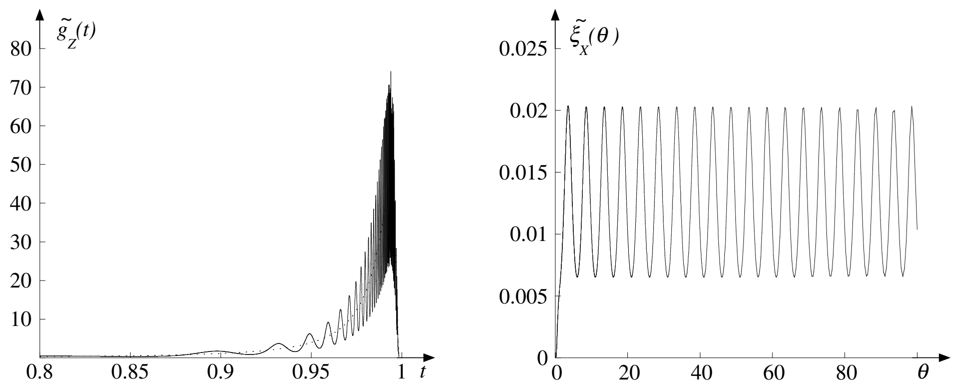


Figure 11: The process $Z(t)$ given in (4.27) with $\beta = 1$ and $\sigma^2 = 2$ and boundary $S(t) = (1-t)\,\{3 + 0.2\,\sin[2\,\pi\,t/(5-5\,t)]\}$. On the left the FPT pdf $\tilde{g}_Z(t) = \tilde{g}_Z[S(t), t \mid 0, 0]$ (solid line) is compared with the density $\alpha\,\exp\{-\alpha\,t/(1-t)\}/(1-t)^2$ (dotted line), with $\alpha$ estimated as $\widehat{\alpha} = 0.012337$. On the right the function $\tilde{\xi}_X(\vartheta)$, computed via (3.20), is plotted for $\vartheta = t/(1-t)$.


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