Scientiae Mathematicae Japonicae Online, **e-2008**, 241–260

ON OPERATION-PREOPEN SETS IN TOPOLOGICAL SPACES *

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Received October 18, 2007;revised January 31, 2008

Abstract. In this paper, we present concepts of pre γ_p-open sets and pre γ_p-closures of a subset in a topological space, where γ_p is an operation on the family of all preopen sets of the topological space, and study some topological properties on them. As its application, we introduce the concept of pre γ_p-T_i spaces ($i = 0, 1/2, 1, 2$) and study some properties of these spaces.

1 Introduction Throughout this paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed, unless otherwise mentioned. The closure and interior of $A \subset X$ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively. The power set of X will be denoted by $\mathcal{P}(X)$. An *operation* γ *on* τ is a function from τ into $\mathcal{P}(X)$ such that $U \subset U^\gamma$ for every set $U \in \tau$, where U^γ denotes the value $\gamma(U)$ of γ at U. In 1979, Kasahara [11] firstly defined and investigated the concept of operations on τ. He used the following symbol "α" as the operation on τ, i.e., a function $\alpha : \tau \to \mathcal{P}(X)$ is called an *operation on* τ if $U \subset \alpha(U)$ holds for any $U \in \tau$. He generalized the notion of compactness with help of operation. After the work of Kasahara, Janković [8] defined the concept of operation-closures (cf. Definition 2.4 below) and investigated some properties of functions with operation-closed graphs. In 1991, Ogata [20] defined and investigated the concept of *operation-open sets*, i.e., γ-open sets, and used it to investigate some new separation axioms. He used the symbol $\gamma : \tau \to \mathcal{P}(X)$ as an operation on τ. Thus, he avoided a confusion between the concept of α-open sets [18] and one of operation "α"-open sets (where the later symbol "α" is operation in the sense of Kasahara [11]). Let $\gamma : \tau \to \mathcal{P}(X)$ be an operation on τ. A nonempty subset A is said to be γ-*open* (*in the sense of Ogata*)[20] if for each point $x \in A$, there exists an open set U containing x such that $U^\gamma \subset A$. An arbitrary union of γ-open sets is also γ-open [20, Proposition 2.3]. Using the concepts of operation-open sets and operation-closures, some operator-approaches to topological properties were studied [20]. Recently, Krishnan et al. [12] investigated operations on the family of all semi-open sets [13].

In the present paper, we shall introduce an alternative operation-open sets, i.e., *pre* γ_p-*open sets* (cf. Definition 2.3), and investigate more operator-approaches to properties of topological spaces. Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be an operation from the family $PO(X, \tau)$ of all preopen sets of (X, τ) into $\mathcal{P}(X)$ (cf. Definition 2.1). The concept of *preopen sets* was introduced and investigated by Mashhour et al. [16]. Next section contains fundamental definitions of γ_p-*open sets* and γ_p-*closures*. In Section 3, the notions of *pre* γ_p-*open sets*

*2000 Math. Subject classification–:54A05, 54A10, 54B10, 54C05, 54C99, 54D10, 54D30.
Key words and phrases — preopen sets, operation-open sets, δ-closed sets, δ-closures, pre γ_p-open sets, pre γ_p-$T_{1/2}$ space, pre γ_p-T_1 spaces, pre γ_p-T_0 spaces, pre γ-T_2 spaces.

and four kinds of operation-closures, τ_{γ_p}-Cl(A), $PO(X)_{\gamma_p}$-Cl(A), pCl$_{\gamma_p}(A)$, Cl$_{\gamma_p}(A)$, are introduced and studied (cf. Definitions 3.9, 3.10, Theorem 3.16). In Section 4, *pre γ_p-generalized closed sets* and *pre γ_p-T_i separation axioms* are introduced and investigated, where i=0, 1/2, 1 or 2. The concept of γ_p-$T_{1/2}$ (resp. pre γ_p-$T_{1/2}$) spaces is charactrized by using γ_p-open singletons and γ_p-closed singletons (resp. pre γ_p-open singletons and pre γ_p-closed singletons) (cf. Theorem 4.6(ii) (resp. (i)). Especially, assume γ_p is the "identity operation"(cf. Example 3.2(i)), then the concept of "*id*"-$T_{1/2}$ spaces coincides with the concept of $T_{1/2}$-spaces due to Levine [14] (cf. [4, Theorem 2.5]). The digital line (Z, κ) is a typical example of $T_{1/2}$-spaces (e.g., [6, p.31 and the list of the references]). We have other examples of operations (cf. [20] [8]; Example 3.2 and Remark 3.4 below). For some undefined or related concepts, we refer the reader to [17] and [7].

2 Preliminiaries A subset A of topological space (X, τ) is said to be *preopen* [16] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ holds. We denote by $PO(X, \tau)$ (sometimes, $PO(X)$) the set of all preopen sets in (X, τ)[16]. The complement of a preopen set is called *preclosed*. The intersection of all preclosed sets of (X, τ) containing a subset A is called the *preclosure* of A and is denoted by pCl(A) [5]. The union of all preopen sets contained in a subset A is called the *preinterior* of A and is denoted by pInt(A). The set pCl(A) is preclosed and pInt(A) is preopen in (X, τ) for any subset A of (X, τ), because an arbitrary union of preopen sets of (X, τ) is preopen [1]. It is well known that [2, Theorem 1.5 (e)(f)] $\mathrm{pCl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}(A))$ and $\mathrm{pInt}(A) = A \cap \mathrm{Int}(\mathrm{Cl}(A))$ hold for any subset A of (X, τ). We note that $\tau \subset PO(X, \tau)$ for any topological space (X, τ) and $PO(X, \tau)$ is not a topology on X in general.

Definition 2.1 Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be a mapping from $PO(X, \tau)$ into $\mathcal{P}(X)$ satisfying the following property: $V \subset \gamma_p(V)$ for any $V \in PO(X, \tau)$. We call the mapping γ_p an *operation on* $PO(X, \tau)$. We denote $V^{\gamma_p} := \gamma_p(V)$ for any $V \in PO(X, \tau)$.

Remark 2.2 For an operation $\gamma_p : PO(X) \to \mathcal{P}(X)$, the *restriction of γ_p onto τ* (say $\gamma_p|\tau : \tau \to \mathcal{P}(X)$) is well defined. Indeed, $\tau \subset PO(X, \tau)$ holds and so $(\gamma_p|\tau)(V) = V^{\gamma_p}$ is well defined for any set $V \in \tau$. This restriction $\gamma_p|\tau : \tau \to \mathcal{P}(X)$ is the operation on τ in the sense of Ogata [20, Definition 2.1] (cf. Section 1 above). By [20, Definition 2.2] (cf. Section 1 above), a nonempty set A is called a *$\gamma_p|\tau$-open set* of (X, τ) if for each point $x \in A$, there exists an open set U containing x such that $U^{\gamma_p|\tau} \subset A$. Moreover, a subset A is said to be *$\gamma_p|\tau$-closed* in (X, τ), if $X \setminus A$ is $\gamma_p|\tau$ -open in (X, τ). We suppose that the empty set is $\gamma_p|\tau$-open and we denote the set of all $\gamma_p|\tau$-open sets of (X, τ) by $\tau_{\gamma_p|\tau}$. We note that:

$(*)$ $U^{\gamma_p} = U^{\gamma_p|\tau}$ holds for any set $U \in \tau$.

Definition 2.3 (cf. [20]) Let (X, τ) be a topological space and $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ an operation on $PO(X, \tau)$. A nonempty subset A of (X, τ) is called a *γ_p-open set* of (X, τ) if for each point $x \in A$, there exists an open set U such that $x \in U$ and $U^{\gamma_p} \subset A$. We suppose that the emptyset $\emptyset$ is also γ_p-open for any operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$. The complement of a γ_p-open set is called *γ_p-closed* in (X, τ). We denote the set of all γ_p-open sets in (X, τ) by τ_{γ_p}.

Definition 2.4 (cf. [8, Definition 2.2]) Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be an operation and A a subset of a topological space (X, τ).

(i) The point $x \in X$ is *in the γ_p-closure of a set A* if $U^{\gamma_p} \cap A \neq \emptyset$ for each open set U containing x. The γ_p-closure of a set A is denoted by $\mathrm{Cl}_{\gamma_p}(A)$.

Namely, $\mathrm{Cl}_{\gamma_p}(A) := \{x \in X | U^{\gamma_p} \cap A \neq \emptyset$ for each open set U containing $x\}$.

(ii) A subset A is said to be γ_p*-closed* (*in the sense of Janković*) in (X, τ) if $A = \mathrm{Cl}_{\gamma_p}(A)$ holds.

Remark 2.5 We note that $\mathrm{Cl}_{\gamma_p}(A) = \mathrm{Cl}_{\gamma_p|\tau}(A)$ holds for any subset A of a topological space (X, τ), where $\mathrm{Cl}_{\gamma_p|\tau}(A) := \{x \in X | U^{\gamma_p|\tau} \cap A \neq \emptyset$ for any open set U containing $x\}$ (cf. [8], e.g., [20, Definition 3.1]). Indeed, $U^{\gamma_p} = U^{\gamma_p|\tau}$ holds for any open set U of (X, τ) (cf. Remark 2.2(ii)). It is obvious that $A \subset \mathrm{Cl}_{\gamma_p}(A)$ for any subset A of (X, τ).

Proposition 2.6 (cf. [20]) *Let* $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ *be an operation on* $PO(X, \tau)$ *and a subset* A *of a topological space* (X, τ).

(i) *A is γ_p-open in (X, τ) if and only if A is $\gamma_p|\tau$-open in (X, τ) (in the sense of Ogata* [20, Definition 2.2]; cf. Section 1 above). *Namely,* $\tau_{\gamma_p} = \tau_{\gamma_p|\tau}$ *holds.*

(ii) *A is γ_p-closed (in the sense of Janković),* i.e., $A = \mathrm{Cl}_{\gamma_p}(A)$, *if and only if A is $\gamma_p|\tau$-closed (in the sense of Janković* [8]), i.e., $A = \mathrm{Cl}_{\gamma_p|\tau}(A)$.

(iii)[20, Theorem 3.7] *The following properties are equivalent:*

(1) *A is $\gamma_p|\tau$-open in (X, τ);*

(2) *$X \setminus A$ is $\gamma_p|\tau$-closed (in the sense of Janković),* i.e., $\mathrm{Cl}_{\gamma_p|\tau}(X \setminus A) = X \setminus A$ *holds;*

(3) $\tau_{\gamma_p|\tau}$-$\mathrm{Cl}(X \setminus A) = X \setminus A$ *holds;*

(4) *$X \setminus A$ is $\gamma_p|\tau$-closed in (X, τ)* (cf. [20, Definition 2.2]).

(iv) *A is γ_p-closed (in the sense of Janković),* i.e., $A = \mathrm{Cl}_{\gamma_p}(A)$, *if and only if $X \setminus A$ is γ_p-open in (X, τ).*

(v) (cf. [20, p.176]) *Every γ_p-open set is open in (X, τ),* i.e., $\tau_{\gamma_p} \subset \tau$.

(vi) *An abitrary union of γ_p-open sets is γ_p-open.*

Proof. **(i) (Necessity)** Let $x \in A$. There exists an open set U such that $x \in U$ and $U^{\gamma_p} \subset A$. By $(*)$ in Remark 2.2, $U^{\gamma_p|\tau} \subset A$ and so A is $\gamma_p|\tau$-open. **(Sufficiency)** It is easy to prove by using $(*)$ in Remark 2.2. **(ii)** This follows from Definition 2.4 and Remark 2.5. **(iii)** By (i), (ii) and [20, Theorem 3.7], (iii) is proved. **(iv)** This is shown by (i), (ii) and (iii). **(v)** Let $A \in \tau_{\gamma_p}$. Then, for each point $x \in A$ there exists an open set $U(x)$ containing x such that $A = \bigcup\{U(x) | x \in A\} = \bigcup\{U(x)^{\gamma_p} | x \in A\}$. Thus, we have that $A \in \tau$. **(vi)** By [20, Proposition 2.3] (cf. Section 1 above), an arbitrary union of $\gamma_p|\tau$-open sets is $\gamma_p|\tau$-open. Thus, using (i), (iv) is obtained. □

3 Pre γ_p-open sets and operation-closures In this section the notion of *pre-γ_p-open sets* is defined and related properties are investigated, where $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is an operation on $PO(X, \tau)$.

Definition 3.1 Let (X, τ) be a topological space and $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ an operation on $PO(X, \tau)$. A nonempty subset A of (X, τ) is called a *pre γ_p-open set* of (X, τ) if for each point $x \in A$, there exists a preopen set U such that $x \in U$ and $U^{\gamma_p} \subset A$. We suppose that the emptyset $\emptyset$ is also pre γ_p-open for any operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$. We denote the set of all pre γ_p-open sets in (X, τ) by $PO(X, \tau)_{\gamma_p}$ (or shortly, $PO(X)_{\gamma_p}$).

Example 3.2 (i) A subset A is a pre "*id*"-open set of (X, τ) if and only if A is preopen in (X, τ). The operation "*id*" $: PO(X, \tau) \to \mathcal{P}(X)$ is defined by $V^{\text{“}id\text{”}} = V$ for any set $V \in PO(X, \tau)$; this operation is called the *identity operation* on $PO(X, \tau)$ (cf. [8]). A subset A is an "*id*"-open set of (X, τ) if and only if A is open in (X, τ). Therefore, we have that $PO(X, \tau)_{\text{“}id\text{”}} = PO(X, \tau)$ and $\tau_{\text{“}id\text{”}} = \tau$.

(ii) (ii-1) We characterize pre "Cl"-open sets, where "Cl": $PO(X,\tau) \to \mathcal{P}(X)$ is the operation defined by $V^{\text{"}Cl\text{"}}$:=Cl(V) for any subset $V \in PO(X,\tau)$. A nonempty subset A is pre "Cl"-open in (X,τ) if and only if, by definition, for each point $x \in A$ there exists a subset $U \in PO(X,\tau)$ such that $x \in U$ and $U^{\text{"}Cl\text{"}} \subset A$; if and only if for each point $x \notin X \setminus A$, there exists a subset $V \in PO(X,\tau)$ such that $x \in V$ and $V^{\text{"}Cl\text{"}} \cap (X \setminus A) = \emptyset$; if and only if $\mathrm{pCl}_{\text{"}Cl\text{"}}(X \setminus A) \subset X \setminus A$, where $\mathrm{pCl}_{\text{"}Cl\text{"}}(B) := \{z \in X | Cl(W) \cap B \neq \emptyset$ for any subset $W \in PO(X,\tau)$ such that $z \in W\}$ for a subset B of (X,τ) (cf. Definition 3.10 below). Thus, a nonempty set A is pre "Cl"-open in (X,τ) if and only if $\mathrm{pCl}_{\text{"}Cl\text{"}}(X \setminus A) = X \setminus A$ hold. The following property holds: a nonempty set A is pre "Cl"-open in (X,τ) if and only if A is θ-open in (X,τ).

Proof. Let A be pre "Cl"-open. For any point x of A, there exists $U \in PO(X,\tau)$ such that $x \in U \subset \mathrm{Cl}(U) \subset A$. Set O =Int(Cl(U)), then we have $x \in U \subset O \in \tau$ and $\mathrm{Cl}(U) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(U)))=\mathrm{Cl}(O) \subset \mathrm{Cl}(U)$. Therefore, we obtain that for each $x \in A$ there exists $O \in \tau$ such that $x \in O \subset \mathrm{Cl}(O) \subset A$. This shows that A is θ-open. The converse is obvious.

(ii-2) The operation "pCl" : $PO(X,\tau) \to \mathcal{P}(X)$ is defined by $V^{\text{"}pCl\text{"}} = pCl(V)$ for any set $V \in PO(X,\tau)$. We note that "pCl"$\neq$"Cl": $PO(X,\tau) \to \mathcal{P}(X)$ in general and

($*$) a subset A is a pre "pCl"-open set if and only if A is pre θ-open [22] in (X,τ).

A closure $\mathrm{pCl}_\theta(B)$ of a subset B is defined by $\mathrm{pCl}_\theta(B) := \{y \in X | \mathrm{pCl}(V) \cap B \neq \emptyset$ for every preopen set V containing $y\}$ [22]. A subset B is said to be *pre θ-closed* in (X,τ) if $B = \mathrm{pCl}_\theta(B)$ holds; a subset A is said to be *pre θ-open* in (X,τ) if $X \setminus A = \mathrm{pCl}_\theta(X \setminus A)$ holds. It is obviously obtained that $A \subset \mathrm{pCl}_\theta(A)$ for any subset A of (X,τ).

The proof of ($*$): A subset A is pre "pCl"-open if and only if, for each point $x \in A$, there exists a subset $U \in PO(X,\tau)$ such that $x \in U$ and $U^{\text{"}pCl\text{"}} \subset A$; if and only if, for each point $x \notin X \setminus A$, there exists a subset $U \in PO(X,\tau)$ such that $x \in U$ and $U^{\text{"}pCl\text{"}} \cap (X \setminus A) = \emptyset$; if and only if $\mathrm{pCl}_\theta(X \setminus A) \subset X \setminus A$ and so A is pre θ-open in (X,τ).

(ii-3) The following example shows that the operations "pCl" and "Cl" are distinct operations on $PO(X,\tau)$ in general. Indeed, let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a,b\}, X\}$. In a topological space (X,τ), $PO(X,\tau) = \{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{a,c\},X\}$ holds and so $\mathrm{Cl}(\{a\}) = X$; $\mathrm{pCl}(\{a\})=\{a\}$. Thus, we have that, for a preopen set $\{a\}$, "pCl"$(\{a\}) \neq$ "Cl"$(\{a\})$.

(iii) The operation "$Int \circ Cl$":$PO(X,\tau) \to \mathcal{P}(X)$ is well defined by $V^{\text{"}Int\circ Cl\text{"}}$:=Int(Cl(V)) for any subset $V \in PO(X,\tau)$. Indeed, $V \subset V^{\text{"}Int\circ Cl''} = Int(Cl(V))$ holds for any $V \in PO(X,\tau)$ by definition of the preopen sets. This is called the *interior-closure operation* on $PO(X,\tau)$ (cf. [8]). For this operation we note that:

($**$) a subset A is pre "$Int \circ Cl$"-open in (X,τ) if and only if A is "$Int \circ Cl$"-open in (X,τ), i.e., A is δ-open in (X,τ) [26].

The proof of ($**$): Suppose that A is pre "$Int \circ Cl$"-open in (X,τ). For a point $x \in A$, there exists a subset $U \in PO(X,\tau)$ such that $x \in U$ and $Int(Cl(U)) \subset A$ if and only if there exists a subset $G \in \tau$ such that $x \in G$ and $Int(Cl(G)) \subset A$ (i.e., by definition, A is "$Int \circ Cl$"-open in (X,τ)). For the proof of necessity of the last equivalence, we can take $G = Int(Cl(U))$. By definitions, A is "$Int \circ Cl$"-open in (X,τ) if and only if A is δ-open in (X,τ). Recall that τ_δ denotes the collection of all δ-open sets in (X,τ). It is well known that τ_δ is a topology of X. By means of ($**$), we conclude that $\tau_\delta = \tau_{\text{"}Int\circ Cl\text{"}} = PO(X,\tau)_{\text{"}Int\circ Cl\text{"}}$ holds and so $PO(X,\tau)_{\text{"}Int\circ Cl\text{"}}$ is a topology of X (cf. Theorem 3.8(iv) below).

(iv) For more examples, operations from $PO(X,\tau)$ into $\mathcal{P}(X)$ are well defined as follows: The operations "Cl_θ", "Cl_δ", "pCl_θ", "αCl", "sCl", "θ-sCl" : $PO(X,\tau) \to \mathcal{P}(X)$

are well defined, respectively, by $V^{\text{"}Cl_\theta\text{"}}$:=$\mathrm{Cl}_\theta(V)$ [26], $V^{\text{"}Cl_\delta\text{"}}$:=$\mathrm{Cl}_\delta(V)$ [26], $V^{\text{"}pCl_\theta\text{"}}$:= $\mathrm{pCl}_\theta(V)$ [16], $V^{\text{"}\alpha Cl\text{"}} := \alpha\ \mathrm{Cl}(V)$ [18], $V^{\text{"}sCl\text{"}}$:=sCl(V) [13], $V^a := \theta$-sCl(V), where $a :=$ "θ-sCl" [10] for every set $V \in PO(X,\tau)$. We recall some definitions as follows: For a subset B of (X,τ), δ-closure $\mathrm{Cl}_\delta(B)$ [26] (resp. θ-closure $\mathrm{Cl}_\theta(B)$ [26]) of B is defined by $\mathrm{Cl}_\delta(B) := \{y \in X|\ \mathrm{Int}(\mathrm{Cl}(U)) \cap B \neq \emptyset$ for every open set U containing $y\}$ (resp. $\mathrm{Cl}_\theta(B) := \{y \in X|\mathrm{Cl}(U) \cap B \neq \emptyset$ for every open set U containing $y\}$). For a subset B of (X,τ), the α-closure $\alpha\mathrm{Cl}(B)$ [18] (resp. semi-closure $\mathrm{sCl}(B)$ [13]) of the set B is the intersection of all α-closed sets (resp. semi-closed sets) containing B; $\alpha\mathrm{Cl}(B)$ (resp. $\mathrm{sCl}(B)$) is α-closed (resp. semi-closed) in (X,τ). For these operations above, the following results are probably unexpected:

"Cl" = "Cl_θ" = "Cl_δ" = "αCl" : $PO(X,\tau) \to \mathcal{P}(X)$, "$pCl$" = "$pCl_\theta$" : $PO(X,\tau) \to \mathcal{P}(X)$ and "sCl" = "θ-sCl" : $PO(X,\tau) \to \mathcal{P}(X)$ hold.

Indeed, it is shown that $\mathrm{Cl}(V)$ =$\mathrm{Cl}_\theta(V)$ =$\mathrm{Cl}_\delta(V) = \alpha\mathrm{Cl}(V)$ hold for any set $V \in PO(X,\tau)$ ([9, Corollary 2.5 (c)], e.g., [19, Lemma 2.1]), $\mathrm{pCl}(V)$ =$\mathrm{pCl}_\theta(V)$ holds for any set $V \in PO(X,\tau)$ ([3, Proposition 4.2]) and $\mathrm{sCl}(V) = \theta$-$\mathrm{sCl}(V)$ holds for any set $V \in PO(X,\tau)$ ([23, Lemma 1]).

(v) Let "Cl"$|\tau$, "pCl"$|\tau$, "$\alpha Cl|\tau$" : $\tau \to \mathcal{P}(X)$ be the restrictions to τ of operations "Cl", "pCl", "αCl" : $PO(X,\tau) \to \mathcal{P}(X)$, respectively. Then, "$Cl$"$|\tau$ = "pCl"$|\tau$ = "αCl"$|\tau : \tau \to \mathcal{P}(X)$ holds over τ, because $\mathrm{Cl}(V)$ =$\mathrm{pCl}(V) = \alpha\mathrm{Cl}(V)$ for any open set V of (X,τ).

(vi) Suppose that $X = \{a,b,c\}, \tau = \{\emptyset, X, \{a\}, \{a,b\}\}$. Then, it is shown that $PO(X,\tau) = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$.

We define an operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ such that $\gamma_p(A) := A$ if $b \in A$, $\gamma_p(A)$:=pCl(A) if $b \notin A$. Then we have that $PO(X,\tau)_{\gamma_p} = \{\emptyset, X, \{a,b\}\}$.

Theorem 3.3 *Let $\gamma_p : PO(X,\tau) \to P(X)$ be any operation on $PO(X,\tau)$.*

(i) *Every pre γ_p-open set of (X,τ) is preopen in (X,τ), i.e., $PO(X,\tau)_{\gamma_p} \subset PO(X,\tau)$.*

(ii) *Every γ_p-open set of (X,τ) is pre γ_p-open, i.e., $\tau_{\gamma_p} \subset PO(X,\tau)_{\gamma_p}$.*

(iii) *If $\{A_i|i \in J\}$ is a collection of pre γ_p-open sets in (X,τ), then $\bigcup\{A_i|\ i \in J\}$ is pre γ_p-open in (X,τ), where J is any index set.*

Proof. **(i)** Suppose that $A \in PO(X)_{\gamma_p}$. Let $x \in A$. Then, there exists a preopen set U such that $x \in U \subset U^{\gamma_p} \subset A$. Because U is a preopen set, this implies $x \in U \subset \mathrm{Int}(\mathrm{Cl}(U)) \subset \mathrm{Int}(\mathrm{Cl}(A))$. Thus we show that $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ and hence $A \in PO(X)$. Thus we have that $PO(X,\tau)_{\gamma_p} \subset PO(X,\tau)$.

An alternative proof: Suppose that $A \in PO(X)_{\gamma_p}$. Let $x \in A$. There exists a preopen set $U(x)$ containing x such that $U(x)^{\gamma_p} \subset A$. Then, $\bigcup\{U(x)|x \in A\} \subset \bigcup\{U(x)^{\gamma_p}|x \in A\} \subset A$ and so $A = \bigcup\{U(x)|x \in A\} \in PO(X,\tau)$ holds. **(ii)** Let A be a γ_p-open set in (X,τ) and $x \in A$. There exists an open set U such that $x \in U \subset U^{\gamma_p} \subset A$. Since every open set is a preopen set, this implies that A is a pre γ_p-open set. Hence, it follows from definitions that $\tau_{\gamma_p} \subset PO(X,\tau)_{\gamma_p}$ holds. **(iii)** Let $x \in \bigcup\{A_i|i \in J\}$, then $x \in A_i$ for some $i \in J$. Since A_i is a pre γ_p-open set, there exists a preopen set U containing x such that $U^{\gamma_p} \subset A_i \subset \bigcup\{A_i|i \in J\}$. Hence $\bigcup\{A_i|i \in J\}$ is a pre γ_p-open set. □

Remark 3.4 (i) The converses of Theorem 3.3 (i) and (ii) above need not be true. Let $X := \{a,b,c,d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then, for a topological space (X,τ), we have $PO(X,\tau) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}, \{a,b,c\}\}$. Define an operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ by putting $\gamma_p(A) := A$ if $a \in A$, $\gamma_p(A)$:=pCl(A) if $a \notin A$. Then it is clearly to see that $\{b\} \in PO(X,\tau)$ but $\{b\}$ is not pre γ_p-open; $\{a,b,d\}$ is a pre γ_p-open set but not a γ_p-open set.

(ii) In general, the intersection of two pre γ_p-open sets need not be a pre γ_p-open set. Let $X := \{a,b,c\}$, $\tau := \{\emptyset, X, \{a\}, \{a,b\}\}$. For a topological space $(X,\tau), PO(X,\tau) = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$. Define an operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ by putting $\gamma_p(A) := A$ if $A \neq \{a\}$, $\gamma_p(A) := \{a,b\}$ if $A = \{a\}$. Then $A = \{a,b\}$ and $B = \{a,c\}$ are pre γ_p-open sets but $A \cap B = \{a\}$ is not a pre γ_p-open set.

Definition 3.5 Let (X,τ) be a topological space and $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ an operation. Then, (X,τ) is said to be *pre γ_p-regular* (resp. *γ_p-regular*) if for each point $x \in X$ and for every preopen (resp. open) set V containing x, there exists a preopen (resp. an open) set U containing x such that $U^{\gamma_p} \subset V$.

Theorem 3.6 *Let (X,τ) be a topological space and $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ an operation on $PO(X,\tau)$.*

(i) *The following properties are equivalent:*

(1) *$PO(X,\tau) = PO(X,\tau)_{\gamma_p}$;*

(2) *(X,τ) is a pre γ_p-regular space;*

(3) *For every $x \in X$ and for every preopen set U of (X,τ) containing x, there exists a pre γ_p-open set W of (X,τ) such that $x \in W$ and $W \subset U$.*

(ii) *A space (X,τ) is γ_p-regular if and only if (X,τ) is $\gamma_p|\tau$-regular (in the sense of Kasahara)*[11], e.g., [20].

(iii) *The following properties are equivalent:*

(1) *$\tau = \tau_{\gamma_p}$ holds;*

(2) *(X,τ) is a γ_p-regular space;*

(3) *For every $x \in X$ and for every open set U of (X,τ) containing x, there exists a γ_p-open set W of (X,τ) such that $x \in W$ and $W \subset U$.*

Proof. **(i) (1)⇒(2)** Let $x \in X$ and V a preopen set containing x. It follows from assumption that V is a pre γ_p-open set. This implies that there exists a preopen set U containing x such that $U^{\gamma_p} \subset V$. Hence, (X,τ) is a pre γ_p-regular space. **(2)⇒(3)** Let $x \in X$ and U be a preopen set containing x. Then, by (2) there is a preopen set W containing x and $W \subset W^{\gamma_p} \subset U$. By using (2) for the set W, it is shown that W is pre γ_p-open. Hence, W is a pre γ_p-open set containing x such that $W \subset U$. **(3)⇒(1)** By (3) and Theorem 3.3(iii), it follows that every preopen set is pre γ_p-open, i.e., $PO(X,\tau) \subset PO(X,\tau)_{\gamma_p}$. It follows from Theorem 3.3(i) that the converse inclusion $PO(X,\tau)_{\gamma_p} \subset PO(X,\tau)$ holds. **(ii)** By definition, $U^{\gamma_p} = U^{\gamma_p|\tau}$ holds for every $U \in \tau$. Thus the proof is obtained. **(iii) (1)⇒(2)** By (1) and Proposition 2.6(i), $\tau = \tau_{\gamma_p|\tau} = \tau_{\gamma_p}$. Using [20, Proposition 2.4], we have that (X,τ) is $\gamma_p|\tau$ -regular and so, by (ii), (X,τ) is γ_p-regular. **(2)⇒(3)** Let $x \in X$ and U be an open set containing x. By (2), there exists a subset $W \in \tau$ such that $x \in W$ and $W^{\gamma_p} \subset U$. Using (2) for the set W and any point $y \in W$, it is shown that W is γ_p-open. Then, U is γ_p-open and so (X,τ) is a γ_p-regular space. **(3)⇒(1)** It is enough to prove $\tau \subset \tau_{\gamma_p}$, because $\tau_{\gamma_p} \subset \tau$ (cf. Proposition 2.6(v)). Let $U \in \tau$. By using (3) for the set U and each point $x \in U$, there exists a subset $W(x) \in \tau_{\gamma_p}$ such that $W(x) \subset U$. Thus we have that $U = \bigcup\{W(x) | x \in U\}$ and $U \in \tau_{\gamma_p}$ (cf. Proposition 2.6(vi)). Therefore, we have that $\tau \subset \tau_{\gamma_p}$. □

Definition 3.7 An operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is called to be *preregular* (resp. *regular*, cf.,[11, p.98], e.g., [20, Definition 2.5]) if for each point $x \in X$ and for every pair of preopen (resp. open) sets U and V containing $x \in X$, there exists a preopen (resp. an open) set W such that $x \in W$ and $W^{\gamma_p} \subset U^{\gamma_p} \cap V^{\gamma_p}$.

Theorem 3.8 (i) *Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be a preregular operation on $PO(X,\tau)$. If A and B are pre γ_p-open in (X,τ), then $A \cap B$ is also pre γ_p-open in (X,τ).*

(ii) *An operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is regular if and only if $\gamma_p|\tau : \tau \to \mathcal{P}(X)$ is regular* (*in the sense of* [11, p.98], e.g., [20, Definition 2.5]).

(iii) *Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be a regular operation on $PO(X,\tau)$. If A and B are γ_p-open in (X,τ), then $A \cap B$ is also γ_p-open in (X,τ).*

(iv) *If $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is a preregular* (resp. *regular*) *operation, then $PO(X,\tau)_{\gamma_p}$* (resp. τ_{γ_p}) *is a topology of X.*

Proof. **(i)** Let $x \in A \cap B$. Since A and B are pre γ_p-open sets, there exists preopen sets U, V such that $x \in U, x \in V$ and $U^{\gamma_p} \subset A$ and $V^{\gamma_p} \subset B$. By preregularity of γ_p, there exists a preopen set W containing x such that $W^{\gamma_p} \subset U^{\gamma_p} \cap V^{\gamma_p} \subset A \cap B$. Therefore, $A \cap B$ is a pre γ_p-open set. **(ii)** Since $U^{\gamma_p} = U^{\gamma_p|\tau}$ for any open subset A of (X,τ), we have the equivalence. **(iii)** It is proved by (ii) above, Proposition 2.6(i) and [20, Proposition 2.9]. **(iv)** It is proved by (i) above and Theorem 3.3(iii) (resp. (iii) above and Proposition 2.6(vi)). □

Definition 3.9 Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation and A a subset of a topological space (X,τ).

(i) A subset A is said to be γ_p*-closed* in (X,τ) if $X \setminus A$ is a γ_p-open set of (X,τ) (cf. Proposition 2.6(iv)).

(ii) A subset A is said to be *pre γ_p-closed* in (X,τ) if $X \setminus A$ is pre γ_p-open in (X,τ).

(iii) The following subsets are well defined as follows:

τ_{γ_p}-Cl$(A) := \bigcap\{F | F$ is a γ_p-closed set of (X,τ) such that $A \subset F\}$(cf. (i) above, Proposition 2.6(ii)(iii));

$PO(X)_{\gamma_p}$-Cl$(A) := \bigcap\{F | F$ is pre-γ_p-closed in (X,τ) such that $A \subset F\}$.

Definition 3.10 Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation and A a subset of a topological space (X,τ). A point $x \in X$ is in the *pre γ_p-closure of a set A* if $U^{\gamma_p} \bigcap A \neq \emptyset$ for each preopen set U containing x. The pre γ_p-closure of A is denoted by $\mathrm{pCl}_{\gamma_p}(A)$. Namely, $\mathrm{pCl}_{\gamma_p}(A) := \{x \in X | U^{\gamma_p} \cap A \neq \emptyset$ for any preopen set U containing $x\}$.

Theorem 3.11 *Let A be a subset of a topological space (X,τ). Then we have the following properties on $PO(X,\tau)_{\gamma_p}$-closures and τ_{γ_p}-closures.*

(i) $PO(X)_{\gamma_p}$-Cl$(A) = \{y \in X | V \cap A \neq \emptyset$ *for every set* $V \in PO(X,\tau)_{\gamma_p}$ *such that* $y \in V\}$.

(ii) τ_{γ_p}-Cl$(A) = \{y \in X | V \cap A \neq \emptyset$ *for every set* $V \in \tau_{\gamma_p}$ *such that* $y \in V\}$.

Proof. **(i)** Denote $E := \{y \in X | V \cap A \neq \emptyset$ for every set $V \in PO(X,\tau)_{\gamma_p}$ such that $y \in V\}$. We shall prove that $PO(X)_{\gamma_p}$-Cl$(A) = E$. Let $x \notin E$. Then there exists a pre γ_p-open set V containing x such that $V \cap A = \emptyset$. This implies that $X \setminus V$ is pre γ_p-closed and $A \subset X \setminus V$. Hence $PO(X)_{\gamma_p}$-Cl$(A) \subset X \setminus V$. It follows that $x \notin PO(X)_{\gamma_p}$-Cl$(A)$. Thus, we have that $PO(X)_{\gamma_p}$-Cl$(A) \subset E$. Conversely, let $x \notin PO(X)_{\gamma_p}$-Cl$(A)$. Then there exists a pre γ_p-closed set F such that $A \subset F$ and $x \notin F$. Then we have that $x \in X \setminus F$, $X \setminus F \in PO(X,\tau)_{\gamma_p}$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin E$. Hence $E \subset PO(X)_{\gamma_p}$-Cl$(A)$. Therefore, we have that $PO(X)_{\gamma_p}$-Cl$(A) = E$. **(ii)** By using Definition 3.9(iii), Proposition 2.6(i) and [20, (3.2), Proposition 3.3], it is obtained that τ_{γ_p}-Cl$(A) = \bigcap\{F | A \subset F, X \setminus F \in \tau_{\gamma_p}\} = \bigcap\{F | A \subset F, X \setminus F \in \tau_{\gamma_p|\tau}\} = \tau_{\gamma_p|\tau}$-Cl$(A) = \{y \in X | V \cap A \neq \emptyset$ for any $V \in \tau_{\gamma_p|\tau}$ such that $y \in V\} = \{y \in X | V \cap A \neq \emptyset$ for any $V \in \tau_{\gamma_p}$ such that $y \in V\}$. □

For $\mathrm{pCl}_{\gamma_p}(A)$ (cf. Definition 3.10) and $PO(X)_{\gamma_p}$-Cl(A)(cf. Definition 3.9(iii)), where A is a subset of a topological space (X,τ), we have the following properties Theorem 3.12 and Theorem 3.13:

Theorem 3.12 *Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation on $PO(X,\tau)$ and A and B subsets of a topological space (X,τ). Then, we have the following properties on* $\mathrm{pCl}_{\gamma_p}(A)$ *and* $\mathrm{pCl}_{\gamma_p}(B)$.

(i) *The set* $\mathrm{pCl}_{\gamma_p}(A)$ *is a preclosed set of* (X,τ) *and* $A \subset \mathrm{pCl}_{\gamma_p}(A)$.

(ii) $\mathrm{pCl}_{\gamma_p}(\emptyset) = \emptyset$ *and* $\mathrm{pCl}_{\gamma_p}(X) = X$.

(iii) *A is pre γ_p-closed* (i.e., *$X\backslash A$ is pre γ_p-open*) *in (X,τ) if and only if* $\mathrm{pCl}_{\gamma_p}(A) = A$ *holds.*

(iv) *If $A \subset B$, then* $\mathrm{pCl}_{\gamma_p}(A) \subset \mathrm{pCl}_{\gamma_p}(B)$.

(v) $\mathrm{pCl}_{\gamma_p}(A)\cup \mathrm{pCl}_{\gamma_p}(B) \subset \mathrm{pCl}_{\gamma_p}(A\cup B)$ *holds.*

(vi) *If $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is preregular, then* $\mathrm{pCl}_{\gamma_p}(A)\cup\mathrm{pCl}_{\gamma_p}(B) = \mathrm{pCl}_{\gamma_p}(A\cup B)$ *holds.*

(vii) $\mathrm{pCl}_{\gamma_p}(A\cap B) \subset \mathrm{pCl}_{\gamma_p}(A) \cap \mathrm{pCl}_{\gamma_p}(B)$ *holds.*

Proof. **(i)** For each point $x \in X\backslash \mathrm{pCl}_{\gamma_p}(A)$, by Definition 3.10, there exists a preopen set $U(x)$ containing x such that $U(x)^{\gamma_p} \cap A = \emptyset$. We set $V := \bigcup\{U(x) | x \in X\backslash\ \mathrm{pCl}_{\gamma_p}(A)\}$. Then, it is shown that $V = X\backslash\ \mathrm{pCl}_{\gamma_p}(A)$ holds. Indeed, for a point $y \in V$, there exists a subset $U(x) \in PO(X,\tau)$ such that $y \in U(x)$ and $U(x)^{\gamma_p} \cap A = \emptyset$. This shows that $y \notin \mathrm{pCl}_{\gamma_p}(A)$ and so $V \subset X\backslash \mathrm{pCl}_{\gamma_p}(A)$. Conversely, let $y \in X\backslash\ \mathrm{pCl}_{\gamma_p}(A)$. There exists a subset $U(y) \in PO(X,\tau)$ such that $U(y)^{\gamma_p} \cap A = \emptyset$ and so $y \in U(y) \subset V$. Thus, we conclude that $X\backslash\ \mathrm{pCl}_{\gamma_p}(A) \subset V$; we have that $V = X\backslash\ \mathrm{pCl}_{\gamma_p}(A)$. Therefore, $\mathrm{pCl}_{\gamma_p}(A)$ is preclosed in (X,τ), because $V \in PO(X,\tau)$. Obviously, by Definition 3.10, we have that $A \subset \mathrm{pCl}_{\gamma_p}(A)$. **(ii) (iv)** They are obtained from Definition 3.10. **(iii) (Necessity)** Suppose that $X \setminus A$ is pre γ_p-open in (X,τ). We claim that $\mathrm{pCl}_{\gamma_p}(A) \subset A$. Let $x \notin A$. There exists a preopen set U containing x such that $U^{\gamma_p} \subset X \setminus A$, i.e., $U^{\gamma_p} \cap A = \emptyset$. Hence, using Definition 3.10, we have that $x \notin \mathrm{pCl}_{\gamma_p}(A)$ and so $\mathrm{pCl}_{\gamma_p}(A) \subset A$. By (i), it is proved that $A = pCl_{\gamma_p}(A)$. **(Sufficiency)** Suppose that $A = \mathrm{pCl}_{\gamma_p}(A)$. Let $x \in X \setminus A$. Since $x \notin pCl_{\gamma_p}(A)$, there exists a preopen set U containing x such that $U^{\gamma_p} \cap A = \emptyset$, i.e., $U^{\gamma_p} \subset X \setminus A$. Namely, $X \setminus A$ is pre γ_p-open in (X,τ) and so A is pre γ_p-closed. **(v) (vii)** They are obtained from (iv). **(vi)** Let $x \notin \mathrm{pCl}_{\gamma_p}(A)\cup \mathrm{pCl}_{\gamma_p}(B)$. Then, there exist two preopen sets U and V containing x such that $U^{\gamma_p} \cap A = \emptyset$ and $V^{\gamma_p} \cap B = \emptyset$. By Definition 3.7, there exists a preopen set W containing x such that $W^{\gamma_p} \subset U^{\gamma_p} \cap V^{\gamma_p}$. Thus, we have that $W^{\gamma_p} \cap (A\cup B) \subset (U^{\gamma_p} \cap V^{\gamma_p}) \cap (A \cup B) \subset [U^{\gamma_p} \cap A] \cup [V^{\gamma_p} \cap B] = \emptyset$, i.e., $W^{\gamma_p} \cap (A\cup B) = \emptyset$. Namely, we have that $x \notin \mathrm{pCl}_{\gamma_p}(A\cup B)$ and so $\mathrm{pCl}_{\gamma_p}(A\cup B) \subset \mathrm{pCl}_{\gamma_p}(A)\cup \mathrm{pCl}_{\gamma_p}(B)$. We can obtain (vi) using (v). □

Theorem 3.13 *Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation on $PO(X,\tau)$ and A and B subsets of a topological space (X,τ). Then, we have the following properties on $PO(X)_{\gamma_p}$-*Cl(A) *and $PO(X)_{\gamma_p}$-*Cl(B).

(i) *The set $PO(X)_{\gamma_p}$-*Cl(A) *is a pre γ_p-closed set of (X,τ) and $A \subset PO(X)_{\gamma_p}$-$Cl(A)$.*

(ii) *$PO(X)_{\gamma_p}$-*Cl$(\emptyset) = \emptyset$ *and $PO(X)_{\gamma_p}$-*Cl$(X) = X$.

(iii) *A subset A is pre γ_p-closed* (i.e., *$X \setminus A$ is pre γ_p-open*) *in (X,τ) if and only if $PO(X)_{\gamma_p}$-*Cl$(A) = A$ *holds.*

(iv) *If $A \subset B$, then $PO(X)_{\gamma_p}$-*Cl$(A) \subset PO(X)_{\gamma_p}$-Cl$(B)$.

(v) $(PO(X)_{\gamma_p}$-Cl$(A)) \cup (PO(X)_{\gamma_p}$-Cl$(B)) \subset PO(X)_{\gamma_p}$-Cl$(A\cup B)$ *holds.*

(vi) *If* $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ *is preregular, then* $(PO(X)_{\gamma_p}\text{-Cl}(A)) \cup (PO(X)_{\gamma_p}\text{-Cl}(B)) = PO(X)_{\gamma_p}\text{-Cl}(A \cup B)$ *holds.*
(vii) $PO(X)_{\gamma_p}\text{-Cl}(A \cap B) \subset (PO(X)_{\gamma_p}\text{-Cl}(A)) \cap (PO(X)_{\gamma_p}\text{-Cl}(B))$ *holds.*
(viii) $PO(X)_{\gamma_p}\text{-Cl}(PO(X)_{\gamma_p}\text{-Cl}(A)) = PO(X)_{\gamma_p}\text{-Cl}(A)$ *holds.*

Proof. **(i)** By Theorem 3.3(iii) and Definition 3.9(ii)(iii), it is obtained that $PO(X)_{\gamma_p}$-Cl(A) is a pre γ-closed set and $A \subset PO(X)_{\gamma_p}\text{-Cl}(A)$. **(ii)** **(iv)** They are obtained from Definition 3.9(iii). **(iii)** By (i) and Definition 3.9, the equivalence is proved. **(v)** **(vii)** They are proved by (iv). **(vi)** Let $x \notin (PO(X)_{\gamma_p}\text{-}Cl(A)) \cup (PO(X)_{\gamma_p}\text{-Cl}(B))$. Then, there exist two pre γ_p-open sets U and V containing x such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. By Theorem 3.8, it is proved that $U \cap V$ is γ_p-open in (X,τ) such that $(U \cap V) \cap (A \cup B) = \emptyset$. Thus, we have that $x \notin PO(X)_{\gamma_p}\text{-Cl}(A \cup B)$ and hence $PO(X)_{\gamma_p}\text{-Cl}(A \cup B) \subset (PO(X)_{\gamma_p}$-Cl$(A)) \cup (PO(X)_{\gamma_p}\text{-Cl}(B))$. Using (v), we have the equality. **(viii)** The proof is obvious from (i) and (iii). □

For $\text{Cl}_{\gamma_p}(A)$ (cf. Definition 2.4(i)) and $\tau_{\gamma_p}\text{-Cl}(A)$(cf. Definition 3.9(iii)), where A is a subset of (X,τ), we have the following properties Theorem 3.14 and Theorem 3.15:

Theorem 3.14 *Let* $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ *be an operation on* $PO(X,\tau)$ *and* A *and* B *subsets of a topological space* (X,τ)*. Then, we have the following properties on* $\text{Cl}_{\gamma_p}(A)$ *and* $\text{Cl}_{\gamma_p}(B)$,(cf. Definition 2.4(i)).
(i) *The set* $\text{Cl}_{\gamma_p}(A)$ *is a closed set of* (X,τ) *and* $A \subset \text{Cl}_{\gamma_p}(A)$.
(ii) $\text{Cl}_{\gamma_p}(\emptyset) = \emptyset$ *and* $\text{Cl}_{\gamma_p}(X) = X$.
(iii)(Proposition 2.6(iv), Definition 3.9(i)) *A subset* A *is* γ_p*-closed* (i.e., $X \setminus A$ *is* γ_p*-open*) *in* (X,τ) *if and only if* A *is* γ_p*-closed in* (X,τ)(*in the sense of Janković*) (i.e., $\text{Cl}_{\gamma_p}(A) = A$ *holds*).
(iv) *If* $A \subset B$*, then* $\text{Cl}_{\gamma_p}(A) \subset \text{Cl}_{\gamma_p}(B)$.
(v) $\text{Cl}_{\gamma_p}(A) \cup \text{Cl}_{\gamma_p}(B) \subset \text{Cl}_{\gamma_p}(A \cup B)$ *holds.*
(vi) *If* $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ *is regular, then* $\text{Cl}_{\gamma_p}(A) \cup \text{Cl}_{\gamma_p}(B) = \text{Cl}_{\gamma_p}(A \cup B)$ *holds.*
(vii) $\text{Cl}_{\gamma_p}(A \cap B) \subset \text{Cl}_{\gamma_p}(A) \cap \text{Cl}_{\gamma_p}(B)$ *holds.*

Proof. **(i)** By Remark 2.5 and [20, Theorem 3.6(i)], respectively, it is known that $\text{Cl}_{\gamma_p}(A) = \text{Cl}_{\gamma_p|\tau}(A)$ and every $\text{Cl}_{\gamma_p|\tau}(A)$ is closed in (X,τ) for any susbset A of (X,τ) and any operation $\gamma_p|\tau : \tau \to P(X)$. **(ii)** **(iv)** They are obtained from Definition 2.4. **(v)** **(vii)** They are obtained from (iv). **(vi)** This follows from Remark 2.5, Theorem 3.8(ii) and [20, Lemma 3.10]. □

Theorem 3.15 *Let* $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ *be an operation on* $PO(X,\tau)$ *and* A *and* B *subsets of a topological space* (X,τ)*. Then, we have the following properties on* $\tau_{\gamma_p}\text{-Cl}(A)$ *and* $\tau_{\gamma_p}\text{-Cl}(B)$.
(i) *The set* $\tau_{\gamma_p}\text{-Cl}(A)$ *is a* γ_p*-closed set of* (X,τ) *and* $A \subset \tau_{\gamma_p}\text{-Cl}(A)$.
(ii) $\tau_{\gamma_p}\text{-Cl}(\emptyset) = \emptyset$ *and* $\tau_{\gamma_p}\text{-Cl}(X) = X$.
(iii) A *is* γ_p*-closed* (i.e., $X \setminus A$ *is* γ_p*-open*) *in* (X,τ) *if and only if* $\tau_{\gamma_p}\text{-Cl}(A) = A$ *holds.*
(iv) *If* $A \subset B$*, then* $\tau_{\gamma_p}\text{-Cl}(A) \subset \tau_{\gamma_p}\text{-Cl}(B)$.
(v) $(\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B)) \subset \tau_{\gamma_p}\text{-Cl}(A \cup B)$ *holds.*
(vi) *If* $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ *is regular, then* $(\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B)) = \tau_{\gamma_p}$-Cl$(A \cup B)$ *holds.*
(vii) $\tau_{\gamma_p}\text{-Cl}(A \cap B) \subset (\tau_{\gamma_p}\text{-Cl}(A)) \cap (\tau_{\gamma_p}\text{-Cl}(B))$ *holds.*
(viii) $\tau_{\gamma_p}\text{-Cl}(\tau_{\gamma_p}\text{-Cl}(A)) = \tau_{\gamma_p}\text{-Cl}(A)$ *holds.*

Proof. **(i)** By Proposition 2.6(iv)(vi) and Definition 3.9(iii), it is obtained that τ_{γ_p}-Cl(A) is a γ_p-closed set. **(ii)-(iv)** They are obtained from Definition 3.9(iii). **(v) (vii)** They are proved by using (iv). **(vi)** Let $x \notin (\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B))$. Then, there exist two γ_p-open sets U and V containing x such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. By Theorem 3.8(iii), it is proved that $U \cap V$ is γ_p-open in (X, τ) such that $(U \cap V) \cap (A \cup B) = \emptyset$. Thus, we have that $x \notin \tau_{\gamma_p}\text{-Cl}(A \cup B)$ and hence $\tau_{\gamma_p}\text{-Cl}(A \cup B) \subset (\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B))$. Using (v), we have the equality. **(viii)** The proof is obvious from (i) and (iii). □

Theorem 3.16 *For a subset A of a topological space (X, τ) and any operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$, the following relations hold.*

(i) $\text{pCl}(A) \subset \text{pCl}_{\gamma_p}(A) \subset PO(X)_{\gamma_p}\text{-Cl}(A) \subset \tau_{\gamma_p}\text{-Cl}(A)$.

(ii) $\text{pCl}(A) \subset \text{Cl}(A) \subset \text{Cl}_{\gamma_p}(A) \subset \tau_{\gamma_p}\text{-Cl}(A)$.

Proof. **(i)** The implication that $\text{pCl}(A) \subset \text{pCl}_{\gamma_p}(A)$ is proved by Definition 3.10 and Definition 2.1; $\text{pCl}_{\gamma_p}(A) \subset PO(X)_{\gamma_p}\text{-Cl}(A)$ is proved by using Theorem 3.11 (i), Definition 3.1 and Definition 3.10; $PO(X)_{\gamma_p}\text{-Cl}(A) \subset \tau_{\gamma_p}\text{-Cl}(A)$ is obtained by Theorem 3.3(ii) and Definition 3.9(iii). **(ii)** The implication that $\text{pCl}(A) \subset \text{Cl}(A)$ is proved by a fact that $\tau \subset PO(X, \tau)$; $\text{Cl}(A) \subset \text{Cl}_{\gamma_p}(A)$ is obtained by Definition 2.4; $\text{Cl}_{\gamma_p}(A) \subset \tau_{\gamma_p}\text{-Cl}(A)$ is proved by using Theorem 3.11(ii), Definition 2.3 and Definition 2.4(i). □

Corollary 3.17 *Let A be a subset of a topological space (X, τ) and $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ an operation on $PO(X, \tau)$.*

(i) *The following properties are equivalent:*

(1) *A subset A is pre γ_p-open in (X, τ)* (cf. Definition 3.1)*;*

(2) $\text{pCl}_{\gamma_p}(X \setminus A) = X \setminus A$*;*

(3) $PO(X)_{\gamma_p}\text{-Cl}(X \setminus A) = X \setminus A$*;*

(4) *$X \setminus A$ is pre γ_p-closed in (X, τ)* (cf. Definition 3.9(ii)).

(ii) *The following properties are equivalent:*

(1) *A subset A is γ_p-open in (X, τ)* (cf. Definition 2.3)*;*

(2) $\text{Cl}_{\gamma_p}(X \setminus A) = X \setminus A$*;*

(3) $\tau_{\gamma_p}\text{-Cl}(X \setminus A) = X \setminus A$*;*

(4) *A subset $X \setminus A$ is γ_p-closed in (X, τ)* (cf. Definition 3.9(ii)).

(5) *A subset $X \setminus A$ is $\gamma_p|\tau$-closed in (X, τ)* (cf. [20, Definition 2.2]).

Proof. **(i) (1)⇔(2) (resp. (3)⇔(4))** It is obtained by Theorem 3.12(iii) (resp. Theorem 3.13(iii)). **(4)⇔(1)** This follows from Definition 3.1 and Definition 3.9(ii). **(ii) (1)⇔(2) (resp. (3)⇔(4))** This follows from Theorem 3.14(iii) (resp. Theorem 3.15(iii)). **(4)⇔(1) (resp. (5)⇔(1))** This follows from Definition 2.3 and Definition 3.9(i) (resp. Proposition 2.6(i)(iii)). □

Corollary 3.18 *Let A be a subset of a topological space (X, τ) and $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ an operation on $PO(X, \tau)$.*

(i) *If (X, τ) is a pre γ_p-regular space, then* $\text{pCl}(A) = \text{pCl}_{\gamma_p}(A) = PO(X)_{\gamma_p}\text{-Cl}(A)$.

(ii) *If (X, τ) is a γ_p-regular space, then* $\text{Cl}(A) = \text{Cl}_{\gamma_p}(A) = \tau_{\gamma_p}\text{-Cl}(A)$.

Proof. **(i)** By Theorem 3.6(i), it is shown that $\text{pCl}(A) = PO(X)_{\gamma_p}\text{-Cl}(A)$. Using Theorem 3.16(i), we have that $\text{pCl}(A) = \text{pCl}_{\gamma_p}(A) = PO(X)_{\gamma_p}\text{-Cl}(A)$. **(ii)** By Theorem 3.6(iii), it is shown that $\text{Cl}(A) = \tau_{\gamma_p}\text{-Cl}(A)$. Using Theorem 3.16(ii), we have that $\text{Cl}(A) = \text{Cl}_{\gamma_p}(A) = \tau_{\gamma_p}\text{-Cl}(A)$. □

In order to investigate the relationship among $PO(X)_{\gamma_p}$-Cl(A), pCl$_{\gamma_p}(A)$, τ_{γ_p}-Cl(A) and Cl$_{\gamma_p}(A)$ for any set $A \in \mathcal{P}(X)$, we introduce the following notions of *preopen operations* and *open operations* (cf. [20, Definition 2.6, Example 2.7]):

Definition 3.19 An operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is called to be *preopen* (resp. *open*, cf. [20]) if for each point $x \in X$ and for every preopen set (resp. open set) U containing x, there exists a pre γ_p-open set (resp. a γ_p-open set) V such that $x \in V$ and $V \subset U^{\gamma_p}$ (cf. Remark 3.21 below for examples etc).

Theorem 3.20 *Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation on $PO(X,\tau)$ and A be a subset of a topological space (X,τ).*

(i) *If $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is a preopen operation, then* pCl$_{\gamma_p}(A) = PO(X)_{\gamma_p}$-Cl$(A)$ *and* pCl$_{\gamma_p}($pCl$_{\gamma_p}(A)) =$ pCl$_{\gamma_p}(A)$ *hold and* pCl$_{\gamma_p}(A)$ *is pre γ_p-closed in (X,τ).*

(ii) $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ *is open if and only if* $\gamma_p|\tau : \tau \to \mathcal{P}(X)$ *is open* (*in the sense of* [20, Definition 4.4]).

(iii) (cf. Remark 3.21(v)) *If $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is an open operation, then* Cl$_{\gamma_p}(A) = \tau_{\gamma_p}$-Cl$(A)$ *and* Cl$_{\gamma_p}($Cl$_{\gamma_p}(A)) =$ Cl$_{\gamma_p}(A)$ *hold and* Cl$_{\gamma_p}(A)$ *is γ_p-closed in (X,τ).*

Proof. **(i)** By Theorem 3.16(i) we have pCl$_{\gamma_p}(A) \subset PO(X)_{\gamma_p}$-Cl$(A)$. Suppose that $x \notin$ pCl$_{\gamma_p}(A)$. Then, there exists a preopen set U containing x such that $U^{\gamma_p} \cap A = \emptyset$. Since γ_p is preopen, by Definition 3.19, there exists a pre γ_p-open set V such that $x \in V \subset U^{\gamma_p}$ and so $V \cap A = \emptyset$. By Theorem 3.11(i), $x \notin PO(X)_{\gamma_p}$-Cl$(A)$. Hence we have that pCl$_{\gamma_p}(A) = PO(X)_{\gamma_p}$-Cl$(A)$. Furthermore, using the above result and Theorem 3.13, we have that pCl$_{\gamma_p}($pCl$_{\gamma_p}(A)) = PO(X)_{\gamma_p}$-Cl$(PO(X)_{\gamma_p}$-Cl$(A)) = PO(X)_{\gamma_p}$-Cl$(A)$ =pCl$_{\gamma_p}(A)$ and so pCl$_{\gamma_p}(A)$ is pre-γ_p-closed in (X,τ). **(ii)** It is proved by Definition 3.19, Proposition 2.6(i) and a fact that $U^{\gamma_p} = U^{\gamma_p|\tau}$ holds for any open set U of (X,τ). **(iii)** By Theorem 3.16(ii), we have Cl$_{\gamma_p}(A) \subset \tau_{\gamma_p}$-Cl$(A)$. Suppose that $x \notin$ Cl$_{\gamma_p}(A)$. Then there exists an open set U containing x such that $U^{\gamma_p} \cap A = \emptyset$. Since γ_p is an open operation, by Definition 3.19, there exists a γ_p-open set V such that $x \in V \subset U^{\gamma_p}$ and so $V \cap A = \emptyset$. By Theorem 3.11(ii), $x \notin \tau_{\gamma_p}$-Cl(A). Hence we have that Cl$_{\gamma_p}(A) = \tau_{\gamma_p}$-Cl$(A)$. Furthermore, using the above result and Theorem 3.15, we have that Cl$_{\gamma_p}($Cl$_{\gamma_p}(A)) = \tau_{\gamma_p}$-Cl$(\tau_{\gamma_p}$-Cl$(A)) = \tau_{\gamma_p}$-Cl$(A)$ =Cl$_{\gamma_p}(A)$ and Cl$_{\gamma_p}(A)$ is γ_p-closed in (X,τ). □

Remark 3.21 (i) Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be the "*Int ∘ Cl*"-operation (cf. Example 3.2(iii)), where (X,τ) is any topological space. Then, it is shown that the operation $\gamma_p =$ "*Int ∘ Cl*" is preopen (resp. open) on $PO(X,\tau)$. Indeed, let $x \in X$ and U_x be a preopen (resp. open) set containing x. Put G =Int(Cl(U_x)). Then it is shown that the set G is a pre γ_p-open (resp. γ_p-open) and $x \in G \subset (U_x)^{\gamma_p}$, because $x \in$ Int(Cl$(U_x)) = G = G^{\gamma_p}$ =Int(Cl(Int(Cl(U_x)))) hold in (X,τ).

(ii) The identity operation $\gamma_p =$ "*id*" : $PO(X,\tau) \to \mathcal{P}(X)$ is preopen and open.

(iii) If (X,τ) is a pre γ_p-regular (resp. a γ_p-regular) space for an operation $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$, then γ_p is preopen (resp. open). Indeed, by Theorem 3.6(i) (2)⇒ (3) (resp. (iii) (2)⇒ (3)), Definition 2.1 and Definition 3.19, it is obtained.

(iv) The following example shows that the converse of (iii) above needs not be true. Let (X,τ) be a topological space, where $X := \{a,b,c\}$ and $\tau := \{\emptyset, X, \{a\}\}$. Define an operation $\gamma_p : PO(X) \to \mathcal{P}(X)$ as follows: $\gamma_p(A) := A$ if $b \in A$, $\gamma_p(A) :=$pCl(A) if $b \notin A$. Then, we have that $PO(X,\tau) = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$, $PO(X,\tau)_{\gamma_p} = \{\emptyset, \{a,b\}, X\}$

and $\tau_{\gamma_p} = \{\emptyset, X\}$. Since $PO(X,\tau) \neq PO(X,\tau)_{\gamma_p}$ (resp. $\tau \neq \tau_{\gamma_p}$) holds, (X,τ) is not pre-γ_p-regular (resp. γ_p-regular), cf. Theorem 3.6 (i) (resp. (iii)). But we can check that the operation γ_p is preopen and open.

(v) We have a non-open operation γ_p and a property that $\mathrm{Cl}_{\gamma_p}(\mathrm{Cl}_{\gamma_p}(A)) \neq \mathrm{Cl}_{\gamma_p}(A)$ for some subset A of a topological space (X,τ) (cf. Theorem 3.20(iii)). Let (X,τ) be a topological space, where $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Define $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ by $A^{\gamma_p} := Cl(A)$ for any $A \in PO(X,\tau)$. Then, $\tau_{\gamma_p} = \{\emptyset, X\} = PO(X,\tau)_{\gamma_p}$ and γ_p is not open and it is also not preopen. For a subset $\{a\}$ of (X,τ), $\mathrm{Cl}_{\gamma_p}(\mathrm{Cl}_{\gamma_p}(\{a\})) = \mathrm{Cl}_{\gamma_p}(\{a,c\}) = X \neq \mathrm{Cl}_{\gamma_p}(\{a\}) = \{a,c\}$.

Corollary 3.22 *Let "$Int \circ Cl$" : $PO(X,\tau) \to \mathcal{P}(X)$ be the Interior-closure operation and "id" : $PO(X,\tau) \to \mathcal{P}(X)$ the identity operation on $PO(X,\tau)$. Then, we have the following properties:*

(i) $PO(X,\tau)_{\text{“}Int\circ Cl\text{”}} = \tau_{\text{“}Int\circ Cl\text{”}} = \tau_\delta$;

$\mathrm{pCl}_{\text{“}Int\circ Cl\text{”}}(A) = PO(X)_{\text{“}Int\circ Cl\text{”}}\text{-}\mathrm{Cl}(A) = \tau_\delta\text{-}\mathrm{Cl}(A) = \mathrm{Cl}_\delta(A) = \mathrm{Cl}_{\text{“}Int\circ Cl\text{”}}(A)$ *hold for any subset A of (X,τ).*

(ii) $PO(X,\tau)_{\text{“}id\text{”}} = PO(X,\tau), \tau_{\text{“}id\text{”}} = \tau$;

$\mathrm{pCl}_{\text{“}id\text{”}}(A) = \mathrm{pCl}(A)$ *and* $\mathrm{Cl}_{\text{“}id\text{”}}(A) = \mathrm{Cl}(A)$ *hold for any subset A of (X,τ).*

Proof. **(i)** It follows from Example 3.2(iii) that $PO(X,\tau)_{\text{“}Int\circ Cl\text{”}} = \tau_{\text{“}Int\circ Cl\text{”}} = \tau_\delta$ and so $\tau_{\text{“}Int\circ Cl\text{”}}\text{-}\mathrm{Cl}(A) = PO(X)_{\text{“}Int\circ Cl\text{”}}\text{-}\mathrm{Cl}(A) = \tau_\delta\text{-}\mathrm{Cl}(A)$ for a subset A of (X,τ). Since "$Int \circ Cl$" and "id" are preopen and also open (cf. Example 3.21(i)), we have that $PO(X,\tau)_{\text{“}Int\circ Cl\text{”}}\text{-}\mathrm{Cl}(A) = \mathrm{pCl}_{\text{“}Int\circ Cl\text{”}}(A)$ (cf. Theorem 3.20(i)), $\tau_{\text{“}Int\circ Cl\text{”}}\text{-}\mathrm{Cl}(A) = \mathrm{Cl}_{\text{“}Int\circ Cl\text{”}}(A)$ (cf. Theorem 3.20(iii)) and, by definitions, $\mathrm{Cl}_\delta(A) = \mathrm{Cl}_{\text{“}Int\circ Cl\text{”}}(A)$, where A is a subset of (X,τ). Therefore, we have the required equalities. **(ii)** We have that $PO(X,\tau)_{\text{“}id\text{”}} = PO(X,\tau)$ and $\tau_{\text{“}id\text{”}} = \tau$ (cf. Example 3.2(i)). Since "id" is preopen and also open (cf. Remark 3.21(ii)), we have that $\mathrm{pCl}_{\text{“}id\text{”}}(A) = PO(X)_{\text{“}id\text{”}}\text{-}\mathrm{Cl}(A) = \mathrm{pCl}(A)$ and $\mathrm{Cl}_{\text{“}id\text{”}}(A) = \mathrm{Cl}(A)$ for any subset A of (X,τ) (cf. Theorem 3.20(i)(iii)). □

4 Pre γ_p-generalized closed sets and pre γ_p-T_i spaces, where i=0, 1/2, 1 or 2

Throughout this section, let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation on $PO(X,\tau)$.

Definition 4.1 Let A be a subset of a topological space (X,τ).

(i) A subset A is said to be *pre γ_p-generalized closed* (shortly, *pre γ_p-g.closed*) in (X,τ) if $\mathrm{pCl}_{\gamma_p}(A) \subset U$ whenever $A \subset U$ and U is a pre γ_p-open set of (X,τ).

(ii) (cf. [20, Definition 4.4]) A subset A is said to be *γ_p-generalized closed* (shortly, *γ_p-g.closed*) in (X,τ) if $\mathrm{Cl}_{\gamma_p}(A) \subset U$ whenever $A \subset U$ and U is γ_p-open in (X,τ).

(iii) A subset A of (X,τ) is said to be *pre γ_p-g.open* (resp. *γ_p-g.open*) in (X,τ) if the complement $X \setminus A$ is pre γ_p-g.closed (resp. γ_p-g.closed) in (X,τ).

Theorem 4.2 *Let $\gamma_p : PO(X;\tau) \to \mathcal{P}(X)$ be an operation and A a subset of a topological space (X,τ).*

(i) *The following properties are equivalent:*

(1) *A subset A is pre γ_p-g.closed in (X,τ);*

(2) $(PO(X)_{\gamma_p}\text{-}\mathrm{Cl}(\{x\})) \cap A \neq \emptyset$ *for every $x \in \mathrm{pCl}_{\gamma_p}(A)$;*

(3)$\mathrm{pCl}_{\gamma_p}(A) \subset PO(X)_{\gamma_p}$ -Ker(A) *holds,where* $PO(X)_{\gamma_p}$-Ker$(E) =: \bigcap\{V | E \subset V, V \in PO(X,\tau)_{\gamma_p}\}$ *for any subset E of (X,τ).*

(ii) (cf. [20, Proposition 4.6] [15, Proposition 4.5]) *The following properties are equivalent:*

(1) *A subset A is γ_p-g.closed in (X,τ);*

(2) $(\tau_{\gamma_p}\text{-Cl}(\{x\})) \cap A \neq \emptyset$ *for every* $x \in \text{Cl}_{\gamma_p}(A)$;

(3) $\text{Cl}_{\gamma_p}(A) \subset \tau_{\gamma_p}\text{-Ker}(A)$ *holds, where* $\tau_{\gamma_p}\text{-Ker}(E) := \bigcap\{V | E \subset V, V \in \tau_{\gamma_p}\}$ *for any subset E of (X,τ).*

(iii) *A subset A is γ_p-g.closed in (X,τ) if and only if A is $\gamma_p|\tau$-g.closed in (X,τ), where $\gamma_p|\tau$ is the restriction of γ_p onto τ* (cf. Remark 2.2(ii)).

Proof. **(i) (1)⇒(2)** Let A be a pre γ_p-g.closed set of (X,τ). Suppose that there exists a point $x \in pCl_{\gamma_p}(A)$ such that $(PO(X)_{\gamma_p}\text{-Cl}(\{x\})) \cap A = \emptyset$. By Theorem 3.13(i), $PO(X)_{\gamma_p}\text{-Cl}(\{x\})$ is a pre γ_p-closed. Put $U = X \setminus (PO(X)_{\gamma_p}\text{-Cl}(\{x\}))$. Then, we have that $A \subset U, x \notin U$ and U is a pre γ_p-open set of (X,τ). Since A is a pre γ_p-g.closed set, $\text{pCl}_{\gamma_p}(A) \subset U$. Thus, we have that $x \notin \text{pCl}_{\gamma_p}(A)$. This is a contradiction. **(2)⇒(3)** Let $x \in \text{pCl}_{\gamma_p}(A)$. By (2), there exists a point z such that $z \in (PO(X)_{\gamma_p}\text{-Cl}(\{x\}))$ and $z \in A$. Let $U \in PO(X,\tau)_{\gamma_p}$ be a subset of X such that $A \subset U$. Since $z \in U$ and $z \in (PO(X)_{\gamma_p}\text{-Cl}(\{x\}))$, we have that $U \cap \{x\} \neq \emptyset$. Namely, we show that $x \in PO(X)_{\gamma_p}$-Ker(A). Therefore, we prove that $\text{pCl}_{\gamma_p}(A) \subset PO(X)_{\gamma_p}\text{-Ker}(A)$. **(3)⇒(1)** Let U be any pre γ_p-open set such that $A \subset U$. Let x be a point such that $x \in \text{pCl}_{\gamma_p}(A)$. By (3), $x \in PO(X)_{\gamma_p}\text{-Ker}(A)$ holds. Namely, we have that $x \in U$, because $A \subset U$ and $U \in PO(X,\tau)_{\gamma_p}$. **(ii) (1)⇒(2)** Let A be a γ_p-g.closed set of (X,τ). Suppose that there exists a point $x \in \text{Cl}_{\gamma_p}(A)$ such that $(\tau_{\gamma_p}\text{-Cl}(\{x\})) \cap A = \emptyset$. By Theorem 3.15(i), $\tau_{\gamma_p}\text{-Cl}(\{x\})$ is γ_p-closed. Put $U := X \setminus \tau_{\gamma_p}\text{-Cl}(\{x\})$. Then, we have that $A \subset U, x \notin U$ and U is a γ_p-open set of (X,τ). Since A is a γ_p-g.closed set, $\text{Cl}_{\gamma_p}(A) \subset U$. Thus, we have that $x \notin \text{Cl}_{\gamma_p}(A)$. This is a contradiction. **(2)⇒(3)** Let $x \in \text{Cl}_{\gamma_p}(A)$. By (2), there exists a point z such that $z \in \tau_{\gamma_p}\text{-Cl}(\{x\})$ and $z \in A$. Let $U \in \tau_{\gamma_p}$ be a subset of X such that $A \subset U$. Since $z \in U$ and $z \in \tau_{\gamma_p}\text{-Cl}(\{x\})$, we have that $U \cap \{x\} \neq \emptyset$. Namely, we show that $x \in \tau_{\gamma_p}\text{-Ker}(A)$ for any point $x \in \text{Cl}_{\gamma_p}(A)$ and so $\text{Cl}_{\gamma_p}(A) \subset \tau_{\gamma_p}\text{-Ker}(A)$. **(3)⇒(1)** Let U be any γ_p-open set such that $A \subset U$. Let x be a point such that $x \in \text{Cl}_{\gamma_p}(A)$. By (3), $x \in \tau_{\gamma_p}\text{-Ker}(A)$ holds. Namely, we have that $x \in U$. **(iii)** It is obtained by Definition 4.1, Remark 2.5, Proposition 2.6(i) and [20, Definition 4.4]. □

Theorem 4.3 *Let A be a subset of a topological space (X,τ).*

(i) *If A is pre γ_p-g.closed in (X,τ), then $\text{pCl}_{\gamma_p}(A) \setminus A$ does not contain any non-empty pre γ_p-closed set.*

(i)′ *If $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is a preopen operation* (cf. Definition 3.19), *then the converse of* (i) *is true.*

(ii) (cf. [20, Remark 4.8] [15, Proposition 4.6(i)]) *If A is γ_p-g.closed in (X,τ), then $\text{Cl}_{\gamma_p}(A) \setminus A$ does not contain any non-empty γ_p-closed set.*

(ii)′ *If $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is an open operation* (cf. Definition 3.19), *then the converse of* (ii) *is true.*

Proof. **(i)** Suppose that there exists a pre γ_p-closed set F such that $F \subset \text{pCl}_{\gamma_p}(A) \setminus A$. Then, we have that $A \subset X \setminus F$ and $X \setminus F$ is pre γ_p-open. It follows from assumption that $\text{pCl}_{\gamma_p}(A) \subset X \setminus F$ and so $F \subset (\text{pCl}_{\gamma_p}(A) \setminus A) \cap (X \setminus \text{pCl}_{\gamma_p}(A))$. Therefore, we have that $F = \emptyset$. **(i)′** Let U be a pre γ_p-open set such that $A \subset U$. Since γ_p is a preopen operaion, it follows from Theorem 3.20(i) that $\text{pCl}_{\gamma_p}(A)$ is pre γ_p-closed in (X,τ). Thus, using Theorem 3.3(iii), Definition 3.9(ii) we have that $\text{pCl}_{\gamma_p}(A) \cap (X \setminus U)$, say F, is a pre γ_p-closed in (X,τ). Since $X \setminus U \subset X \setminus A, F \subset \text{pCl}_{\gamma_p}(A) \setminus A$. Using the assumption of the converse of (i) above, $F = \emptyset$ and hence $\text{pCl}_{\gamma_p}(A) \subset U$. **(ii)** Suppose that there exists an γ_p-closed set F such that $F \subset \text{Cl}_{\gamma_p}(A) \setminus A$. Then, we have that $A \subset X \setminus F$ and $X \setminus F$ is

γ_p-open. It follows from assumption that $\mathrm{Cl}_{\gamma_p}(A) \subset X \setminus F$ and so $F \subset (\mathrm{Cl}_{\gamma_p}(A) \setminus A) \cap (X \setminus \mathrm{Cl}_{\gamma_p}(A))$. Therefore, we have $F = \emptyset$. **(ii)′** Let U be a γ_p-open set such that $A \subset U$. Since γ_p is an open operaion, it follows from Theorem 3.20(iii) that $\mathrm{Cl}_{\gamma_p}(A)$ is γ_p-closed in (X, τ). Thus, using Proposition 2.6(vi), Corollary 3.17(ii) and Definition 3.9(i), we have that $\mathrm{Cl}_{\gamma_p}(A) \cap (X \setminus U)$, say F, is a γ_p-closed in (X, τ). Since $X \setminus U \subset X \setminus A, F \subset \mathrm{Cl}_{\gamma_p}(A) \setminus A$. Using the assumption of the converse of (ii) above, $F = \emptyset$ and hence $\mathrm{pCl}_{\gamma_p}(A) \subset A$. □

We define the following new classes of topological spaces called as γ_p-$T_{1/2}$ *spaces* and *pre* γ_p-$T_{1/2}$ *spaces*. We recall that every γ_p-closed (resp. pre γ_p-closed) set is γ_p-g.closed (resp. pre γ_p-g.closed) (cf. Definition 4.1).

Definition 4.4 (i) A topological space (X, τ) is said to be a *pre* γ_p-$T_{1/2}$ *space* if every pre γ_p-g.closed set of (X, τ) is pre γ_p-closed.

(ii) (cf. [20, Definition 4.5]) A topological space (X, τ) is said to be a γ_p-$T_{1/2}$ *space* if every γ_p-g.closed set of (X, τ) is γ_p-closed.

We prove a lemma needed later.

Lemma 4.5 *For any operation* $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$*, the following properties hold.*

(i) *For each point* $x \in X$*,* $\{x\}$ *is pre* γ_p*-closed or* $X \setminus \{x\}$ *is pre* γ_p*-g.closed in a topological space* (X, τ)*.*

(ii) (cf. [20, Proposition 4.9]) *For each point* $x \in X$*,* $\{x\}$ *is* γ_p*-closed or* $X \setminus \{x\}$ *is* γ_p*-g.closed in a topological space* (X, τ)*.*

Proof. **(i)** Suppose that $\{x\}$ is not a pre γ_p-closed set. Then, by Corollary 3.17 (or definitions), $X \setminus \{x\}$ is not a pre γ_p-open set. Let U be any pre γ_p-open set such that $X \setminus \{x\} \subset U$. Then, $U = X$ and so we have that $\mathrm{pCl}_{\gamma_p}(X \setminus \{x\}) \subset U$. Therefore, $X \setminus \{x\}$ is a pre γ_p-g.closed set in (X, τ). **(ii)** Suppose that $\{x\}$ is not γ_p-closed in (X, τ). By Definition 3.9(i), $X \setminus \{x\}$ is not γ_p-open. Then, it is shown that $X \setminus \{x\}$ is γ_p-g.closed. □

Theorem 4.6 (i) *A topological space* (X, τ) *is pre* γ_p-$T_{1/2}$ *if and only if, for each point* $x \in X$*,* $\{x\}$ *is pre* γ_p*-open or pre* γ_p*-closed in* (X, τ)*.*

(ii) (cf. [20, Proposition 4.10]) *The following properties on a topological space* (X, τ) *are equivalent:*

(1) (X, τ) *is* γ_p-$T_{\frac{1}{2}}$*;*

(2) *For each point* $x \in X$*,* $\{x\}$ *is* γ_p*-open or* γ_p*-closed in* (X, τ)*;*

(3) (X, τ) *is* $\gamma_p|\tau$-$T_{1/2}$ *(in the sense of Ogata)* [20, Definition 4.5].

Proof. **(i) (Necessity)** Suppose that $\{x\}$ is not a pre γ_p-closed set, by Lemma 4.5 (i), $X \setminus \{x\}$ is a pre γ_p-g.closed set. Since (X, τ) is a pre γ_p-$T_{\frac{1}{2}}$ space, this implies that $X \setminus \{x\}$ is pre γ_p-closed. Hence $\{x\}$ is a pre γ_p-open set. **(Sufficiency)** Let F be a pre γ_p-g.closed set in (X, τ). We shall prove that $\mathrm{pCl}_{\gamma_p}(F) = F$ (cf. Corollary 3.17(i)). It is sufficient to show that $\mathrm{pCl}_{\gamma_p}(F) \subset F$. Assume that there exits a point x such that $x \in \mathrm{pCl}_{\gamma_p}(F) \setminus F$. Then, by assumption, $\{x\}$ is pre γ_p-closed or pre γ_p-open. **Case 1**. $\{x\}$ is a pre γ_p-closed set: For this case, we have a pre γ_p-closed set $\{x\}$ such that $\{x\} \subset \mathrm{pCl}_{\gamma_p}(F) \setminus F$. This is a contradiction to Theorem 4.3(i). **Case 2**. $\{x\}$ is a pre γ_p-open set: Using Theorem 3.16(i), we have $x \in PO(X)_{\gamma_p}$-$\mathrm{Cl}(F)$. Since $\{x\}$ is pre γ_p-open, it implies that $\{x\} \cap F \neq \emptyset$ (cf. Theorem 3.11(i)). This is a contradiction. Thus, we have that

$\mathrm{pCl}_{\gamma_p}(F) = F$ and so, by Corollary 3.17(i), F is pre γ_p-closed. **(ii) (1)⇒(2)** Suppose that $\{x\}$ is not a γ_p-closed set. By Lemma 4.5(ii), $X \setminus \{x\}$ is a γ_p-g.closed set. Thus, we have that $X \setminus \{x\}$ is γ_p-closed (i.e., $\{x\}$ is a γ_p-open set, cf. Definition 3.9(i)). **(2)⇒(1)** Let F be a γ_p-g.closed set in (X, τ). We claim that $\mathrm{Cl}_{\gamma_p}(F) \subset F$ (cf. Corollary 3.17(ii)). Assume that there exists a point x such that $\{x\} \subset \mathrm{Cl}_{\gamma_p}(F) \setminus F$. **Case 1**. $\{x\}$ is a γ_p-closed set: For this case, we have a contradiction to Theorem 4.3(ii). **Case 2**. $\{x\}$ is a γ_p-open set: Using Theorem 3.16(ii), we have $x \in \tau_{\gamma_p}\text{-Cl}(F)$. This shows that $\{x\} \cap F \neq \emptyset$ (cf. Theorem 3.11(ii)). This is a contradiction. Thus, we have $\mathrm{Cl}_{\gamma_p}(F) = F$. Hence, using Corollary 3.17(ii) every γ_p-g.closed set is γ_p-closed. **(1)⇔(3)** This follows from Theorem 4.2(iii), Corollary 3.17(ii), Definition 4.4(ii) and [20, Definition 4.5]. □

Remark 4.7 By Theorem 4.6(ii) and Proposition 2.6(i), it is obtained that *a topological space* (X, τ) *is* $\gamma_p|\tau$-$T_{1/2}$ *(in the sense of Ogata)*[20] *if and only if for each point* $x \in X, \{x\}$ *is* $\gamma|\tau$*-open or* $\gamma|\tau$*-closed in* (X, τ). Therefore, this shows that the regularity on γ in [20, Proposition 4.10(ii)] can be omitted (cf. [25, Corollary 4.12 (ii)⇔(iii), Remark 4.13], [15, Remark 5.14 (ii) (5.15)]).

In the end of this section, we introduce further new operation-separation axioms on topological spaces called as pre-γ_p-T_i , where $i \in \{0, 1, 2\}$.

Definition 4.8 A topological space (X, τ) is said to be

(a) a *pre* γ_p-T_0 *space* if for any two distinct points $x, y \in X$, there exists a pre open set U such that either $x \in U$ and $y \notin U^{\gamma_p}$ or $y \in U$ and $x \notin U^{\gamma_p}$;

(a)′ a *pre* γ_p-T_0' *space* if for any two distinct points $x, y \in X$, there exists a pre γ_p-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$;

(b) a *pre* γ_p-T_1 *space* if for any two distinct points $x, y \in X$, there exists two pre open sets U and V containing x and y, respectively, such that $y \notin U^{\gamma_p}$ and $x \notin V^{\gamma_p}$;

(b)′ a *pre* γ_p-T_1' *space* if for any two distinct points $x, y \in X$, there exists two pre γ_p-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$;

(c) a *pre* γ_p-T_2 *space* if for any two distinct points $x, y \in X$, there exists two pre open set U and V such that $x \in U$, $y \in V$ and $U^{\gamma_p} \cap V^{\gamma_p} = \emptyset$;

(c)′ a *pre* γ-T_2' *space* if for any two distinct points $x, y \in X$, there exists two pre γ_p-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 4.9 (i) *A topological space* (X, τ) *is a pre* γ_p-T_0' *space if and only if, for every pair* $x, y \in X$ *with* $x \neq y$, $PO(X)_{\gamma_p}\text{-Cl}(\{x\}) \neq PO(X)_{\gamma_p}\text{-Cl}(\{y\})$ *holds.*

(ii) *Suppose that* $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ *is preopen. A topological space* (X, τ) *is a pre* γ_p-T_0 *space if and only if, for every pair* $x, y \in X$ *with* $x \neq y$, $\mathrm{pCl}_{\gamma_p}(\{x\}) \neq \mathrm{pCl}_{\gamma_p}(\{y\})$ *holds.*

(iii) *Suppose that* $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ *is preopen. A topological space* (X, τ) *is pre* γ_p-T_0 *if and only if* (X, τ) *is pre* γ_p-$T_0^{'}$.

Proof. **(i) (Necessity)** Let x and y be any two distinct points of a pre γ_p-T_0' space (X, τ). Then, by definition, we assume that there exists a pre γ_p-open set U such that $x \in U$ and $y \notin U$. Hence $y \in X \setminus U$. Because $X \setminus U$ is a pre γ-closed set, we obtain that $PO(X)_{\gamma_p}\text{-}Cl(\{y\}) \subset X \setminus U$ (cf. Theorem 3.13(iii)(iv)) and so $PO(X)_{\gamma_p}$-$\mathrm{Cl}(\{x\}) \neq PO(X)_{\gamma_p}\text{-Cl}(\{y\})$. **(Sufficiency)** Suppose that for any $x, y \in X$, $x \neq y$. Then we have $PO(X)_{\gamma_p}\text{-Cl}(\{x\}) \neq PO(X)_{\gamma_p}\text{-Cl}(\{y\})$. Thus we assume that there exists

$z \in PO(X)_{\gamma_p}$-Cl$(\{x\})$ but $z \notin PO(X)_{\gamma_p}$-Cl$(\{y\})$. We shall prove that $x \notin PO(X)_{\gamma_p}$-Cl$(\{y\})$. Indeed, if $x \in PO(X)_{\gamma_p}$-Cl$(\{y\})$, then we get $PO(X)_{\gamma_p}$-Cl$(\{x\}) \subset PO(X)_{\gamma_p}$-Cl$(\{y\})$ (cf. Theorem 3.13(iv)(viii)). This implies that $z \in PO(X)_{\gamma_p}$-Cl$(\{y\})$. This contradiction shows that $X \setminus PO(X)_{\gamma_p}$-Cl$(\{y\})$ is a pre γ_p-open set containing x but not y (cf. Theorem 3.13(i)). Hence (X, τ) is a pre γ_p-T_0' space. **(ii) (Necessity)** Let x and y be any two distinct points of a pre γ_p-T_0 space (X, τ). Then, by definition, we assume that there exists a preopen set U such that $x \in U$ and $y \notin U^{\gamma_p}$. It follows from assumption that there exists a pre γ_p-open set S such that $x \in S$ and $S \subset U^{\gamma_p}$. Hence, $y \in X \setminus U^{\gamma_p} \subset X \setminus S$. Because $X \setminus S$ is a pre γ_p-closed set, we obtain that pCl$_{\gamma_p}(\{y\}) \subset X \setminus S$ and so pCl$_{\gamma_p}(\{x\}) \neq$ pCl$_{\gamma_p}(\{y\})$. **(Sufficiency)** Suppose that $x \neq y$ for any $x, y \in X$. Then, we have that pCl$_{\gamma_p}(\{x\}) \neq$ pCl$_{\gamma_p}(\{y\})$. Thus, we assume that there exists $z \in$ pCl$_{\gamma_p}(\{x\})$ but $z \notin$ pCl$_{\gamma_p}(\{y\})$. If $x \in$ pCl$_{\gamma_p}(\{y\})$, then we get pCl$_{\gamma_p}(\{x\}) \subset$ pCl$_{\gamma_p}(\{y\})$ (cf. Theorem 3.20(i)). This implies that $z \in$ pCl$_{\gamma_p}(\{y\})$. This contradiction shows that $x \notin$ pCl$_{\gamma_p}(\{y\})$, i.e., by Definition 3.10, there exists a preopen set W such that $x \in W$ and $W^{\gamma_p} \cap \{y\} = \emptyset$. Thus, we have that $x \in W$ and $y \notin W^{\gamma_p}$. Hence, (X, τ) is a pre γ_p-T_0. **(iii)** This follows from (i), (ii) above and a fact that, for any subset A of (X, τ), $PO(X)_{\gamma_p}$-Cl$(A) =$ pCl$_{\gamma_p}(A)$ holds under the assumption that γ_p is preopen (cf. Theorem 3.20(i)). □

Theorem 4.10 *For a topological space (X, τ) and an operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$, the following properties hold.*

(i) *The following properties are equivalent:*

(1) *(X, τ) is pre γ_p-T_1;*

(2) *For every point $x \in X$, $\{x\}$ is a pre γ_p-closed set;*

(3) *(X, τ) is pre γ_p-T_1'.*

(ii) *Every pre γ_p-T_i' space is pre γ_p-T_i, where $i \in \{2, 0\}$.*

(iii) *Every pre γ_p-T_2 space is pre γ_p-T_1.*

(iv) *Every pre γ_p-T_1 space is pre γ_p-$T_{1/2}$.*

(v) *Every pre γ_p-$T_{1/2}$ space is pre γ_p-T_0'.*

(vi) *Every $\gamma_p|\tau$-T_i space (in the sense of Ogata)[20] is pre γ_p-T_i, where $i \in \{2, 1, 1/2, 0\}$.*

(vii) *Every pre γ_p-T_i' space is pre γ_p-T_{i-1}', where $i \in \{2, 1\}$.*

Proof. **(i) (1)⇒(2)** Let $x \in X$ be a point. For each point $y \in X \setminus \{x\}$, there exists a preopen set U_y such that $y \in U_y$ and $x \notin (U_y)^{\gamma_p}$. Then, $X \setminus \{x\} = \bigcup\{(U_y)^{\gamma_p} | y \in X \setminus \{x\}\}$. It is shown that $X \setminus \{x\}$ is pre γ_p-open in (X, τ). **(2)⇒(3)** Let x and y be any distinct points of X. By (2), $X \setminus \{x\}$ and $X \setminus \{y\}$ are the required pre γ_p open sets such that $y \in X \setminus \{x\}, x \notin X \setminus \{x\}$ and $x \in X \setminus \{y\}, x \notin X \setminus \{y\}$. **(3)⇒(1)** It is shown that if $x \in U$, where $U \in PO(X, \tau)_{\gamma_p}$, then there exists a preopen set V such that $x \in V \subset V^{\gamma_p} \subset U$. Using (3), we have that (X, τ) is pre γ_p-T_1. **(ii) (iii) (vii)** The proofs are obvious by Definition 4.8. **(iv)** This follows from (i) above and Theorem 4.6(i). **(v)** This follows from Theorem 4.6(i) and Definition 4.8(a)′. **(vi)** For an open set U of (X, τ), $U^{\gamma_p|\tau} = U^{\gamma_p}$ and $U \in PO(X, \tau)$ hold. Thus, the proofs of (vi) for $i \in \{2, 1, 0\}$ are obvious from definitions (cf. [20, Definitions 4.1, 4.2, 4.3], Definition 4.8). The proof for $i = 1/2$, is obtained by Remark 4.7, Proposition 2.6(i), Theorem 3.3(ii) and Theorem 4.6(i). □

Remark 4.11 By Theorem 4.10, [20], [21] and [25, Proposition 5.8], we obtain the following diagram of implications. Moreover, the following Examples 4.12, 4.13, 4.14, 4.15, 4.16, 4.17 below and [21] [25, Section 5] show that some of these implications are not reversible.

$$
\begin{array}{ccccccc}
\text{pre } \gamma_p\text{-}T_2^{'} & \stackrel{\rightarrow}{\not\leftarrow} & \text{pre } \gamma_p\text{-}T_1^{'} & & \stackrel{\rightarrow}{\not\leftarrow} & & \text{pre } \gamma_p\text{-}T_0^{'} \\
\downarrow & & \downarrow \ \uparrow & & & \nearrow & \downarrow \ \not\uparrow \\
\text{pre } \gamma_p\text{-}T_2 & \stackrel{\rightarrow}{\not\leftarrow} & \text{pre } \gamma_p\text{-}T_1 & \stackrel{\rightarrow}{\not\leftarrow} & \text{pre } \gamma_p\text{-}T_{1/2} & \stackrel{\rightarrow}{\not\leftarrow} & \text{pre } \gamma_p\text{-}T_0 \\
\uparrow & & \uparrow \ \not\swarrow & & \uparrow \ \not\swarrow & & \uparrow \ \not\swarrow \\
\gamma_p|\tau\text{-}T_2 & \stackrel{\rightarrow}{\not\leftarrow} & \gamma_p|\tau\text{-}T_1 & \stackrel{\rightarrow}{\not\leftarrow} & \gamma_p|\tau\text{-}T_{1/2} & \stackrel{\rightarrow}{\not\leftarrow} & \gamma_p|\tau\text{-}T_0
\end{array}
$$

Example 4.12 The converse of Theorem 4.10(iii) is not true in general. Let (X,τ) be the double origin topological space, where $X := \mathbf{R}^2 \cup \{O^*\}$ and O^* denotes an additional point (eg., [24, p.92]). Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be the closure operation, i.e., $U^{\gamma_p} :=\mathrm{Cl}(U)$ for every preopen set U of (X,τ). We first prove that (X,τ) *is not pre* γ_p*-*T_2. Let U be a preopen set containing $O := (0,0) \in \mathbf{R}^2$ and V be a preopen set containing O^*. Then, $O \in U \subset \mathrm{Int}(\mathrm{Cl}(U))$ and $O^* \in V \subset \mathrm{Int}(\mathrm{Cl}(V))$ hold in (X,τ). By the definition of τ, there exists an open neighbourhood of O, say $B^+_\varepsilon(O) := \{O\} \cup \{(x,y) \in \mathbf{R}^2 |\ x^2 + y^2 < \varepsilon^2, y > 0\}$ for some positive real number ε, such that $B^+_\varepsilon(O) \subset \mathrm{Cl}(U)$. Similarly, for the point O^*, there exists an open neighbourhood of O^*, say $B^-_\delta(O^*) := \{O^*\} \cup \{(x,y) \in \mathbf{R}^2 |\ x^2 + y^2 < \delta^2, y < 0\}$ for some positive real number δ, such that $B^-_\delta(O^*) \subset \mathrm{Cl}(V)$. Then, we have that $\mathrm{Cl}(B^+_\varepsilon(O)) \cap \mathrm{Cl}(B^-_\delta(O^*)) = \{(x,0) \in \mathbf{R}^2 |\ -Min\{\varepsilon,\delta\} \leq x \leq Min\{\varepsilon,\delta\}\} \neq \emptyset$ and $\mathrm{Cl}(B^+_\varepsilon(O)) \cap \mathrm{Cl}(B^-_\delta(O^*)) \subset U^{\gamma_p} \cap V^{\gamma_p}$ hold and so $U^{\gamma_p} \cap V^{\gamma_p} \neq \emptyset$. This shows that (X,τ) is not pre γ_p-T_2 for the closure operation γ_p. Finally we have that (X,τ) *is pre* γ_p*-*T_1. Indeed, $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ is the closure operation on $PO(X,\tau)$ if and only if $\gamma_p|\tau : \tau \to \mathcal{P}(X)$ is the closure operation on τ (eg., [8], [20], [21]). Then, it follows from [21, (a) in Proof of Theorem 1] that the space (X,τ) is $\gamma_p|\tau$-T_1 for the closure operation $\gamma_p|\tau$ on τ. By Theorem 4.10(vi), it is obtained that (X,τ) is pre γ_p-T_1.

Example 4.13 The following example shows that (X,τ) *is pre* γ_p*-*T_0*; this is not pre* γ_p*-*$T_{1/2}$. Let (X,τ) be the double origin space of Example 4.12. We note that $\mathrm{Int}(\mathrm{Cl}(\{O^*\})) = \emptyset$ holds in (X,τ). Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation defined by $\gamma_p(A) := A \cup \{O^*\}$ for every set $A \in PO(X,\tau)$ (cf. [25, Example 5.11]). First we show that (X,τ) *is not pre* γ_p*-*$T_{1/2}$ *for this operation* γ_p. The singleton $\{O^*\}$ is neither pre γ_p-open nor pre γ_p-closed in (X,τ). Indeed, supposed that $\{O^*\}$ is pre γ_p-open. There exists a preopen set U such that $O^* \in U \subset U^{\gamma_p} \subset \{O^*\}$. Then, we have that $U = \{O^*\} \subset \mathrm{Int}(\mathrm{Cl}(\{O^*\}))$. This shows a contradiction that $\{O^*\} = \emptyset$. Suppose that $\{O^*\}$ is pre γ_p-closed. For an origin $O = (0,0) \in \mathbf{R}^2 \subset X \setminus \{O^*\}$, there exists a preopen set V such that $O \in V$ and $V^{\gamma_p} \subset X \setminus \{O^*\}$. We have also a contradiction that $O^* \subset X \setminus \{O^*\}$. By Theorem 4.6 (i), it is shown that (X,τ) *is not pre* γ_p*-*$T_{1/2}$ *for this operation* γ_p.

Finally, we show that (X,τ) *is pre* γ_p*-*T_0. Indeed, let x and y be distinct points of X. In the below argument, let $d(z,w)$ denote the distance of two point z and w of the Euclidean plane $\mathbf{R}^2$.

Case 1. $x, y \notin \{O, O^*\}$: Let ε be a positive real number such that $\varepsilon < (1/2)d(x,y)$. Then, there exists a subset $B_\varepsilon(x) \in PO(X,\tau)$ such that $x \in B_\varepsilon(x)$ and $y \notin B_\varepsilon(x)^{\gamma_p}$, where $B_\varepsilon(x) := \{z \in R^2 |\ d(x,z) < \varepsilon\}$. **Case 2**. $x = O, y \neq O^*$: Let ε be a positive real number such that $\varepsilon < (1/2)d(O,y)$. Then, there exists a subset $B^+_\varepsilon(O)$ (cf. Example 4.12) such that $x = O \in B^+_\varepsilon(O)$ and $y \notin B^+_\varepsilon(O)^{\gamma_p} = B^+_\varepsilon(O) \cup \{O^*\}$. **Case 3**. $x = O^*, y \neq O$: Let ε be a positive real number such that $\varepsilon < (1/2)d(O,y)$. Then, there exists a subset $B^-_\varepsilon(O^*) \in PO(X,\tau)$ (cf. Example 4.12) such that $x \in B^-_\varepsilon(O^*)$ and $y \notin B^-_\varepsilon(O^*)^{\gamma_p} = \{(a,b) \in R^2 |\ a^2 + b^2 < \varepsilon^2, b < 0\} \cup \{O^*\}$. **Case 4**. $x = O^*, y = O$: There exists a subset

$B_\varepsilon^-(O^*) \in PO(X,\tau)$ such that $x \in B_\varepsilon^-(O^*)$ and $y \notin B_\varepsilon^-(O^*)^{\gamma_p} = \{(a,b) \in R^2 |\ a^2+b^2 < \varepsilon^2, b<0\} \cup \{O^*\}$, where $\varepsilon \in \mathbf{R}$. Therefore, (X,τ) is pre γ_p-T_0 for the operation γ_p.

Example 4.14 The converse of Theorem 4.10(iv) is not true in general. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. For a topological space (X,τ), we have that $PO(X,\tau) = \tau$. Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation defined by $\gamma_p(A) :=\mathrm{Int}(\mathrm{Cl}(A))$ for every set $A \in PO(X,\tau)$. Then, *the space* (X,τ) *is pre* γ_p-$T_{1/2}(= \gamma_p$-$T_{1/2})$. Indeed, singletons $\{a\}$ and $\{b\}$ are pre γ_p-open sets in (X,τ); a singleton $\{c\}$ is pre γ_p-closed in (X,τ). By Theorem 4.6(i), (X,τ) is pre γ_p-$T_{1/2}$ for the operation γ_p. However, (X,τ) *is not pre* γ_p-T_1. Indeed, $X \setminus \{a\}$ is not pre γ_p-open. For a point $c \in X \setminus \{a\}$, any preopen set containing c is only X and so $X^{\gamma_p} = X \not\subset X \setminus \{a\}$. Thus, $\{a\}$ is not pre γ_p-closed; by Theorem 4.10(i), (X,τ) is not pre γ_p-T_1.

Example 4.15 Some converses of Theorem 4.10(vi) are not true in general.

(i) For $i = 0$, the following example shows that (X,τ) *is pre* γ_p-T_0*; this is not* $\gamma_p|\tau$-T_0 (*in the sense of Ogata*) [20], [21]. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{a,b\}, X\}$. For a topological space (X,τ), we have that $PO(X,\tau) = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, X\}$. Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be defined by $\gamma_p(A) := A$ for every set A such that $A \neq \{a\}; \gamma_p(\{a\}) =: \{a,b\}$. Then, we have that $PO(X,\tau)_{\gamma_p} = \{\emptyset, \{a,b\}, \{a,c\}, X\}$ and so (X,τ) is pre γ_p-T_0'. Hence, (X,τ) is pre γ_p-T_0 (cf. Theorem 4.10(ii)). However, it is shown that (X,τ) is not $\gamma_p|\tau$-T_0.

(ii) For $i = 1/2$, the following example shows that (X,τ) *is pre* γ_p-$T_{1/2}$*; this is not* $\gamma_p|\tau$ -$T_{1/2}$ (*in the sense of Ogata*) [20], [21]. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b,c\}, X\}$. For a topological space (X,τ), we have that $PO(X,\tau) = \mathcal{P}(X)$. Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be defined by $\gamma_p(A) := A$ for every set $A \in \{\{a\}, \{b\}, \{b,c\}\}$, $\gamma_p(\emptyset) := \emptyset$ and $\gamma_p(B) := X$ for every nonempty set $B \in \mathcal{P}(X) \setminus \{\{a\}, \{b\}, \{b,c\}\}$. We have that $PO(X,\tau)_{\gamma_p} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X, \}$ and so singletons $\{a\}$ and $\{b\}$ are pre γ_p-open and $\{c\}$ is pre γ_p-closed in (X,τ). By Theorem 4.6(i), (X,τ) is pre γ_p- $T_{1/2}$. However, (X,τ) is not $\gamma_p|\tau$-$T_{1/2}$ (in the sense of Ogata). Indeed, $\tau_{\gamma_p|\tau} = \{\emptyset, \{a\}, \{b,c\}, X\}$ and so $\{c\}$ is neither $\gamma_p|\tau$-open nor $\gamma_p|\tau$-closed in (X,τ). By Remark 4.7, (X,τ) is not $\gamma_p|\tau$-$T_{1/2}$.

(iii) For $i = 1$, the same space (X,τ) of (ii) above shows that, for the identity operation γ_p, (X,τ) *is pre* γ_p-T_1*; this is not* $\gamma_p|\tau$-T_1 (*in the sense of Ogata*) [20], [21].

Example 4.16 The converses of Theorem 4.10(vii) are not true in general. For $i = 2$, Example 4.12 shows that *the space* (X,τ) *is not pre* γ_p-T_2' (cf. Theorem 4.10(ii) for $i = 2$); *this is pre* γ_p-T_1' (cf. Theorem 4.10(i)). For $i = 1$, Example 4.15(i) shows that *the space* (X,τ) *is pre* γ_p-T_0'. *The space* (X,τ) *is not pre* γ_p-T_1', because $\{a\}$ is not pre γ_p-closed in (X,τ).

Example 4.17 The converse of Theorem 4.10(ii) for $i = 0$ is not true in general. Let (X,τ) be the same topological space of Example 4.14, i.e., $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then, $PO(X,\tau) = \tau$ holds. Let $\gamma_p : PO(X,\tau) \to \mathcal{P}(X)$ be an operation defined newly by $\{a\}^{\gamma_p} := \{a,c\}, \{b\}^{\gamma_p} := \{a,b\}, \{a,b\}^{\gamma_p} := \{a,b\}, \emptyset^{\gamma_p} := \emptyset$ and $X^{\gamma_p} := X$. It is obtained that $PO(X,\tau)_{\gamma_p} = \{\emptyset, \{a,b\}, X\}$ and γ_p is neither preregular nor preopen. Then, (X,τ) *is not pre* γ_p-T_0'. Indeed, for every pre γ_p-open set V_a containing a, we have $b \in V_a$; for every pre γ_p-open set V_b containing b, we have $a \in V_b$. By Definition 4.8(a)$'$, the space (X,τ) is not pre γ_p-T_0'. Moreover, *the space* (X,τ) *is pre* γ_p-T_0.

Remark 4.18 Example 4.17 shows that, in Theorem 4.9(iii), the assumption of preopenness on γ_p cannot be removed.

Acknowledgements. The authors would like to express their thanks to the referee for her/his detailed examinations of this paper and her/his helpful comments.

REFERENCES

[1] M.E. Abd El-Monsef, Studies on some pretopological concepts, Ph. D.Thesis, Fac. Sci. Tanta University, Egypt, (1980).

[2] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24-32.

[3] J. Dontchev, M. Ganster and T. Noiri, *On p-closed spaces*, Internat. J. Math. Math. Sci., **24**(2000), 203 - 212.

[4] W. Dunham, $T_{1/2}$*-spaces*, Kyungpook Math. J., **17** (1977), 161-169.

[5] N.El-Deeb, I.A. Hasanein, A.S. Mashour and T. Noiri, *On p-regular spaces*, Math. Soc. Sci. Math. R. S. Roumanie (N.S), **27**(1983), 311-315.

[6] M. Fujimoto, T. Takigawa, J. Dontchev, T. Noiri and H. Maki, *The topological structures and groups of digital n-spaces*, Kochi J. Math., **1**(2006), 31-55.

[7] K.P. Hart, J. Nagata and J.E. Vaughan, Encyclopedia of General Topology, Elsevier (2004).

[8] D.S. Janković, *On functions with* α*-closed graphs*, Glas. Mat., **18**(1983), 141 - 148.

[9] D.S. Janković, *A note on mappings of extremally disconnected spaces*, Acta. Math. Hung., **46**(1985), 83-92.

[10] J.E. Joseph and M.H. Kwack, *On S-closed spaces*, Proc. Amer. Math. Soc.,**80**(1980), 341-348.

[11] S. Kasahara, *Operation-compact spaces*, Math. Japon., **24** (1979), 97 - 105.

[12] G. S. S. Krishnan, M. Ganster and K. Balachandran, *Operation approaches on semi open sets and applications*, Kochi J. Math., **2**(2007), 21-33.

[13] N. Levine, *Semi-open sets and semi continuity in topological spaces*, Amer. Math. Monthly, **70**(1963), 36-41.

[14] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo (2), **19**(1970), 89-96.

[15] H. Maki and T. Noiri, *Bioperations and some separation axioms*, Sci. Math. Jpn., **53**(1) (2001), 9-24; Online **e4** (2001), 165 - 180.

[16] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt., **53** (1982), 47 - 53.

[17] G. B. Navalagi, *"Definition bank" in general topology*, (2000), http://at.yorku.ca/i/d/e/b/75.htm.

[18] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961 - 970.

[19] T. Noiri, *Characterizations of extremally disconnected spaces*, Indian J. Pure Appl. Math., **19**(1998), 325-329.

[20] H. Ogata, *Operations on topological spaces and associated topology* , Math. Japon., **36**(1991), 175-184.

[21] H. Ogata, *Remarks on some operation-separation axioms*, Bull. Fukuoka Univ. Ed. Part III, **40**(1991), 41-43.

[22] M.C. Pal and P. Bhattacharyya, *Feeble and strong formes of preirresolute functions*, Bull. Malaysian Math. Sci. Soc. (2), **19**(1996), 63-75.

[23] V. Popa and T. Noiri, *On upper and lower θ-irresolute multifunctions*, Indian J. Math., **34**(1992), 237-257.

[24] L.A. Steen and J.A. Seebach Jr, Counterexamples in Topology, Springer, New York, (1978).

[25] J. Umehara, H. Maki and T. Noiri, *Bioperations on topological spaces and some separation axioms*, Mem. Fac. Sci. Kochi Univ. Ser. A (Math.), **13**(1992), 45-59.

[26] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. (2), **78**(1968), 103-118.

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