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ON SOME GENERALIZED CLOSED SETS IN TOPOLOGOCAL SUM *

Huilan Jin, Haruo Maki and Fumikazu Tamari

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Abstract. In the topological sum of pairwise disjoint topological spaces, we investigate topological properties of the following generalized closed sets: preclosed sets, semi-closed sets, α-closed sets, g-closed sets, sg-closed sets, gs-closed sets, gp-closed sets. Moreover, we have a characterization of some generalized compact topological sum. The results are proved by using a unified concept in the sense of [23] [13] [22] (eg., [17]) (cf. Section 2 and Section 3 below); we have generalized versions of main results.

1 Introduction and main results Throughout this paper, let $\{(X_i, \tau_i)|i \in \Omega\}$ be a family of topological spaces satisfying $X_i \cap X_j = \emptyset$ for any different $i, j \in \Omega$ (i.e., it is called a family of pairwise disjoint topological spaces). Let $X := \cup_{i\in\Omega} X_i$ and a family τ of subsets on X as follows: $\tau := \{U : U \subset X, U \cap X_i \in \tau_i$ for every $i \in \Omega\}$. Sometimes, in this paper, τ is denoted by $\bigoplus_{i\in\Omega} \tau_i$ and X is denoted by $\bigoplus_{i\in\Omega} X_i$. Then, it is well known that a pair (X, τ) is a topological space. This space is called the *topological sum* of topological spaces $\{(X_i, \tau_i)|i \in \Omega\}$ and it is denoted by $\bigoplus_{i\in\Omega}(X_i, \tau_i)$ or $\bigoplus_{i\in\Omega} X_i$ (eg., [27, p.66] [14, p.74, 2.2] [16, p.5, p.44]; in [14, p.74, 2.2], this space (X, τ) is called the *sum of spaces* $\{(X_i, \tau_i)|i \in \Omega\}$).

The purpose of the present paper is to investigate some generalized closures and kernels in the topological sum (cf. Theorem 1.2, Theorem 1.3) and more on generalized closed sets in the topological sum (cf. Lemma 1.5, Theorem 1.6, Theorem 3.5). The main results are proved, in Sections 2 and 3, by a generalized point of views in the light of [13, Sections 4 and 5] [23, Section 2] [22] (eg., [17, Chapters 4, 6, 7]). We first recall the following definitions and properties on a topological space and Definition 1.1 below on some kernels of a subset of a topological space. For a subset A of a topological space (Y, σ), the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset S of a topological spaces (Y, σ) is called *semi-open* [18] (resp. *preopen* [24], *α-open* [28]) if $S \subset Cl(Int(S))$ (resp. $S \subset Int(Cl(S)), S \subset Int(Cl(Int(S)))$) holds. Moreover, a subset F is said to be *semi-closed* (resp. *preclosed, α-closed*) if the complement $Y \setminus F$ is semi-open (resp. preopen, α-open). The family of all semi-open sets (resp. preopen sets, α-open sets) of (Y, σ) is denoted by $SO(Y, \sigma)$ (resp. $PO(Y, \sigma), \sigma^\alpha$ (or $\alpha(Y, \sigma)$)). It is well known that $SO(Y, \sigma) \cap PO(Y, \sigma) = \sigma^\alpha$ [29, Lemma 3.1] holds and the family σ^α is a topology of Y[28] (eg., [31]) and $\sigma \subset \sigma^\alpha$ holds for any topological space (Y, σ). The families $SO(Y, \sigma)$ and $PO(Y, \sigma)$ are not topologies of Y in general. The arbitrary union of semi-open sets (resp. preopen sets) is semi-open (resp. preopen). The intersection of all semi-closed sets (resp. preclosed sets, α-closed sets) containing a set A of (Y, σ) is called the *semi-closure* (resp. *preclosure, α-closure*) of the set A and denoted by $sCl(A)$ (resp. $pCl(A)$, $\alpha Cl(A)$). It is well known that a subset F is semi-closed (resp. preclosed, α-closed) in (Y, σ) if and only if $sCl(F) = F$ (resp. $pCl(F) = F, \alpha Cl(F) = F$) holds in (Y, σ).

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Definition 1.1 Let (Y,σ) be a topological space and A a subset of (Y,σ).

(i) (eg., [26] [21]) $Ker(A) := \bigcap\{V|A \subset V$ and $V \in \sigma\}$.

(ii) [12] [6] $sKer(A) := \bigcap\{V|A \subset V$ and $V \in SO(Y,\sigma)\}$.

(iii) [7] $pKer(A) := \bigcap\{V|A \subset V$ and $V \in PO(Y,\sigma)\}$.

In [21] (resp. [6], [7]), $Ker(A)$ (resp. $sKer(A)$, $pKer(A)$) is denoted by A^{Λ} (resp. A^{Λ_s}, A^{Λ_p}).

For a subset A of $\bigoplus_{i\in\Omega}(X_i,\tau_i)$, it is well known that $Cl(A) = \bigcup_{i\in\Omega} Cl(A\cap X_i)$ and $Int(A) = \bigcup_{i\in\Omega} Int(A\cap X_i)$.

Theorem 1.2 *For a subset A of $\bigoplus_{i\in\Omega}(X_i,\tau_i)$, the following properties hold:*

(i) $sCl(A) = \bigcup_{i\in\Omega} sCl(A\cap X_i)$; (ii) $pCl(A) = \bigcup_{i\in\Omega} pCl(A\cap X_i)$,

(iii) $\alpha Cl(A) = \bigcup_{i\in\Omega} \alpha Cl(A\cap X_i)$.

Theorem 1.3 *For a subset A of $\bigoplus_{i\in\Omega}(X_i,\tau_i)$, the following properties hold:*

(i) $Ker(A) = \bigcup_{i\in\Omega} Ker(A\cap X_i)$; (ii) $sKer(A) = \bigcup_{i\in\Omega} sKer(A\cap X_i)$,

(iii) $pKer(A) = \bigcup_{i\in\Omega} pKer(A\cap X_i)$; (iv) $\alpha Ker(A) = \bigcup_{i\in\Omega} \alpha Ker(A\cap X_i)$.

After preparing some generalizations (cf. Lemma 2.1, Lemma 2.4, Theorem 2.5, Theorem 2.6), the proofs of Theorem 1.2 and Theorem 1.3 are given in the end of Section 2.

We need the following definitions and characterizations on some generalized closed sets in a topological space.

Definition 1.4 Let (Y,σ) be a topological space and A a subset of (Y,σ).

(i) The set A is said to be *g-closed* [19] (resp. *gs-closed* [2],*gp-closed* [30]) in (Y,σ), if $Cl(A) \subset U$ (resp. $sCl(A) \subset U, pCl(A) \subset U$) whenever $A \subset U$ and U is open in (Y,σ).

(ii) The set A is said to be *sg-closed* [3], if sCl$(A) \subset U$ whenever $A \subset U$ and U is semi-open in (Y,σ).

Complements of g-closed (resp. gs-closed, gp-closed, sg-closed) sets are called *g-open* (resp. *gs-open, gp-open, sg-open*).

It is well known that, for a subset A of a topological space (Y,σ),

(1a) A is g-closed in (Y,σ) if and only if $Cl(A) \subset Ker(A)$,

(1b) [12, Lemma 1.1 (i)] A is sg-closed in (Y,σ) if and only if $sCl(A) \subset sKer(A)$.

Lemma 1.5 *Let (Y,σ) be a topological space and A a subset of (Y,σ). Then, we have the following properties:*

(i) *A is gs-closed in (Y,σ) if and only if $sCl(A) \subset Ker(A)$,*

(ii) *A is gp-closed in (Y,σ) if and only if $pCl(A) \subset Ker(A)$.*

Theorem 1.6 *Let A be a subset of $\bigoplus_{i\in\Omega}(X_i,\tau_i)$. Then the following properties hold:*

(i) *A is g-closed in $\bigoplus_{i\in\Omega}(X_i,\tau_i)$ if and only if $A\cap X_i$ is g-closed in (X_i,τ_i) for every $i\in\Omega$,*

(ii) *A is gs-closed in $\bigoplus_{i\in\Omega}(X_i,\tau_i)$ if and only if $A\cap X_i$ is gs-closed in (X_i,τ_i) for every $i\in\Omega$,*

(iii) *A is gp-closed in $\bigoplus_{i\in\Omega}(X_i,\tau_i)$ if and only if $A\cap X_i$ is gp-closed in (X_i,τ_i) for every $i\in\Omega$,*

(iv) *A is sg-closed in $\bigoplus_{i\in\Omega}(X_i,\tau_i)$ if and only if $A\cap X_i$ is sg-closed in (X_i,τ_i) for every $i\in\Omega$.*

The proofs of Lemma 1.5 and Theorem 1.6 are stated in Section 3 after preparing generalizations (cf. Lemma 3.2, Theorem 3.3). As corollary, we have a characterization of a gs-compact [9, Definition 5.3] (resp. gp-compact [1, Definition 7], go-compact [4, Definition 6], sg-compact [5] [9] [10]) topological sum (cf. Theorem 3.5, Definition 3.4 and Proof of Theorem 1.7).

Theorem 1.7 *Let $\{(X_i,\tau_i)|i\in\Omega\}$ be a family of pairwise disjoint non-empty topological spaces. For the topological sum $(X,\tau):=\bigoplus_{i\in\Omega}(X_i,\tau_i)$, we have the following properties:*

(X,τ) is gs-compact (resp. gp-compact, go-compact, sg-compact ([11, Proposition 2.11])*) if and only if each topological space (X_i,τ_i) is gs-compact (resp. gp-compact, g-compact, sg-compact) and Ω is finite, where $i\in\Omega$.*

Throughout this paper, all topological spaces lack any separation axioms unless explicitely stated.

2 Generalizations and proofs of Theorems 1.2 and 1.3 Let $P(Y)$ be the power set of a nonempty set Y and $\mathcal{E}_Y$ a non-empty subfamily of the power set $P(Y)$ of Y satisfying a property that $\emptyset\in\mathcal{E}_Y$ and $Y\in\mathcal{E}_Y$ (say $(\mathcal{A})_{\mathcal{E}_Y}$) . Sometimes, $\mathcal{E}_Y$ is abbreviated by $\mathcal{E}$. We will use the following notation (2a), (2b), (2b') due to [23],[13],[22] (eg., [17]): for a subset B of Y and a family $\mathcal{E}_Y$ satisfying $(\mathcal{A})_{\mathcal{E}_Y}$,

(2a) $\mathcal{E}_Y\text{-}Cl(B):=\bigcap\{F|B\subset F \text{ and } Y\setminus F\in\mathcal{E}_Y\}$,

(2b) $\mathcal{E}_Y\text{-}Ker(B):=\bigcap\{U|B\subset U \text{ and } U\in\mathcal{E}_Y\}$,

(2b') $\mathcal{E}_Y\text{-}Int(B):=\bigcup\{U|U\subset B \text{ and } U\in\mathcal{E}_Y\}$.

(2c) For a non-empty subfamily $\mathcal{E}_Y$, let $\mathcal{E}_Y^c:=\{F|Y\setminus F\in\mathcal{E}_Y\}$. Then, $(\mathcal{E}_Y^c)_Y^c=\mathcal{E}_Y$ holds. Note that the proof is obtained without assuming the property $(\mathcal{A})_{\mathcal{E}_Y}$.

Lemma 2.1 *Let (Y,σ) be a topological space and $\mathcal{E}_Y$ be a subfamily of $P(Y)$ satisfying property $(\mathcal{A})_{\mathcal{E}_Y}$, i.e., $\emptyset\in\mathcal{E}_Y$ and $Y\in\mathcal{E}_Y$.*

Then, the following properties hold:

(i) *$\emptyset\in\mathcal{E}_Y^c$ and $Y\in\mathcal{E}_Y^c$.*

(ii) *$\mathcal{E}_Y\text{-}Ker(B)=\mathcal{E}_Y^c\text{-}Cl(B)$ and $\mathcal{E}_Y\text{-}Cl(B)=\mathcal{E}_Y^c\text{-}Ker(B)$ for a subset B of Y,*

(iii) (eg., [22, Lemma 2.2], [32, Lemma 3.1]) *$Y\setminus(\mathcal{E}_Y\text{-}Cl(B))=\mathcal{E}_Y\text{-}Int(Y\setminus B)$ for a subset B of Y,*

(iv) (cf.[32, Lemma 3.2], [15, Lemma 2.7]) *The following properties are equivalent:*

(1) *$\bigcup\{A_k|k\in\mathcal{K}\}\in\mathcal{E}_Y$ holds, whenever $A_k\in\mathcal{E}_Y$ for each $k\in\mathcal{K}$, where $\mathcal{K}$ is any index set. Namely, $\mathcal{E}_Y$ is closed under an arbitrary union, (sometimes, we say that $\mathcal{E}_Y$ satisfies property $(\mathcal{B})_{\mathcal{E}_Y}$, cf. Definition 2.3 below);*

(2) *$\bigcap\{F_i|i\in\mathcal{K}\}\in\mathcal{E}_Y^c$ holds, whenever $F_i\in\mathcal{E}_Y^c$ for each $i\in\mathcal{K}$, where $\mathcal{K}$ is any index set. Namely, $\mathcal{E}_Y^c$ is closed under an arbitrary intersection;*

(3) *If $\mathcal{E}_Y\text{-}Cl(F)=F$ for a subset F of Y, then $Y\setminus F\in\mathcal{E}_Y$.*

Proof. **(i)** Its proof is obvious from definitions. **(ii)** The first part is proved by using a property that $U\in\mathcal{E}_Y$ if and only if $Y\setminus U\in\mathcal{E}_Y^c$, where $U\subset Y$, and (2a) (2b) above. For the second part, $\mathcal{E}\text{-}Cl(B)=(\mathcal{E}^c)^c\text{-}Cl(B)=\mathcal{E}^c\text{-}Ker(B)$ by the first one and (i); hence $\mathcal{E}\text{-}Cl(B)=\mathcal{E}^c\text{-}Ker(B)$ holds. **(iv)** The equivalence of (1) and (3) was proved firstly by [32, Lemma 3.2]; the below proof is an alternative one by using the concept of kernels of subsets.

(1)⇒(2) Let $F_i, i\in\mathcal{K}$, be subsets of Y such that $F_i\in\mathcal{E}_Y^c$ for every $i\in\mathcal{K}$. Then, $Y\setminus F_i\in\mathcal{E}_Y$ and so, by (1), $\bigcup\{Y\setminus F_i|i\in\mathcal{K}\}\in\mathcal{E}_Y$. Thus, we have that $\bigcap\{F_i|i\in\mathcal{K}\}=Y\setminus\bigcup\{Y\setminus F_i|i\in\mathcal{K}\}\in\mathcal{E}_Y^c$ and so $\bigcap\{F_i|i\in\mathcal{K}\}\in\mathcal{E}_Y^c$. **(2)⇒(3)** Assume that $\mathcal{E}_Y\text{-}Cl(F)=F$. We recall that $\mathcal{E}_Y\text{-}Cl(F)$ is the intersection of all subsets G of Y such that $F\subset G$ and

$Y \setminus G \in \mathcal{E}_Y$, i.e., $G \in \mathcal{E}_Y^c$. Thus, by (2) and the assumption, it is shown that $F \in \mathcal{E}_Y^c$ and so $Y \setminus F \in \mathcal{E}_Y$. **(3)⇒(1)** Let $U_i, i \in \mathcal{K}$, be subsets of Y such that $U_i \in \mathcal{E}_Y$ for every $i \in \mathcal{K}$. Let $V := \bigcup\{U_i | i \in \mathcal{K}\}$. Let $x \in \mathcal{E}_Y\text{-}Cl(Y \setminus V)$; by (ii) above, $x \in \mathcal{E}_Y^c\text{-}Ker(Y \setminus V)$. Then, we have that $Y \setminus U_i \in \mathcal{E}_Y^c, Y \setminus V \subset Y \setminus U_i$ for every $i \in \mathcal{K}$ and so $x \in Y \setminus U_i$ for every $i \in \mathcal{K}$. Thus we have that $x \in \bigcap\{Y \setminus U_i | i \in \mathcal{K}\} = Y \setminus \bigcup\{U_i | i \in \mathcal{K}\} = Y \setminus V$. Therefore, we obtain that $\mathcal{E}_Y\text{-}Cl(Y \setminus V) \subset Y \setminus V$ and so $Y \setminus V = \mathcal{E}_Y\text{-}Cl(Y \setminus V)$. By (3), $Y \setminus (Y \setminus V) \in \mathcal{E}_Y$ and so $V \in \mathcal{E}_Y$. □

Remark 2.2 Let $\mathcal{E}_Y$ be a subfamily of $P(Y)$. Then, $\mathcal{E}_Y$ is called:

(1) a *generalized topology* (in the sense of Lougojan [20]) if $\emptyset \in \mathcal{E}_Y$, $Y \in \mathcal{E}_Y$ and $\mathcal{E}_Y$ is closed under an arbitrary union,

(2) a *supratopology* [25] if $Y \in \mathcal{E}_Y$ and $\mathcal{E}_Y$ is closed under an arbitrary union,

(3) a *generalized topology* (in the sense of Császár [8]) if $\emptyset \in \mathcal{E}_Y$ and $\mathcal{E}_Y$ is closed under an arbitrary union,

(4) a *minimal structure* [32] on Y if $\emptyset \in \mathcal{E}_Y$ and $Y \in \mathcal{E}_Y$ (in [32], $\mathcal{E}_Y$ is denoted by m_Y).

Throughout this section, let $(X, \tau) := \bigoplus_{i\in\Omega}(X_i, \tau_i)$ be the topological sum of pairwise disjoint topological spaces $\{(X_i, \tau_i) | i \in \Omega\}$. Let $X := \bigoplus_{i\in\Omega} X_i$ and $\tau := \bigoplus_{i\in\Omega} \tau_i$.

Definition 2.3 Let $\mathcal{E}_i$ be a non-empty subfamily of $P(X_i)$ for each $i \in \Omega$ and $\mathcal{E}$ a non-empty subfamily of $P(X) := P(\bigoplus_{i\in\Omega} X_i)$. The following properties on subfamilies $\mathcal{E}$ and $\mathcal{E}_i (i \in \Omega)$, which are stated below, are called properties $(\mathcal{A})_\mathcal{E}, (\mathcal{A})_{\mathcal{E}_i}, (\mathcal{B})_\mathcal{E}, (\mathcal{B})_{\mathcal{E}_i}$ and $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$, respectively:

$(\mathcal{A})_\mathcal{E}$: $\emptyset \in \mathcal{E}$ and $X \in \mathcal{E}$;

$(\mathcal{A})_{\mathcal{E}_i}$: $\emptyset \in \mathcal{E}_i$ and $X_i \in \mathcal{E}_i$ for every $i \in \Omega$;

$(\mathcal{B})_\mathcal{E}$: the union of any family of subsets belonging to $\mathcal{E}$ belongs to $\mathcal{E}$, i.e., $\bigcup\{A_k | k \in \mathcal{K}\} \in \mathcal{E}$ whenever $A_k \in \mathcal{E}$ for each $k \in \mathcal{K}$, where $\mathcal{K}$ is any index set;

$(\mathcal{B})_{\mathcal{E}_i}$: the union of any family of subsets belonging to $\mathcal{E}_i$ belongs to $\mathcal{E}_i$; i.e., $\bigcup\{B_k | k \in \mathcal{K}_i\} \in \mathcal{E}_i$ whenever $B_k \in \mathcal{E}_i$ for each $k \in \mathcal{K}_i$, where $\mathcal{K}_i$ is any index set;

$(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$: for any subset B of $\bigoplus_{i\in\Omega}(X_i, \tau_i)$, $B \in \mathcal{E}$ if and only if $B \cap X_i \in \mathcal{E}_i$ for every $i \in \Omega$.

Throughout the present paper, we use the above abbreviated notation.

In particular, for the topologies $\tau_i (i \in \Omega)$ and the sum topology $\tau := \bigoplus_{i\in\Omega} \tau_i$, it is obvious that $(\mathcal{A})_\tau, (\mathcal{A})_{\tau_i}, (\mathcal{B})_\tau, (\mathcal{B})_{\tau_i}$ and $(\mathcal{S})_{\tau,\tau_i} (i \in \Omega)$ hold. Indeed, $(\mathcal{S})_{\tau,\tau_i}$ is obtained by the definition of the topological sum.

We need the following lemma.

Lemma 2.4 *Let $\mathcal{E}$ and $\mathcal{E}^c$ be two non-empty subfamilies of $P(X)$ and $\mathcal{E}_i$ and $\mathcal{E}_i^c$ two non-empty subfamilies of $P(X_i)$ for each $i \in \Omega$. Then, the following properties hold* (*cf. Definition 2.3*)*:*

(i) $(\mathcal{A})_\mathcal{E}$ (resp. $(\mathcal{A})_{\mathcal{E}_i}$) *holds if and only if* $(\mathcal{A})_{\mathcal{E}^c}$(resp. $(\mathcal{A})_{\mathcal{E}_i^c}$) *holds.*

(ii) $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ *holds if and only if* $(\mathcal{S})_{\mathcal{E}^c,\mathcal{E}_i^c}$ *holds.*

Proof. **(i)** The proof of Necessity is obvious. Sufficiency is proved using the property of Necessity and (2c) above. **(ii) (Necessity)** Let E be a subset of $X := \bigoplus_{i\in\Omega} X_i$. Assume that $E \in \mathcal{E}^c$. Then, $X \setminus E \in \mathcal{E}$ and, by the assumption of Necessity for the set $X \setminus E$, it is shown that $(X \setminus E) \cap X_i \in \mathcal{E}_i$ for every $i \in \Omega$. Thus we have that $E \cap X_i \in \mathcal{E}_i^c$ for every $i \in \Omega$, because $X_i \setminus (X_i \cap E) = (X \setminus E) \cap X_i \in \mathcal{E}_i$ holds for every $i \in \Omega$. Namely, if $E \in \mathcal{E}^c$, then $E \cap X_i \in \mathcal{E}_i^c$ for every $i \in \Omega$. Conversely, suppose that $E \subset \bigoplus_{i\in\Omega} X_i$ and $E \cap X_i \in \mathcal{E}_i^c$ for every $i \in \Omega$. Then, we have that $X_i \cap (X \setminus E) = X_i \setminus (E \cap X_i) \in \mathcal{E}_i$ for every $i \in \Omega$. By the assumption of Necessity for the set E of $\bigoplus_{i\in\Omega} X_i$, it is shown that $X \setminus E \in \mathcal{E}$ and so $E \in \mathcal{E}^c$. Namely, if $E \cap X_i \in \mathcal{E}_i^c$ for every $i \in \Omega$, then $E \in \mathcal{E}^c$. We have that $(\mathcal{S})_{\mathcal{E}^c,\mathcal{E}_i^c}$ holds. **(Sufficiency)** Suppose that $(\mathcal{S})_{\mathcal{E}^c,\mathcal{E}_i^c}$ holds. Using Necessity above for the subfamilies $\mathcal{E}^c$ and $\mathcal{E}_i^c$, $(\mathcal{S})_{(\mathcal{E}^c)^c,(\mathcal{E}_i^c)^c}$ holds. By (2c) above, it is shown that $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ holds. □

Theorem 2.5 *Let $X := \bigoplus_{i\in\Omega} X_i$ and $\tau := \bigoplus_{i\in\Omega} \tau_i$. Let $\mathcal{E}_i (i \in \Omega)$ and $\mathcal{E}$ be subfamilies of $P(X_i)$ and $P(X)$, respectively, satisfying the properties $(\mathcal{A})_{\mathcal{E}}, (\mathcal{A})_{\mathcal{E}_i}$ (cf. Definition 2.3). We denote the following properties by (1), (2), (3) and (4), respectively:*

(1) $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ *holds (cf. Definition 2.3);*

(2) $(\mathcal{B})_{\mathcal{E}}$ *and* $(\mathcal{B})_{\mathcal{E}_i}$ *hold for every $i \in \Omega$ (cf. Definition 2.3, Lemma 2.1(iv));*

(3) $\mathcal{E}\text{-}Cl(A) = \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl(A \cap X_i)$ *holds for any subset A of X;*

(4) $\mathcal{E}\text{-}Ker(A) = \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Ker(A \cap X_i)$ *holds for any subset A of X.*

Then, we have the following implications:

(i) (1) $\Rightarrow$ (3),

(ii) (2) *and* (3) $\Rightarrow$ (1),

(iii) (1) $\Rightarrow$ (4).

Proof. **(i)** We first show that $\bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl(A \cap X_i) \subset \mathcal{E}\text{-}Cl(A)$. Let x be a point of X such that $x \notin \mathcal{E}\text{-}Cl(A)$. There exists a subset F of X such that $X \setminus F \in \mathcal{E}, x \notin F$ and $A \subset F$. By using the assumption (1), it is noted that $X_i \setminus (F \cap X_i) = (X \setminus F) \cap X_i \in \mathcal{E}_i$ for every $i \in \Omega$. Then, for every $i \in \Omega, X_i \setminus (F \cap X_i) \in \mathcal{E}_i, x \notin F \cap X_i$ and $(X_i \setminus (F \cap X_i)) \cap (A \cap X_i) = \emptyset$. Thus, for every $i \in \Omega$, there exist subsets $F \cap X_i$ such that $x \notin F \cap X_i$, $A \cap X_i \subset F \cap X_i$ and $X_i \setminus (F \cap X_i) \in \mathcal{E}_i$. Namely, we have that $x \notin \cup_{i\in\Omega}\mathcal{E}_i\text{-}Cl(A \cap X_i)$.

Conversely, we assume that $x \notin \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl(A \cap X_i)$ and $x \in X_j$ for some $j \in \Omega$. There exists a subset F_j of X_j such that $x \notin F_j, X_j \setminus F_j \in \mathcal{E}_j$ and $A \cap X_j \subset F_j$. Put $F_i := X_i$ for every $i \in \Omega$ with $i \neq j$ and $F := \bigcup_{i\in\Omega} F_i$. Then, $A \subset F$ and $(X \setminus F) \cap X_i = X_j \setminus F_j$ or $\emptyset$ according to $i = j$ or $i \neq j$ and so $(X \setminus F) \cap X_i \in \mathcal{E}_i$ for every $i \in \Omega$. Using the assumption (1):$(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ for the set $(X \setminus F) \cap X_i$, we have $X \setminus F \in \mathcal{E}$. Thus, we have that $x \notin F, X \setminus F \in \mathcal{E}$ and $A \subset F$ and so $x \notin \mathcal{E}\text{-}Cl(A)$. Therefore, we show that $\mathcal{E}\text{-}Cl(A) \subset \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl(A \cap X_i)$.

(ii) Let B a subset of X such that $B \in \mathcal{E}$. Then, $X \setminus B \in \mathcal{E}^c$ and so $\mathcal{E}^c\text{-}Ker(X \setminus B) = X \setminus B$. By Lemma 2.1(i)(ii) and (2c) above for $Y := X$, it is shown that $\mathcal{E}\text{-}Cl(X \setminus B) = X \setminus B$. Using the assumption (3), we have that, for an element $j \in \Omega, X_j \cap (X \setminus B) = X_j \bigcap(\bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl((X \setminus B) \cap X_i))$ and so $X_j \setminus (B \cap X_j) = \mathcal{E}_j\text{-}Cl(X_j \setminus (B \cap X_j))$. Hence, by $(\mathcal{B})_{\mathcal{E}_i}$ for every $i \in \Omega$ (cf. Lemma 2.1(iv) (1)$\Leftrightarrow$(3)), it is shown that $B \cap X_i = X_i \setminus (X_i \setminus (B \cap X_i)) \in \mathcal{E}_i$. Namely, we have that if $B \in \mathcal{E}$ then $B \cap X_i \in \mathcal{E}_i$ for every $i \in \Omega$.

Conversely, suppose that for a subset B of $\bigoplus_{i\in\Omega} X_i, B \cap X_i \in \mathcal{E}_i$ holds for every $i \in \Omega$. Then, using (3) we have that $\mathcal{E}\text{-}Cl(X \setminus B) = \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl(X_i \cap (X \setminus B)) = \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Cl(X_i \setminus (B \cap X_i)) = \bigcup_{i\in\Omega}(X_i \setminus (B \cap X_i)) = X \setminus B$ and so $\mathcal{E}\text{-}Cl(X \setminus B) = X \setminus B$. Hence, using $(\mathcal{B})_{\mathcal{E}}$ (cf. Lemma 2.1(iv) (1)$\Leftrightarrow$(3) for $Y = X$ and $\mathcal{E}_Y = \mathcal{E}$), we have that $B = X \setminus (X \setminus B) \in \mathcal{E}$. Namely, we prove that if $B \cap X_i \in \mathcal{E}_i$ holds for every $i \in \Omega$, then $B \in \mathcal{E}$.

(iii) It follows from assumptions and Lemma 2.4 that $(\mathcal{A})_{\mathcal{E}^c}, (\mathcal{A})_{\mathcal{E}_i^c}$ and $(\mathcal{S})_{\mathcal{E}^c,\mathcal{E}_i^c}$ hold. By using (i) above for the set A and the subfamilies $\mathcal{E}^c, \mathcal{E}_i^c$, it is obtained that $\mathcal{E}^c\text{-}Cl(A) = \bigcup_{i\in\Omega} \mathcal{E}_i^c\text{-}Cl(A \cap X_i)$. Therefore, using Lemma 2.1(ii), we have that $\mathcal{E}\text{-}Ker(A) = \bigcup_{i\in\Omega} \mathcal{E}_i\text{-}Ker(A \cap X_i)$. □

In the end of this section, we give the proofs of main results, i.e., Theorems 1.2 and 1.3, by generalized points of views in the light of [13, Sections 4 and 5] [23, Section 2] [22] (eg., [17]). We need the following theorem (Theorem 2.6 below) : let $(X, \tau) := \bigoplus_{i\in\Omega}(X_i, \tau_i)$ be the topological sum of the pairwise disjoint topological spaces $\{(X_i, \tau_i)|i \in \Omega\}$. We denote $X := \bigoplus_{i\in\Omega} X_i$ and $\tau := \bigoplus_{i\in\Omega} \tau_i$. We recall the following well known properties in (X, τ):

(2d) $(\mathcal{S})_{\tau,\tau_i}$ holds, i.e., for a subset U of $(X,\tau), U \in \tau$ if and only if $U \cap X_i \in \tau_i$ for every $i \in \Omega$ (by definition of the topology τ);

(2e) for a subset A of $(X,\tau), Cl(A) = \bigcup_{i\in\Omega} Cl(A \cap X_i)$ holds;

(2f) for a subset A of $(X,\tau), Int(A) = \bigcup_{i\in\Omega} Int(A \cap X_i)$ holds.

Theorem 2.6 *For an ordered pair $(\mathcal{E},\mathcal{E}_i)$ of $\mathcal{E}$ and $\mathcal{E}_i$, where $i \in \Omega$, let $(\mathcal{E},\mathcal{E}_i) = (SO(X,\tau), SO(X_i,\tau_i)), (PO(X,\tau), PO(X_i,\tau_i))$ or $(\tau^\alpha, (\tau_i)^\alpha)$. Then, for each $(\mathcal{E},\mathcal{E}_i)$ above $(i \in \Omega)$, the following property $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ holds:*

$(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$: for any subset U of $(X,\tau) := \bigoplus_{i\in\Omega}(X_i,\tau_i), U \in \mathcal{E}$ if and only if $U \cap X_i \in \mathcal{E}_i$ for every $i \in \Omega$.

Proof. We need the following properties $(*1)$, $(*2)$ and $(*3)$: for a subset A of $(X,\tau) := \bigoplus_{i\in\Omega}(X_i,\tau_i)$,

$(*1)$ $Cl(Int(A)) = \bigcup_{j\in\Omega} Cl(Int(A \cap X_j))$ holds;

$(*2)$ $Int(Cl(A)) = \bigcup_{j\in\Omega} Int(Cl(A \cap X_j))$ holds;

$(*3)$ $Int(Cl(Int(A))) = \bigcup_{j\in\Omega} Int(Cl(Int(A \cap X_j)))$ holds.

Indeed, by (2e) and (2f) above, it is shown that $Cl(Int(A)) = Cl(\bigcup_{i\in\Omega} Int\ (A \cap X_i)) = \bigcup_{j\in\Omega}(Cl(X_j \bigcap(\bigcup_{i\in\Omega} Int(A\cap X_i)))) = \bigcup_{j\in\Omega}(Cl(Int(A\cap X_j))), Int(Cl(A)) = Int(\bigcup_{i\in\Omega} Cl(A\cap X_i)) = \bigcup_{j\in\Omega}(Int(X_j \bigcap(\bigcup_{i\in\Omega} Cl\ (A\cap X_i)))) = \bigcup_{j\in\Omega}\ (Int(Cl(A\cap X_j)))$ and $Int(Cl(Int(A))) = \bigcup_{j\in\Omega} Int(Cl(Int(A))\cap X_j\ ) = \bigcup_{j\in\Omega} Int(\ Cl(Int(A\cap X_j)))$ hold (cf. $(*1)$).

Case 1. $\mathcal{E} = SO(X,\tau), \mathcal{E}_i = SO(X_i,\tau_i)$, where $i \in \Omega$: We prove that $U \in SO(X,\tau)$ if and only if $U \cap X_i \in SO(X_i,\tau_i)$ for every $i \in \Omega$.

Indeed, we suppose that $U \in SO(X,\tau)$. It follows from the assumption and $(*1)$ that $\bigcup_{j\in\Omega} Cl(Int(U \cap X_j)) \supset U$ holds in (X,τ). Then, for an element $i \in \Omega$, we have that $X_i \bigcap(\bigcup_{j\in\Omega} Cl(Int(U \cap X_j))) \supset X_i \cap U$ holds in (X_i,τ_i). Thus, we show that $Cl(Int(U \cap X_i)) \supset X_i \cap U$ holds in (X_i,τ_i). Namely, $U \cap X_i$ is semi-open in (X_i,τ_i) for every $i \in \Omega$. Conversely, suppose that $U \cap X_i \in SO(X_i,\tau_i)$ for every $i \in \Omega$. By using $(*1)$, $Cl(Int(U)) = \bigcup_{j\in\Omega} Cl(Int(U \cap X_j)) \supset \bigcup_{j\in\Omega} U \cap X_j = U$ hold. Hence, this conclude that $U \in SO(X,\tau)$.

Case 2. $\mathcal{E} = PO(X,\tau), \mathcal{E}_i = PO(X_i,\tau_i)$, where $i \in \Omega$: We prove that $U \in PO(X,\tau)$ if and only if $U \cap X_i \in PO(X_i,\tau_i)$ for every $i \in \Omega$.

Indeed, suppose that $U \in PO(X,\tau)$. It follows from the assumption and $(*2)$ that $U \subset \bigcup_{j\in\Omega} Int(Cl(U\cap X_j))$ holds in (X,τ). Then, we have that $X_i \cap U \subset X_i \bigcap(\bigcup_{j\in\Omega} Int(Cl(U\cap X_j))) = Int(Cl(U \cap X_i))$ holds in (X_i,τ_i), where $i \in \Omega$. Thus, we show that $U \cap X_i$ is preopen in (X_i,τ_i) for every $i \in \Omega$. Conversely, suppose that $U \cap X_i \in PO(X_i,\tau_i)$ for every $i \in \Omega$. By using $(*2)$, $Int(Cl(U)) = \bigcup_{j\in\Omega} Int(Cl(U \cap X_j)) \supset \bigcup_{j\in\Omega} U \cap X_j = U$ hold. Hence, this conclude that $U \in PO(X,\tau)$.

Case 3. $\mathcal{E} = \tau^\alpha, \mathcal{E}_i = (\tau_i)^\alpha$, where $i \in \Omega$: We prove that $U \in \tau^\alpha$ if and only if $U \cap X_i \in (\tau_i)^\alpha$ for every $i \in \Omega$.

Indeed, suppose that $U \in \tau^\alpha$. Using $(*3)$, we have that $U \subset \bigcup_{j\in\Omega} Int\ (Cl(Int(U\cap X_j)))$ holds in (X,τ). Then, we have that $X_i \cap U \subset X_i \bigcap\ (\bigcup_{j\in\Omega} Int\ (Cl(Int((U \cap X_j)))) = Int(Cl(Int(U \cap X_i)))$ holds in (X_i,τ_i), where $i \in \Omega$. Namely, $U \cap X_i$ is α-open in (X_i,τ_i) for every $i \in \Omega$. Conversely, suppose that $U \cap X_i \in (\tau_i)^\alpha$ for every $i \in \Omega$. By using $(*3)$, $Int(Cl(Int(U))) = \bigcup_{j\in\Omega} Int(Cl(Int(U \cap X_j))) \supset \bigcup_{j\in\Omega} U \cap X_j = U$ hold. Hence, this conclude that $U \in \tau^\alpha$. $\square$

Proof of Theorem 1.2 and Theorem 1.3 (ii)(iii)(iv). We recall the following notation: $X := \bigoplus_{i\in\Omega} X_i$ and $\tau := \bigoplus_{i\in\Omega} \tau_i$. In Theorem 2.5, assume that $\mathcal{E}_i = SO(X_i,\tau_i)$ (resp. $PO(X_i,\tau_i), (\tau_i)^\alpha$) for every $i \in \Omega$ and $\mathcal{E} = SO(X,\tau)$ (resp. $PO(X,\tau), \tau^\alpha$). We note that $(\mathcal{A})_\mathcal{E}$ and $(\mathcal{A})_{\mathcal{E}_i}$ hold for the families $\mathcal{E}, \mathcal{E}_i (i \in \Omega)$ above; for a subset A of X, $SO(X,\tau)$-$Cl(A) = sCl(A)$ (resp. $PO(X,\tau)$-$Cl(A) = pCl(A), \tau^\alpha$-$Cl(A) = \alpha Cl(X)$). Then, using Theorem 2.6, we have property $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ for an odered pair of the families $(\mathcal{E},\mathcal{E}_i) =$

$(SO(X,\tau), SO(X_i,\tau_i))$ (resp. $(PO(X,\tau), PO(X_i,\tau_i)), (\tau^\alpha, (\tau_i)^\alpha)$). Using Theorem 2.5 (i), we have that Theorem 1.2 (i) (resp. (ii), (iii)). Moreover, using Theorem 2.5 (iii), we have that Theorem 1.3 (ii) (resp. (iii), (iv)). □

Proof of Theorem 1.3 (i). In Theorem 2.5, let $\mathcal{E}_i = \tau_i$ for every $i \in \Omega$ and $\mathcal{E} = \tau$. Since the property $(\mathcal{S})_{\tau,\tau_i}$ is valid (cf. (2d) above), Theorem 1.3 (i) is obtained (cf. Theorem 2.5(iii)). □

3 Generalizations and proofs of Lemma 1.5, Theorems 1.6 and 1.7 In [23, Definition 2.10], the notion of $(\mathcal{E}_Y, \mathcal{E}'_Y)$-*g.closed sets* is defined in general.

Definition 3.1 [23, Definition 2.10] Let $\mathcal{E}_Y$ and $\mathcal{E}'_Y$ be two families of subsets of a topological space (Y,σ) satisfying the properties $(\mathcal{A})_\mathcal{E}$ and $(\mathcal{A})_{\mathcal{E}'}$, respectively, i.e., $\emptyset, Y \in \mathcal{E}_Y$ and $\emptyset, Y \in \mathcal{E}'_Y$. For an ordered pair $(\mathcal{E}_Y, \mathcal{E}'_Y)$, a subset A of (Y,σ) is said to be $(\mathcal{E}_Y, \mathcal{E}'_Y)$-*g.closed in* (Y,σ), if $\mathcal{E}'_Y\text{-}Cl(A) \subset U$ whenever $A \subset U$ and $U \in \mathcal{E}_Y$. The complement of a $(\mathcal{E}_Y, \mathcal{E}'_Y)$-g.closed set is said to be $(\mathcal{E}_Y, \mathcal{E}'_Y)$-*g.open* in (Y,σ).

Lemma 3.2 *Let $\mathcal{E}_Y$ and $\mathcal{E}'_Y$ be two families of subsets of a topological space (Y,σ) satisfying the properties $(\mathcal{A})_\mathcal{E}$ and $(\mathcal{A})_{\mathcal{E}'}$, respectively, i.e., $\emptyset, Y \in \mathcal{E}_Y$ and $\emptyset, Y \in \mathcal{E}'_Y$. For a subset A of (Y,σ), the following properties are equivalent:*
(1) *A is $(\mathcal{E}_Y, \mathcal{E}'_Y)$-g.closed in (Y,σ);*
(2) *$\mathcal{E}'_Y\text{-}Cl(A) \subset \mathcal{E}_Y\text{-}Ker(A)$ holds;*
(3) *$\{U | A \subset U, U \in \mathcal{E}_Y\} \subset \{V | \mathcal{E}'_Y\text{-}Cl(A) \subset V\}$ holds.*

Proof. **(1)⇒(2)** Let $x \notin \mathcal{E}_Y$-Ker(A). There exists a subset $U \in \mathcal{E}_Y$ such that $A \subset U$ and $x \notin U$. Thus, by (1), $\mathcal{E}'_Y\text{-}Cl(A) \subset U$ and $x \notin \mathcal{E}'_Y\text{-}Cl(A)$. Hence, we prove that (2), i.e., $\mathcal{E}'_Y$-Cl$(A) \subset \mathcal{E}_Y\text{-}Ker(A)$. **(2)⇒(3)** We set $\mathcal{K} := \{V | \mathcal{E}'_Y\text{-}Cl(A) \subset V\}$ and $\mathcal{J} := \{U | A \subset U, U \in \mathcal{E}_Y\}$. We claim that $\mathcal{J} \subset \mathcal{K}$ holds. Let $W \in \mathcal{J}$, i.e., $W \in \mathcal{E}_Y$ and $A \subset W$. Namely, $\mathcal{E}_Y\text{-}Ker(A) \subset W$. By (2), $\mathcal{E}'_Y\text{-}Cl(A) \subset W$, i.e., $W \in \mathcal{K}$. This conclude that $\mathcal{J} \subset \mathcal{K}$ holds. **(3)⇒(1)** Let W be a subset such that $A \subset W$ and $W \in \mathcal{E}_Y$, i.e., $W \in \mathcal{J}$ (cf. notations in the proof of (2)⇒(3) above). By (3), $W \in \mathcal{K}$. Namely, $\mathcal{E}'_Y\text{-}Cl(A) \subset W$ and hence A is $(\mathcal{E}_Y, \mathcal{E}'_Y)$-g.closed in (Y,σ). □

Let $(X,\tau) := \bigoplus_{i\in\Omega}(X_i,\tau_i)$ be the topological sum and, for each $i \in \Omega$, $\mathcal{E}_i$ and $\mathcal{E}'_i$ non-empty subfamilies of $P(X_i)$ and two non-empty subfamilies $\mathcal{E}$ and $\mathcal{E}'$ of $P(X)$, where $X := \bigoplus_{i\in\Omega} X_i$ and $\tau := \bigoplus_{i\in\Omega} \tau_i$.

Theorem 3.3 *Suppose that families $\mathcal{E}_i, \mathcal{E}'_i (i \in \Omega)$, $\mathcal{E}$ and $\mathcal{E}'$ satisfy the properties $(\mathcal{A})_\mathcal{E}, (\mathcal{A})_{\mathcal{E}_i}$, $(\mathcal{A})_{\mathcal{E}'}, (\mathcal{A})_{\mathcal{E}'_i}$, $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ and $(\mathcal{S})_{\mathcal{E}',\mathcal{E}'_i}$. For a subset A of the topological sum $(X,\tau) := \bigoplus_{i\in\Omega}(X_i,\tau_i)$, the following properties hold:*
(i) *$A = \mathcal{E}\text{-}Cl(A)$ holds in (X,τ) if and only if $A \cap X_i = \mathcal{E}_i\text{-}Cl(A \cap X_i)$ for every $i \in \Omega$.*
(ii) *A is $(\mathcal{E}, \mathcal{E}')$-g.closed in (X,τ) if and only if $A \cap X_i$ is $(\mathcal{E}_i, \mathcal{E}'_i)$-g.closed in (X_i,τ_i) for every $i \in \Omega$.*
(iii) *A is $(\mathcal{E}, \mathcal{E}')$-g.open in (X,τ) if and only if $A \cap X_i$ is $(\mathcal{E}_i, \mathcal{E}'_i)$-g.open in (X_i,τ_i) for every $i \in \Omega$.*

Proof. **(i) (Necessity)** By Theorem 2.5 (i), it is shown that $A \cap X_i = \mathcal{E}\text{-}Cl(A) \cap X_i = \mathcal{E}_i\text{-}Cl(A \cap X_i)$ holds for every $i \in \Omega$, because $\{(X_i,\tau_i) | i \in \Omega\}$ is a family of pairwise disjoint topological spaces. **(Sufficiency)** By Theorem 2.5 (i), it is also obtained that $\mathcal{E}\text{-}Cl(A) = \cup_{i\in\Omega}\mathcal{E}_i\text{-}Cl(A \cap X_i) = \cup_{i\in\Omega} A \cap X_i = A \cap (\cup_{i\in\Omega} X_i) = A \cap X = A$ and hence $A = \mathcal{E}\text{-}Cl(A)$. **(ii) (Necessity)** Since A is an $(\mathcal{E}, \mathcal{E}')$-g.closed set, $\mathcal{E}'\text{-}Cl(A) \subset \mathcal{E}\text{-}Ker(A)$ by Lemma 3.2.

Using Theorem 2.5 (i)(iii), we have that $\mathcal{E}'_j$-Cl$(A\cap X_j) = (\cup_{i\in\Omega}\mathcal{E}'_i\text{-}Cl(A\cap X_i))\cap X_j = (\mathcal{E}'\text{-}Cl(A))\cap X_j \subset (\mathcal{E}\text{-}Ker(A))\cap X_j = (\cup_{i\in\Omega}\mathcal{E}_i\text{-}Ker(A\cap X_i))\cap X_j = \mathcal{E}_j\text{-}Ker(A\cap X_j)$. Therefore, by Lemma 3.2, $A\cap X_j$ is $(\mathcal{E}_j,\mathcal{E}'_j)$-g.closed for each $j\in\Omega$. **(Sufficiency)** It follows from assumptions that $\mathcal{E}'_i\text{-}Cl(A\cap X_i)\subset\mathcal{E}_i\text{-}Ker(A\cap X_i)$ holds for every $i\in\Omega$. Then, using Theorem 2.5 (i) (iii), we have $\mathcal{E}'\text{-}Cl(A) = \cup_{i\in\Omega}\mathcal{E}'_i\text{-}Cl(A\cap X_i)\subset\cup_{i\in\Omega}\mathcal{E}_i\text{-}Ker(A\cap X_i) = \mathcal{E}\text{-}Ker(A)$. Therefore, by Lemma 3.2, A is $(\mathcal{E},\mathcal{E}')$-g.closed. **(iii)** We note that, for a subset B of X, $(X\setminus B)\cap X_i = X_i\setminus(B\cap X_i)$ holds in (X_i,τ_i) for every $i\in\Omega$. Then, the set A is $(\mathcal{E},\mathcal{E}')$-g.open if and only if $(X\setminus A)\cap X_i$ is $(\mathcal{E}_i,\mathcal{E}'_i)$-g.closed for every $i\in\Omega$ (cf. (ii) above); it holds if and only if $X_i\setminus(A\cap X_i)$ is $(\mathcal{E}_i,\mathcal{E}'_i)$-g.closed in (X_i,τ_i), i.e., $A\cap X_i$ is $(\mathcal{E}_i,\mathcal{E}'_i)$-g.open in (X_i,τ_i). Therefore, we have the required equivalence. □

Proof of Lemma 1.5 (i) (resp. (ii)). In Lemma 3.2, assume that $\mathcal{E}=\sigma$ and $\mathcal{E}' = SO(Y,\sigma)$ (resp. $PO(Y,\sigma)$). Then, a subset A of Y is gs-closed (resp. gp-closed) in (Y,σ) if and only if A is $(\sigma, SO(Y,\sigma))$-g.closed (resp. $(\sigma, PO(Y,\sigma))$-g.closed) in (Y,σ). By Lemma 3.2 for $\mathcal{E}=\sigma$ and $\mathcal{E}'=SO(Y,\sigma)$ (resp. $PO(Y,\sigma)$), Lemma 1.5 (i) (resp. (ii)) is obtained. □

Proof of Theorem 1.6 (i) (resp. (ii), (iii)). In Theorem 3.3 (ii), assume that $\mathcal{E}=\tau$ and $\mathcal{E}_i=\tau_i$ (resp. $SO(X_i,\tau_i), PO(X_i,\tau_i)$) for every $i\in\Omega$. Using Theorem 3.3 (ii) for $\mathcal{E}=\tau$ and $\mathcal{E}_i=\tau_i$ (resp. $SO(X_i,\tau_i), PO(X_i,\tau_i)$), we obtain Theorem 1.6 (i) (resp. (ii), (iii)).

Proof of Theorem 1.6 (iv). In Theorem 3.3 (ii), assume that $\mathcal{E} = SO(X,\tau)$ and $\mathcal{E}_i = SO(X_i,\tau_i)$ for every $i\in\Omega$. Then, we obtain Theorem 1.6 (iv). □

In the end of this section, we investigate a characterization of a kind of a compact topological sum of topological spaces.

Definition 3.4 Let $\mathcal{E}_Y$ and $\mathcal{E}'_Y$ be two families of subsets of a non-empty topological space (Y,σ) satisfying properties $(\mathcal{A})_{\mathcal{E}}$ and $(\mathcal{A})_{\mathcal{E}'}$, respectively, i.e., $\emptyset, Y\in\mathcal{E}_Y$ and $\emptyset, Y\in\mathcal{E}'_Y$. A topological space (Y,σ) is said to be $(\mathcal{E}_Y,\mathcal{E}'_Y)$-*g-compact* if every cover $\{V_j|j\in\nabla\}$ of Y by $(\mathcal{E}_Y,\mathcal{E}'_Y)$-g-open sets of (Y,σ), there exists a finite subset ∇_0 of ∇ such that $Y = \bigcup\{V_j|j\in\nabla_0\}$. The family $\{V_j|j\in\nabla\}$ is an $(\mathcal{E}_Y,\mathcal{E}'_Y)$-*g.open cover of* Y and the subfamily $\{V_j|j\in\nabla_0\}$ is called a *finite subcover* of ∇. We recall that $\emptyset$ and Y are $(\mathcal{E}_Y,\mathcal{E}'_Y)$-g.open in (Y,σ).

Let $(X,\tau) := \bigoplus_{i\in\Omega}(X_i,\tau_i)$ be the topological sum of pairwise disjoint non-empty topological spaces $\{(X_i,\tau_i)|i\in\Omega\}$. We denote $X := \bigoplus_{i\in\Omega}X_i = \bigcup\{X_i|i\in\Omega\}$ and $\tau := \bigoplus_{i\in\Omega}\tau_i$.

Theorem 3.5 *Let $\mathcal{E}_i$ and $\mathcal{E}'_i(i\in\Omega)$* (resp. *$\mathcal{E}$ and $\mathcal{E}'$*) *be families of subsets of a non-empty topological space $(X_i,\tau_i)(i\in\Omega)$* (resp. $X := \bigoplus_{i\in\Omega}(X_i,\tau_i)$). *Suppose that $\mathcal{E}_i$ and $\mathcal{E}'_i(i\in\Omega),\mathcal{E}$ and $\mathcal{E}'$ satisfy the properties $(\mathcal{A})_{\mathcal{E}},(\mathcal{A})_{\mathcal{E}_i}$, $(\mathcal{A})_{\mathcal{E}'},(\mathcal{A})_{\mathcal{E}'_i}$, $(\mathcal{S})_{\mathcal{E},\mathcal{E}_i}$ and $(\mathcal{S})_{\mathcal{E}',\mathcal{E}'_i}$. Then the following properties are equivalent:*

(1) *The topological sum $(X,\tau) =: \bigoplus_{i\in\Omega}(X_i,\tau_i)$ is $(\mathcal{E},\mathcal{E}')$-g-compact, where $X_i\neq\emptyset$ for every $i\in\Omega$;*

(2) *Each (X_i,τ_i) is $(\mathcal{E}_i,\mathcal{E}'_i)$-g-compact, where $i\in\Omega$, and Ω is finite.*

Proof. **(1)⇒(2)** By definition, it is shown that X_i is $(\mathcal{E},\mathcal{E}')$-g.open in (X,τ). Then, a family $\mathcal{V} := \{X_i|i\in\Omega\}$ is an $(\mathcal{E},\mathcal{E}')$-g.open cover of (X,τ). The cover $\mathcal{V}$ has a finite subcover and so Ω is finite. To prove the first part of (2), for each $i\in\Omega$, let $\mathcal{U}^{(i)} := \{U^{(i)}_j|j\in\nabla^{(i)}\}$ be an $(\mathcal{E}_i,\mathcal{E}'_i)$-g.open cover of (X_i,τ_i). Namely, $X_i = \bigcup\{U^{(i)}_j|j\in\nabla^{(i)}\}$. We put $V_j(i) := U^{(i)}_j\cup(\bigcup\{X_k|k\in\Omega, k\neq i\})$, where $j\in\nabla^{(i)}$, and $\mathcal{V}'_{(i)} := \{V_j(i)|j\in\nabla^{(i)}\}$. Then, $\mathcal{V}'_{(i)}$ is an $(\mathcal{E},\mathcal{E}')$-g.open cover of (X,τ). It follows from (1) that there exists a finite subset $\nabla^{(i)}_0$ of $\nabla^{(i)}$

such that $X = \bigcup\{V_j(i)|j \in \nabla_0^{(i)}\}$. Namely, $\mathcal{V}'_{(i)}$ has a finite subcover, say $\{V_j(i)|j \in \nabla_0^{(i)}\}$. Then, $X = \bigcup\{V_j(i)|j \in \nabla_0^{(i)}\} = (\bigcup\{U_j^{(i)}|j \in \nabla_0^{(i)}\}) \cup (\bigcup\{X_k|k \in \Omega, k \neq i\})$ and so $X_i = X \cap X_i = ((\bigcup\{U_j^{(i)}|j \in \nabla_0^{(i)}\}) \cup (\bigcup\{X_k|k \in \Omega, k \neq i\})) \cap X_i = \bigcup\{U_j^{(i)}|j \in \nabla_0^{(i)}\}$. Therefore, (X_i, τ_i) is $(\mathcal{E}_i, \mathcal{E}'_i)$-g-compact for every $i \in \Omega$. **(2)⇒(1)** Let $\mathcal{U} := \{U_j|j \in \nabla\}$ be an $(\mathcal{E}, \mathcal{E}')$-g.open cover of (X, τ). Then, by Theorem 3.3 (iii), for each $i \in \Omega, \mathcal{U}^{(i)} := \{U_j \cap X_i|j \in \nabla\}$ is an $(\mathcal{E}_i, \mathcal{E}'_i)$-g.open cover of (X_i, τ_i). There exists a finite subset $\nabla_0^{(i)}$ of ∇ such that $X_i = \bigcup\{U_j \cap X_i|j \in \nabla_0^{(i)}\}$. Thus we have $X = \cup_{i\in\Omega}(\bigcup\{U_j \cap X_i|j \in \nabla_0^{(i)}\}) \subset \bigcup_{i\in\Omega}(\bigcup\{U_j|j \in \nabla_0^{(i)}\}) = \bigcup\{U_j|j \in \bigcup_{i\in\Omega} \nabla_0^{(i)}\}$. The set $\bigcup_{i\in\Omega} \nabla_0^{(i)}$ is a finite subset of ∇, because Ω is finite. Therefore, (X, τ) is $(\mathcal{E}, \mathcal{E}')$-g-compact. □

Using Theorem 3.5, we have the proof of Theorem 1.7.

Proof of Theorem 1.7. In general, for an ordered pair of two families $(\mathcal{E}_Y, \mathcal{E}'_Y) = (\tau, SO(Y, \sigma))$ (resp. $(\sigma, PO(Y, \sigma)), (\sigma, \sigma), (SO(Y, \sigma), SO(Y, \sigma)))$, an $(\mathcal{E}_Y, \mathcal{E}'_Y)$-g.compact topological space (Y, σ) is called a gs-compact (resp. gp-compact, go-compact, sg-compact) space. Then, by Theorem 3.5, Theorem 1.7 is obtained. □

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Huilan JIN
Basic Science Department
Tianjin Agricultural University
Jinjing Road 22
Xiqing Dist; Tianjin city, 300384 P.R. China

Haruo MAKI
Wakagi-dai 2-10-13; Fukutsu-shi
Fukuoka-ken;811-3221 Japan
e-mail: makih@pop12.odn.ne.jp

Fumikazu Tamari
Department of Information Education
Faculty of Education
Fukuoka University of Education
Munakata, Fukuoka-ken, 811-4192 Japan