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ON THE EXISTENCE, STABILITY, AND STRUCTURE OF PERIODIC BIFURCATING SETS OF PERIODIC DIFFERENTIAL EQUATIONS *

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ABSTRACT. This paper concerns the bifurcation problem from equilibrium to invariant s-compact periodic sets in $\mathbf{R} \times \mathbf{R}^n$, for one parameter families of periodic ordinary differential equations. The analysis is accomplished by using appropriate families of discrete autonomous dynamical systems and some previous results of the authors on the relationship between conditional and unconnditional stability properties of sets in $\mathbf{R} \times \mathbf{R}^n$.

1. Introduction. Consider a one parameter family of differential systems $\dot{x} = f(t, x, \mu)$, $x \in \mathbf{R}^n$, $\mu \geq 0$, where f is smooth and periodic in t for a constant $\omega > 0$ (in particular t–independent). Moreover we assume that the origin 0 is an equilibrium for each μ. The present paper concerns the bifurcation from the origin into invariant, ω–periodic, asymptotically stable, s–compact sets in $\mathbf{R} \times \mathbf{R}^n$, under a drastic change of the stability properties of the origin through the value $\mu = 0$ of the parameter.

For bifurcation phenomena the stability theory plays an important role not only in the analysis of the stability of the bifurcating sets, but also in the analysis of their existence and structure. An example is Hopf bifurcation (see for instance [3],[6] and references within).

The bifurcation problem from the origin into t–independent sets was described in [5] for continuous or discrete autonomous dynamical systems. This problem is analyzed here for periodic differential systems allowing, as we already pointed out, to periodic bifurcating sets. The results are obtained by using stability arguments and appropriate discrete dynamical systems. The interest of periodic bifurcating sets is due to the fact that the cyclic processes in Natural Sciences are often connected to the periodicity of motion of sets rather than the periodicity of motion of single points.

In this paper we have considered the two different situations that the bifurcating sets are of dimension $n+1$ or of a dimension $\nu + 1$, $\nu < n$. The last situation is presented in the case that there exists a ω–periodic invariant manifold Φ_μ in the time space on which each bifurcating set M_μ lies. To connect the stability properties of the bifurcating sets M_μ with respect to the perturbations lying on Φ_μ, the stability properties of Φ_μ, and the unconditional stability properties of M_μ, we have applied some results that we obtained in [12]. These results concern the general case that neither the differential equations nor the invariant manifold are necessarily periodic. They will be summarized in Section 2 in their general formulation as well as in the periodic case.

We have tried to pick out the essential ingredients of the bifurcation phenomenon and then we have assumed the existence of the invariant manifold Φ_μ without specifying any particular condition on f which allows to this existence. In this respect papers [10] and [1] are revisited and enriched.

More detailed information on the structure of the bifurcating sets M_μ are given in Section 4 in the cases $\nu = 1$ and $\nu = 2$. In the first case, for any t, the section $\Phi_\mu(t)$ is homeomorphic to a straigthline y

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passing through the origin. Each section $M_\mu(t)$ is homeomorphic to the union of two segments located in the regions $y > 0$ and $y < 0$ respectively. The end points are fixed with respect to the discrete dynamical system induced on $\Phi_\mu(t)$, while their motion with respect to the differential system is periodic with the same period ω of the system. If $\nu = 2$, in the autonomous case we find the results already known in the usual treatment of Hopf bifurcation although now the asymptotic stability of Φ_μ near the origin is not necessarily exponential. In the nonautonomous periodic case, under some additional assumption we find that the sections $M_\mu(t)$ are homeomorphic to Jordan curves and then the sets M_μ are homeomorphic to tori by interpreting t as an angular variable.

2 Preliminaries.

Denote by $\|\cdot\|$ the Euclidean norm in $\mathbf{R}^n$ and by ρ the induced distance. Let D be a nonempty set in $\mathbf{R}^n$ and for $a > 0$ let $B^n(D,a) = \{x \in \mathbf{R}^n : \rho(x,D) < a\}$, $B^n[D,a] = \{x \in \mathbf{R}^n : \rho(x,D) \leq a\}$, $S^n(D,a) = \{x \in \mathbf{R}^n : \rho(x,D) = a\}$. Consider a set A in $\mathbf{R} \times \mathbf{R}^n$. We say that A is s–nonempty if for any $t \in \mathbf{R}$ the section $A(t) = \{x \in \mathbf{R}^n : (t,x) \in A\}$ is nonempty. If A is s–nonempty and there exists a compact set Q in $\mathbf{R}^n$ such that $A(t) \subseteq Q$ for all $t \in \mathbf{R}$, then A is said to be s–bounded. In this case the intersection of all these sets Q will be denoted by $Q^\star(A)$. If A is s–bounded and each $A(t)$ is compact, we say that A is s–compact. When the mapping $t \to A(t)$ is ω–periodic for some $\omega > 0$ or in particular t–independent, we say that A is ω–periodic or t–independent respectively.

We denote by $\mathcal{L}(x)$ the class of functions $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $(t,x) \to f(t,x)$, which are locally Lipschitzian in x. We will write $f \in \mathcal{L}_u(x)$ if f satisfies the condition that for every compact set $K \subset \mathbf{R}^n$ there exists a constant $L(K) > 0$ such that $\|f(t,x) - f(t,y)\| \leq L(K)\|x - y\|$ for all x, y in K and t in $\mathbf{R}$, and write $f \in \mathcal{L}_{ub}(x)$ if in addition for every compact $K \subset \mathbf{R}^n$ the function f is bounded. Trivially $f \in \mathcal{L}_u(x)$ implies $f \in \mathcal{L}_{ub}(x)$ if there exists at least one $x \in \mathbf{R}^n$ such that the function $f(\cdot, x)$ is bounded.

Consider the system of differential equations

$$\dot{x} = f(t,x), \qquad (\dot{}) = \frac{d}{dt} \tag{2.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ and satisfies conditions ensuring the uniqueness of solutions. Moreover f is supposed to be such that (2.1) admits an invariant s–compact set M in $\mathbf{R} \times \mathbf{R}^n$. For any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ let us denote by $x(t, t_0, x_0)$ the solution through (t_0, x_0) and by $j^+(t_0, x_0)$, $j^-(t_0, x_0)$ its maximal interval of existence in the future and in the past respectively.

Let now A be an invariant set in $\mathbf{R} \times \mathbf{R}^n$. The stability concepts of A are supposed to be known. Assuming that M is contained in A, we limit ourselves to define some stability properties of A "near M" in the sense specified in [12]:

Definition 2.1 *For any $\gamma > 0$ let $I[M,\gamma] = \{(t,x) : t \in \mathbf{R}, x \in B^n[M(t),\gamma]\}$. Then we will say that A has a stability property near M if there exists $\gamma > 0$ such that the property is satisfied with respect to the perturbations $(t_0, x_0) \in I[M,\gamma]$.*

For instance: A is said to be: (i) stable near M if there exists $\gamma > 0$ such that for any $t_0 \in \mathbf{R}$ and $\varepsilon > 0$ one may find $\delta(t_0, \varepsilon) > 0$ with the property that $x_0 \in B^n[M(t_0), \gamma]$ and $\rho(x_0, A(t_0)) < \delta(t_0, \varepsilon)$ imply $\rho(x(t, t_0, x_0), A(t)) < \varepsilon$ for any $t \in J^+(t_0, x_0)$; (ii) attracting near M if there exists $\gamma > 0$ such that one may find $\alpha(t_0) > 0$ for which $x_0 \in B^n[M(t_0), \gamma]$ and $\rho(x_0, A(t_0)) < \alpha(t_0)$ imply

$j^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), A(t)) \to 0$ as $t \to +\infty$. Similarly one may define the other stability or attractivity properties of A near M.

Remark 2.1 *Since M is contained in A, and then $\rho(x_0, A(t_0)) \leq \rho(x_0, M(t_0))$ for any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$, the uniform attractivity of A near M may be defined as follows: There exists $\sigma > 0$ such that $t_0 \in \mathbf{R}$ and $x_0 \in B^n[M(t_0), \sigma]$ implies that $x(t, t_0, x_0)$ exists for all $t \geq t_0$ and satisfies $\rho(x(t, t_0, x_0), A(t)) \to 0$ as $t \to +\infty$, uniformly in (t_0, x_0).*

We need some of the results given in [12]. We summarize them in the following theorem.

Theorem 2.1 *Let us assume that for some integer $\nu \in [1, n)$ there exists for system (2.1) an invariant set in $\mathbf{R} \times \mathbf{R}^n$ defined by*

$$\Phi = \{(t, y, z) : \, t \in \mathbf{R}, \, y \in \mathbf{R}^\nu, \, z \in \mathbf{R}^{n-\nu}, \, z = g(t, y)\},$$

$g \in C^1$, $g \in \mathcal{L}'_{ub}(y)$ such that (i) $M \subset \Phi$; (ii) M is uniformly asymptotically stable on Φ, that is with respect to the initial perturbations $(t_0, x_0) \in \Phi$. Here by $g \in \mathcal{L}'_{ub}(y)$ we want to mean that g belongs to $\mathcal{L}_{ub}(y)$ together with its first partial derivatives. We have:

(a) *If $f \in \mathcal{L}_u(x)$ and Φ is stable (asymptotically stable) near M, then M is stable (asymptotically stable);*

(b) *If $f \in \mathcal{L}_{ub}(x)$ and M is uniformly stable (uniformly asymptotically stable), then Φ is uniformly stable (uniformly asymptotically stable) near M.*

Let us give now the version of Theorem 2.1 when f and M are both ω–periodic in t for the same constant $\omega > 0$, f is continuous and $f \in \mathcal{L}(x)$. The conditions $f \in \mathcal{L}_{ub}(x)$ and then $f \in \mathcal{L}_u(x)$ are clearly satisfied. Moreover in this case the stability and the asymptotic stability of M when occurring are uniform. Consequently even the stability or the asymptotic stability of Φ when occurring are uniform. Indeed if Φ is stable near M, M is stable by virtue of statement (a), then uniformly stable, and then Φ is uniformly stable by virtue of statement (b). Similarly one may proceed for asymptotic stability. Thus the following corollary of Theorem 2.1 holds.

Corollary 2.1 *Assume that f and M are both ω–periodic in t for the same constant $\omega > 0$, f is continuous and $f \in \mathcal{L}(x)$. Then under the assumptions of Theorem 2.1, M is stable (asymptotically stable) if and only if Φ is stable (asymptotically stable) near M.*

3. Bifurcation from equilibrium to s–compact sets for one - parameter families of periodic differential equations. Consider the one-parameter family $\mathcal{S}$ of differential systems

$$(S)_\mu \qquad \dot{x} = f(t, x, \mu), \qquad (\dot{\,}) = \frac{d}{dt},$$

with $f \in C^1(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^+, \mathbf{R}^n)$ and periodic in t for some constant $\omega > 0$. Moreover we assume $f(t, 0, \mu) \equiv 0$ so that $(S)_\mu$ admits the null solution for every $\mu \geq 0$. We denote by M_0 the so–called null set, $M_0 = \{(t, x) : t \in \mathbf{R}, \, x = 0\}$. Given any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ and $\mu > 0$ let us denote by $x_{(S)_\mu}(t, t_0, x_0)$ the solution of $(S)_\mu$ through (t_0, x_0) and by $j^+(t_0, x_0, \mu)$, $j^-(t_0, x_0, \mu)$ its maximal intervals of existence in the future and in the past respectively.

Definition 3.1 *We say that $\mu = 0$ is a bifurcation value on the right for the family $\mathcal{S}$ at $x = 0$ if there exist $\mu^\star > 0$ and a family $\{M_\mu\}$, $\mu \in (0, \mu^\star)$, of s–compact and ω–periodic subsets of $(\mathbf{R} \times \mathbf{R}^n) - M_0$ having the following properties:*

(a) *for each $\mu \in (0, \mu^\star)$, M_μ is invariant under $(S)_\mu$;*

(b) *$M_\mu(t) \to \{0\}$ as $\mu \to 0$ uniformly in t.*

We give now a very general theorem concerning the bifurcation from the origin into bifurcating s–compact sets of dimension $n + 1$. The case that the bifurcating sets lie on an invariant manifold of dimension $\nu + 1$, $\nu < n$, will be treated afterwords. In the sequel a set H in $\mathbf{R} \times \mathbf{R}^n$ will be said to be a s–compact neighborhood of M_0 if H is s–compact and each section $H(t)$ is a compact neighborhood of $x = 0$.

Theorem 3.1 *Suppose that the origin $x = 0$ is asymptotically stable for $\mu = 0$ and completely unstable (i.e. asymptotically stable in the past) for $\mu > 0$. Then $\mu = 0$ is a bifurcation value on the right. Precisely there exist $\mu^\star > 0$ and an s–compact neighborhood H of M_0 such that for each $\mu \in (0, \mu^\star)$ the largest s–compact invariant set of $(S)_\mu$ contained in $H - M_0$, say M_μ, is nonempty, ω–periodic, and the family $\{M_\mu\}$ satisfies (b) in Definition 3.1. Moreover each M_μ is asymptotically stable under $(S)_\mu$.*

Proof. Let r, r' be two numbers such that $0 < r < r'$. For each $\mu \geq 0$ consider the family $\mathcal{S}'$ of systems

$$(S')_\mu \qquad\qquad \dot{x} = f(t, x, \mu)\alpha(x),$$

where $\alpha \in C^\infty(\mathbf{R}^n, [0, 1])$ is such that $\alpha(x) = 1$ for $\|x\| < r$ and $\alpha(x) = 0$ for $\|x\| \geq r'$. Because of the local character of our problem, and because for each μ system $(S)_\mu$ coincides with $(S')_\mu$ in $B^n(r)$, the new family $\mathcal{S}'$ satisfies all the stability assumptions in Theorem 3.1 and may replace the original family $\mathcal{S}$. For any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ and $\mu \geq 0$ let us denote by $x_{(S')_\mu}(t, t_0, x_0)$ the solution of $(S')_\mu$ through (t_0, x_0). This solution clearly exists for all t in $\mathbf{R}$. The proof is divided into four steps.

(i) Since the origin is an asymptotically stable solution of $(S')_0$ there exists a number $\gamma \in (0, r)$ and a function $V \in C^\infty(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$, ω–periodic in t, such that

$$a(\|x\|) \leq V(t, x) \leq b(\|x\|), \tag{3.1}$$

$$\dot{V}_{(S')_0}(t, x) \leq -c(\|x\|), \tag{3.2}$$

for all $t \in \mathbf{R}$ and $x \in B^n(\gamma)$ [7, 8]. Here a, b, c are continuous strictly increasing functions from $\mathbf{R}^+$ into $\mathbf{R}^+$ with $a(0) = b(0) = c(0) = 0$, and the left hand side of (3.2) is the derivative of V along the solutions of $(S')_0$. We determine now a number $\mu^\star > 0$ and an s–compact neighborhooh H of M_0 such that for all $\mu \in (0, \mu^\star)$ H is asymptotically stable under $(S')_\mu$ and invariant only in the future. Precisely choose a number $\lambda \in (0, a(\gamma))$ and consider the subset of $\mathbf{R} \times \mathbf{R}^n$

$$H = \{(t, x) : \|x\| \leq \gamma, \, V(t, x) \leq \lambda\}.$$

Clearly $(t,x) \in H$ implies $\|x\| < \gamma$. Moreover we see that H contains the points for which $t \in \mathbf{R}$ and $|\|x\| \leq b^{-1}(\lambda)$. Thus each section $H(t)$ is a compact neighborhood of $x = 0$ and is contained in the open ball $B^n(\gamma)$. By (3.2) and continuity arguments we may select $\mu^\star$ so that

$$\dot{V}_{(S')_\mu}(t,x) \leq -c(\frac{b^{-1}(\lambda)}{2})$$

for all $\mu \in (0,\mu^\star)$, $t \in \mathbf{R}$ and $x \in B^n(\gamma) - B^n(b^{-1}(\lambda))$. Hence H has all the properties we have required above and in addition its region of attraction under $(S')_\mu$ contains a fixed neighborhood $H^\star$ of H. We will choose $H^\star = \{(t,x) : \|x\| \leq \gamma, V(t,x) \leq \lambda^\star\}$, for some $\lambda^\star > \lambda$.

(ii) Let $\mathbf{Z}$ be the set of the integers. For any fixed $t_0 \in \mathbf{R}$ and $\mu \in (0,\mu^\star)$ consider the map $\Pi := \Pi_{t_0\mu} : \mathbf{Z} \times \mathbf{R}^n \to \mathbf{R}^n$ defined by $\Pi(i,x) = x_{(S')_\mu}(t_0 + i\omega, t_0, x)$. Clearly we have $\Pi(0,x) = x$ and $\Pi(i_1 + i_2, x) = \Pi(i_1, \Pi(i_2,x))$ for every $i_1, i_2 \in \mathbf{Z}$ and $x \in \mathbf{R}^n$. Hence Π defines an autonomous discrete dynamical system depending on t_0, μ. For $x \in \mathbf{R}^n$ let $J^+(x) := J^+_{t_0\mu}(x)$, $J^-(x) := J^-_{t_0\mu}(x)$ be the positive and the negative prolongational limit set of x under Π. Precisely:

$$\begin{aligned} J^+(x) = \{\xi \in \mathbf{R}^n \ : \ & \text{there exist two sequences } (i_k),\ i_k \to +\infty, (x_k), x_k \to x, \\ & \text{such that } \Pi(i_k, x_k) \to \xi\}, \\ J^-(x) = \{\xi \in \mathbf{R}^n \ : \ & \text{there exist two sequences } (i_k),\ i_k \to -\infty, (x_k), x_k \to x, \\ & \text{such that } \Pi(i_k, x_k) \to \xi\}. \end{aligned}$$

For the characterization of asymptotic stability by means of prolongational limit sets see for instance [2]. This characterization is usually given for continuous dynamical systems but, with slight modifications in the proof, it remains valid also for discrete dynamical systems. The set $H(t_0)$ is a uniform attractor under Π and its region of uniform attraction contains $H^\star(t_0)$. Therefore, $x \in H^\star(t_0)$ implies $J^+(x) \neq \emptyset$ and $J^+(x) \subseteq H(t_0)$. Let $\phi := \phi_{t_0\mu}$ be the largest invariant set under Π contained in $H(t_0)$) and denote by $A^-(0) := A^-_{t_0\mu}(0)$ the region of negative attraction of the origin of $\mathbf{R}^n$ under Π. Clearly, because of our assumptions, $A^-(0)$ is an open neighborhood of $x = 0$. We show that $\phi \supset A^-(0)$. Since $A^-(0)$ is invariant for Π, it is sufficient to show that $H(t_0) \supset A^-(0)$. Indeed $x \in A^-(0)$ implies the existence of $i \in \mathbf{Z}^-$ such that $\Pi(i,x) \in H(t_0)$ and then $x = \Pi(-i, \Pi(i,x)) \in H(t_0)$ by virtue of the positive invariance of $H(t_0)$. Define now the set M_μ with the condition that its section at any time $t_0 \in \mathbf{R}$ is $M_\mu(t_0) = \phi - A^-(0)$. We prove that $M_\mu(t_0)$ is a uniform attractor under Π and its region of uniform attractivity contains $H^\star(t_0) - \{0\}$. Indeed, assume $x \in H^\star(t_0) - \{0\}$, then $J^+(x) \neq \emptyset$ and $J^+(x) \subseteq H(t_0)$. Since $J^+(x)$ is invariant under Π, we have $J^+(x) \subseteq \phi$. Hence it remains to prove that $y \in J^+(x)$ implies $y \notin A^-(0)$. Indeed we have $x \in J^-(y)$. Therefore if $y \in A^-(0)$, then $J^-(y) = \{0\}$ and consequently $x = 0$. This is a contradiction and the assert is proved. Since $H(t_0) = H(t_0 + \omega)$ and Π remains unchanged when t_0 is replaced by $t_0 + \omega$, we have $M_\mu(t_0) = M_\mu(t_0 + \omega)$. Then M_μ is a ω–periodic set in $\mathbf{R} \times \mathbf{R}^n$. It is immediate to recognize that M_μ is s–compact and that $M_\mu(t) \to \{0\}$ as $\mu \to 0$ uniformly in t.

(iii) We prove now that M_μ is invariant under $(S')_\mu$. Clearly it is sufficient to show that for each t_0 the set $M_\mu(t_0)$ is the image of $M_\mu(0)$ under the flow generated by $(S')_\mu$. Let $G = x_{(S')_\mu}(t_0, 0, M_\mu(0))$.

We have

$$\Pi(i,G) = x_{(S')_\mu}(t_0+i\omega, t_0, G) = x_{(S')_\mu}(t_0+i\omega, t_0, x_{(S')_\mu}(t_0,0,M_\mu(0)))$$
$$= x_{(S')_\mu}(t_0+i\omega, i\omega, x_{(S')_\mu}(i\omega,0,M_\mu(0))) = x_{(S')_\mu}(t_0+i\omega, i\omega, M_\mu(i\omega))$$
$$= x_{(S')_\mu}(t_0,0,M_\mu(i\omega)) = x_{(S')_\mu}(t_0,0,M_\mu(0)) = G.$$

Thus G is a compact invariant set for Π which does not contain the origin. Therefore $G \subseteq M_\mu(t_0)$. By using the same argument we find that $M_\mu(0) \supseteq W$, where $W = x_{(S')_\mu}(0,t_0,M_\mu(t_0))$. Then $x_{(S')_\mu}(t_0,0,W) \supseteq G$ and since $x_{(S')_\mu}(t_0,0,W) = M_\mu(t_0)$, we have $M_\mu(t_0) \subseteq G$. Hence $G = M_\mu(t_0)$ and this completes the proof of the invariance of M_μ. It is also clear that M_μ is the largest s–compact invariant set of $(S')_\mu$ contained in $H - M_0$.

(iv) Finally we prove that M_μ is an asymptotically stable set under $(S')_\mu$. For this it is sufficient to prove that M_μ is a uniform attractor under $(S')_\mu$. Let σ denote any positive number such that the set $\{x : \rho(x, M_\mu(0)) < \sigma\}$ is contained in $H^\star(0) - \{0\}$. It is easy to see that given any $\beta > 0$ we can find a number $\delta(\beta) \in (0,\sigma)$ such that for $j \in \mathbf{Z}$ and $t_0 = j\omega$, $\rho(x, M_\mu(t_0)) < \delta(\beta)$ implies $\rho(x_{(S')_\mu}(t,t_0,x), M_\mu(t)) < \beta$ for all $t \in [t_0, t_0+\omega]$. The existence of the number $\delta(\beta)$ follows by using the compactness of $M_\mu(t_0)$, continuity arguments and the ω–periodicity of $(S')_\mu$. Moreover, since $M_\mu(0)$ is a uniform attractor under $\Pi_{0\mu}$ we have that relatively to $\delta(\beta)$ there exists an integer $h(\beta) = \tau(\delta(\beta))$ such that $\rho(x, M_\mu(0)) < \sigma$ implies $\rho(\Pi_{0\mu}(j,x), M_\mu(j\omega)) < \delta(\beta)$ for $j \in \mathbf{Z}$ and $j \geq h(\beta)$. Taking into account that our choice implies $M_\mu(t_0) = M_\mu(j\omega) = M_\mu(0)$, in conclusion we obtain that $\rho(x, M_\mu(0)) < \sigma$ implies $\rho(x(t,0,x,\mu), M_\mu(t)) < \beta$ for all real $t \geq h(\beta)$. The proof is complete. ■

In the autonomous case the sets M_μ are t–independent. Precisely one has:

Proposition 3.1 *Let us assume that $(S)_\mu$ is autonomous for each $\mu > 0$. Then the bifurcating sets M_μ are t–independent, that is $M_\mu = \mathbf{R} \times C_\mu$, where C_μ is the largest compact invariant set of $\mathbf{R}^n$ disjoint from the origin, contained in a fixed positively invariant neighborhood of the origin.*

Clearly, since in Proposition 3.1 $M_\mu(t) \equiv C_\mu$ for any t, as observed before, we may consider all the properties associated with M_μ as properties of the sets C_μ of $\mathbf{R}^n$. Precisely we may say that the sets C_μ are asymptotically stable and that $C_\mu \to \{0\}$ as $\mu \to 0$.

We treat now the case that the bifurcating sets lie on an invariant manifold. Consider again the above family $\mathcal{S}$ of differential systems and assume in addition that each $(S)_\mu$ admits an invariant manifold

$$(3.3)_\mu \qquad \Phi_\mu = \{(t,y,z) : t \in \mathbf{R},\, y \in \mathbf{R}^\nu,\, z \in \mathbf{R}^{n-\nu},\, z = g(t,y,\mu)\},$$

where $\nu < n$, $(y,z) = x$, g is C^1 and ω–periodic in t, $(\partial g/\partial t)$ and $(\partial g/\partial y)$ are locally Lipschitzian in y, and $g(t,0,\mu) \equiv 0$. We notice that the above conditions ensure $g \in \mathcal{L}'_{ub}(y)$. Let $u = z - g(t,y,\mu)$. In terms of y, u the family $\mathcal{S}$ may be identified by the family Σ:

$$(\Sigma)_\mu \qquad \begin{aligned} \dot{y} &= Y(t,y,u,\mu), \\ \dot{u} &= U(t,y,u,\mu), \end{aligned}$$

where Y, U are continuous and locally Lipschitzian in (y, u), $Y(t, 0, 0, \mu) \equiv 0$, $U(t, y, 0, \mu) \equiv 0$. Moreover in the (t, y, u)–space the manifolds Φ_μ coincide all with the manifold

$$\Phi = \{(t, y, u) \,:\, u = 0\}. \tag{3.4}$$

Thus, $\Phi = \mathbf{R} \times \bar{\bar{\Phi}}$, with $\bar{\bar{\Phi}} \subseteq \mathbf{R}^\nu$. For the solutions of $(\Sigma)_\mu$ lying on the invariant manifold Φ the y–part of x satisfies the ν–dimensional system

$$\dot{y} = Y(t, y, 0, \mu). \tag{$(\Sigma)_{y\mu}$}$$

The bifurcating sets of Σ are homeomorphic to those of the original family while the stability properties involved are clearly the same. We continue to denote by M_0 the null set of $(\Sigma)_\mu$, that is the set $M_0 = \{(t, y, u) \,:\, , t \in \mathbf{R},\, y = 0,\, u = 0\}$, and we now indicate by m_0 the null set of $(\Sigma)_{y\mu}$, $m_0 = \{(t, y) \,:\, t \in \mathbf{R},\, y = 0\}$.

Theorem 3.2 *Suppose that:* (1) *the solution* $y = 0$ *of* $(\Sigma)_{y\mu}$ *is asymptotically stable if* $\mu = 0$ *and completely unstable if* $\mu > 0$ *small;* (2) Φ *is asymptotically stable near the origin of* $\mathbf{R}^n$. *Then* $\mu = 0$ *is a bifurcation value on the right for the family* Σ. *Precisely there exist* $\mu^\star > 0$ *and an* s*–compact neighborhood* H *of* M_0 *such that for each* $\mu \in (0, \mu^\star)$ *the largest* s*–compact invariant set of* Σ *contained in* $H - M_0$, *say* M_μ, *is nonempty, lies on* Φ, *is* ω*–periodic, asymptotically stable, and the family* $\{M_\mu\}$ *satisfies* (b) *in Definition* 3.1.

Proof. By a suitable redefinition of (Y, U) outside of a neighborhood of $(0, 0)$ (as indicated in the proof of Theorem 3.1), we may assume without loss of generality that for any μ the solutions of $(\Sigma)_\mu$ are globally existing for all $t \in \mathbf{R}$. This redefinition does not modify the invariant character of Φ (see the proof of Theorem 3.2 in [12]). Assumption (1) is equivalent to say that the origin $(0, 0)$ of $(\Sigma)_\mu$ is asymptotically stable on $(\Sigma)_{y\mu}$. Taking into account assumption (2), we recognize then by virtue of Corollary 2.1 that $(0, 0)$ is (unconditionally) asymptotically stable. This latter property implies as in the proof of Theorem 3.1 the existence of a function $V \in C^\infty(\mathbf{R} \times \mathbf{R}^\nu \times \mathbf{R}^{n-\nu}, R)$, ω–periodic in t, and of three functions a, b, c such that

$$a(\|(y, u)\|) \le V(t, y, u) \le b(\|(y, u)\|), \tag{3.5}$$
$$\dot{V}_{(\Sigma)_0}(t, y, u)) \le -c(\|(y, u)\|), \tag{3.6}$$

for all $t \in \mathbf{R}$ and $(y, u) \in B^n(\gamma)$ for some $\gamma \in (0, \sigma_1)$. Here we denote by σ_1 the number σ in Remark 2.1 relative to the uniform attractivity of Φ near $(0, 0)$. Moreover a, b, c are again continuous strictly increasing functions from $\mathbf{R}^+$ into $\mathbf{R}^+$ with $a(0) = b(0) = c(0) = 0$. Letting $v(t, y) \equiv V(t, y, 0)$, we see that v is a Liapunov function associated with the asymptotic stability of the solution $y = 0$ of $(\Sigma)_{y\mu}$. Assuming $\lambda \in (0, a(\gamma))$, consider the two sets

$$H = \{(t, y, u) \,:\, \|(y, u)\| \le \gamma,\, V(t, y, u) \le \lambda\},$$
$$h = \{(t, y) \,:\, \|y\| \le \gamma,\, , v(t, y) \le \lambda\}.$$

As in the proof of Theorem 3.1 we recognize that H is an s–compact neighborhood of M_0 which is invariant only in the future for $(\Sigma)_\mu$ and whose sections $H(t)$ are contained in $B^n(\gamma)$. Similarly h is a compact neighborhood of m_0 for $(\Sigma)_{y\mu}$, invariant only in the future, and whose sections $h(t)$ are contained in $B^\nu(\gamma)$. Moreover $\{(y,u)\,:\,y\in h(t),\,u=0\}=H(t)\cap\bar{\Phi}$. If $\mu^\star>0$ is sufficiently small then for all $\mu\in(0,\mu^\star)$ the sets $H,\,h$ are asymptotically stable for $(\Sigma)_\mu$ and $(\Sigma)_{y\mu}$ and their regions of attractions contain fixed neighborhoods $H^\star,\,h^\star$, of $H,\,h$ respectively. By virtue of Theorem 3.1 applied to system $(\Sigma)_{y\mu}$ we recognize that if $\mu^\star>0$ is sufficiently small then for each $\mu\in(0,\mu^\star)$ the largest s–compact invariant set of $(\Sigma)_{y\mu}$ contained in $h-m_0$, say m_μ, is nonempty, ω–periodic, and $m_\mu\to m_0$ as $\mu\to 0$. Thus if $\mu\in(0,\mu^\star)$ the set

$$M_\mu=\{t,y,u)\,:\,(t,y)\in m_\mu,\,u=0\}$$

is for $(\Sigma)_\mu$ nonempty, ω–periodic, invariant, contained in $H-M_0$, and asymptotically stable with respect to the solutions lying on Φ. Since for every t in $\mathbf{R}$ the section $H(t)$ and then the section $M_\mu(t)$ are contained in $B^n(\gamma)$, we see that Φ is asymptotically stable near each $M_\mu,\,\mu\in(0,\mu^\star)$. By using again Corollary 2.1 and taking into accoun our choice of γ, it follows that M_μ is asymptotically stable even for $(\Sigma)_\mu$ and the region of uniform attractivity contains $H^\star-M_0$. Moreover $M_\mu\to M_0$ as $\mu\to 0$. To complete the proof it remains only to prove that M_μ is the largest invariant set of $(\Sigma)_\mu$ contained in $H-M_0$. For any fixed $t_0\in\mathbf{R}$ and $\mu\in(0,\mu^\star)$ consider as in the proof of Theorem 3.1 the autonomous discrete dynamical system $\Pi:=\Pi_{t_0\mu}\,:\mathbf{Z}\times\mathbf{R}^n\to\mathbf{R}^n$ relative to system $(\Sigma)_\mu$. Clearly it is sufficient to prove that any section $M_\mu(t_0)$ is the largest compact invariant set of Π contained in $H(t_0)-\{0,0\}$, and consequently in $H(t_0)-A^-(\{0,0\})$. Let $x_0=(y_0,u_0)\in H(t_0)-A^-(\{0,0\})$ and $x_0\notin M_\mu(t_0)$. Suppose that the complete orbit of x_0 under Π is contained in $H(t_0)-A^-(\{0,0\})$. Denote by $\Lambda^-(x_0)$ the negative limit set of x_0 under Π. The set $\Lambda^-(x_0)$ is nonempty, contained in $H(t_0)-\{0,0\}$, invariant under Π, and compact. Clearly $\Lambda^-(x_0)\cap M_\mu(t_0)=\emptyset$ otherwise x_0 would be weakly attracted from $M_\mu(t_0)$ in the past and then $M_\mu(t_0)$ could not be stable. Let $\delta>0$ be the distance between the two compact sets $\Lambda^-(x_0)$, $M_\mu(t_0)$ and let ξ be any point in $\Lambda^-(x_0)$. Because of the invariance of $\Lambda^-(x_0)$ we have

$$\rho(\Pi(i,\xi),M_\mu(t_0))>\delta\quad\text{for every integer}\quad i\geq 0. \tag{3.7}$$

Since $M_\mu(t_0)$ is asymptotically stable under Π, it follows that $\Lambda^+(\xi)\subseteq M_\mu(t_0)$, a contradiction. The proof is complete. ■

4 On the structure of the bifurcatings sets for $\nu=1$ and $\nu=2$. We need preliminarly to recall a concept of asymptotic stability that we have used elsewhere for autonomous as well as for periodic differential systems [9, 11]:

Definition 4.1 *Let Γ^r_τ, $\tau\geq 0$, be the set of functions $W\,:\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$ such that (1) $W(t,w)$ has continuous partial derivatives up to order $r\geq 1$, (2) W is τ–periodic in t and satisfies $\tau=$ inf $\{\lambda>0\,:\,W$ is $\lambda-$ periodic in $t\}$. Consider the n–dimensional system $\dot{w}=W(t,w)$, where $W\in\Gamma^r_\tau$ and $W(t,0)\equiv 0$. Let h, $0<h\leq r$, be an integer. The solution $w=0$ of the given system is said to be h–asymptotically stable if: (i) for any $\zeta\in\Gamma^r_\tau$ and $\zeta=o(\|w\|^h)$ as $w\to 0$, the solution $w=0$ of system $\dot{w}=W(t,w)+\zeta(t,w)$ is asymptotically stable; (ii) property (i) is not satisfied when h is replaced by any $k=1,2,...,h-1$.*

On the basis of the results in Section 3 we may give some general information on the structure of the bifurcating sets in the cases $\nu=1$ and $\nu=2$.

A) $\nu = 1$. In this case y is a scalar variable and $(\Sigma)_{y\mu}$ is a scalar differential equation. For any fixed t and μ let π the restriction of $\Pi \equiv \Pi_{t\mu}$ to $\bar{\Phi}$. Moreover denote by $\varphi(i, y)$ the y coordinate of $\pi(i, y)$; hence $\pi(i, y) = (\varphi(i, y), 0)$. Clearly φ is a discrete dynamical system defined on $\mathbf{Z} \times \mathbf{R}$. Assume now that the null solution of $(\Sigma)_{y\mu}$ is asymptotically stable and analyze the structure of the bifurcating sets of Theorem 3.2. The set $m_\mu(t)$ in the proof of Theorem 3.2 is now the largest invariant compact set of φ disjoint from $y = 0$ and contained in a fixed interval $(-y^\star, y^\star) \subset \mathbf{R}$. Moreover, the mapping $t \to m_\mu(t)$ is ω–periodic. The two sets $m_\mu(t) \cup \mathbf{R}^+$ and $m_\mu(t) \cup \mathbf{R}^-$ are both compact and nonempty. For fixed μ, we set $a^+(t) = \min\{y \,:\, y \in m_\mu(t) \cup \mathbf{R}^+\}$ and $b^+(t) = \max\{y \,:\, m_\mu(t) \cup R^+\}$. Similarly we define $a^-(t)$ and $b^-(t)$. Clearly $a^+(t)$ and $a^+(t)$ (and similarly $a^-(t)$ and $b^-(t)$) are distinct for all $t \in \mathbf{R}$ or always coincident. It is easy to prove that $m_\mu(t) = [a^-(t), b^-(t)] \cap [a^+(t), b^+(t)]$ and that the end points of these two intervals are fixed points under $\varphi(1, y)$. This latter property will imply that if $\lambda(t)$ is anyone of the above points the set $\{(t, \lambda(t), 0),\, t \in \mathbf{R}\}$ is a periodic trajectory of $(\Sigma)_\mu$.

Thus we have the following theorem.

Theorem 4.1 *Suppose that all the assumptions in Theorem* 3.2 *are satisfied with* $\nu = 1$. *Then each bifurcating set* M_μ *of Theorem* 3.2 *is the union of two nonempty disjoint* s*–compact sets* M_μ^-, M_μ^+ *lying on* Φ. *Their sections* $M_\mu^-(t)$, $M_\mu^+(t)$ *are segments located in some regions, say* σ^-, σ^+, *of* $\bar{\Phi}$ *in which* $y \in (-y^\star, 0)$ *and* $y \in (0, y^\star)$ *respectively. Here* $y^\star$ *is a fixed (independent of* μ, t*) positive number. Each one of the sets* $\mathbf{R} \times \sigma^-$, $\mathbf{R} \times \sigma^+$ *contains at least one* ω*–periodic trajectory. If in* $\mathbf{R} \times \sigma^-$ *(resp.* $\mathbf{R} \times \sigma^+$*) there exists only one* ω*–periodic trajectory, then* M_μ^- *(resp.* M_μ^+*) reduces to this trajectory which will be asymptotically stable.*

We observe that if $(\Sigma)_\mu$ is autonomous for each $\mu > 0$, the sets M_μ are t–independent and $a^-(t)$, $b^-(t)$, $a^+(t)$, $b^+(t)$ defined above are independent of t. Hence we have $M_\mu(t) \equiv [a^-, b^-] \cap [a^+, b^+]$, that is the sets $M_\mu(t)$ are constituted by two fixed segments, each one being bounded by two equilibrium points. When $a^- = b^-$ (respectively $a^+ = b^+$) the corresponding segment reduces to an asymptotically stable equilibrium point.

We notice that Theorem 4.1 holds without any other special requirement on f and g. In order to have the uniqueness (in $\mathbf{R} \times \sigma^-$ and $\mathbf{R} \times \sigma^+$) of the ω–periodic trajectories we need to strengthen the hypotheses on the regularity of f and on the stability properties of the null solution of $(\Sigma)_{y\mu}$. We assume that $(\Sigma)_{y\mu}$ (which is now one dimensional) may be written as

$$(4.1)_{y\mu} \qquad \dot{y} = a(\mu)y + q(t, y, 0, \mu),$$

where $a(0) = 0$ and $a(\mu) > 0$ for $\mu > 0$ and q is a C^k function, $k \geq 3$, ω–periodic in t, vanishing together with its first derivatives at the origin of $\mathbf{R}$ for any $\mu \in (0, \mu^\star)$. The following theorem holds.

Theorem 4.2 *Suppose that system* $(4.1)_{y\mu}$ *satisfies one of the following conditions:* (1) *the null solution of* $(4.1)_{y0}$ *is* 3*–asymptotically stable;* (2) *the null solution of* $(4.1)_{y0}$ *is* h*–asymptotically stable with* $3 < h \leq r$ *and* $a'(0) > 0$, *Then each one of the two sets* M_μ^-, M_μ^+ *reduces to a* ω*–periodic asymptotically stable trajectory of* $(\Sigma)_\mu$.

Proof. In the hypothesis (1), Theorem 3.1 in [11] asserts that system $(\Sigma)_\mu$ cannot have more than two ω–periodic nontrivial bifurcating solutions. Hence by Theorem 4.1 it follows that the number of these solutions is exactly two. In the hypothesis (2), the result consists essentially in the statement of Theorem 7.4 in [1]. ■

B) $\nu = 2$. We first consider the case that f and g in Section 3 are t–independent. The following statement holds:

Proposition 4.1 *Suppose that f and g are time independent. Then we have: If $\nu = 2$ and there exists a neighborhood of the origin $y = 0$ on $\bar{\Phi}$ in which there are not equilibrium positions different from $y = 0$, then each bifurcating set M_μ is t–independent. Its constant section is homeomorphic to a closed annulus and is bounded by two periodic orbits c_1,c_2.*

Clearly the result expressed in this proposition follows immediately from the known Poincaré-Bendixon arguments relative to the limit cycles. Once again we notice that we do not need to assume any additional hypotheses on f, g. Information on the uniqueness of the periodic orbits, may be obtained under the stronger assumption that system $(\Sigma)_{y\mu}$ (which is now 2–dimensional) may be written as

$$(4.2)_{y\mu} \qquad \dot{y} = A(\mu)y + P(y, 0, \mu),$$

where $A(\mu)$ is a square matrix whose eigenvalues $\alpha(\mu) \pm i\beta(\mu)$ satisfy $\alpha(0) = 0$, $\alpha'(0) > 0$, $\beta(0) \neq 0$, and P is a C^k function, $k \geq 3$, which vanish together with its first derivatives at the origin of $\mathbf{R}^2$ for any $\mu \in (0, \mu^\star)$. In this case for each $\mu > 0$ small we have $c_1 = c_2$ if the solution $y = 0$ of $(4.2)_{y0}$ is h–asymptotically stable, $h \in [3, k]$, in the sense specified in Definition 4.1 (see [9]).

In the periodic case we restrict ourselves to examine the situation that $(\Sigma)_{y\mu}$ may be written as

$$(4.3)_{y\mu} \qquad \dot{y} = A(\mu)y + Q(t, y, 0, \mu),$$

where: (1) $A(\mu)$ is a square matrix whose eigenvalues $\alpha(\mu) \pm i\beta(\mu)$ satisfy $\alpha(0) = 0$, $\alpha'(0) > 0$ and $\beta(0) \neq 0$; (2) Q is a C^3 function ω–periodic in t, vanishing together with its first derivatives at the origin of $\mathbf{R}^2$ for any $\mu \in (0, \mu^\star)$ and $t \in [0, \omega]$.

Theorem 4.3 *Suppose that system $(4.3)_{y0}$ is autonomous and the null solution of $(4.3)_{y0}$ is 3–asymptotically stable. Then the sections $M_\mu(t)$ of the bifurcating sets M_μ are Jordan curves around the origin.*

Proof. Taking into account that $\alpha(\mu) > 0$ for $\mu > 0$, (iii) implies that $\alpha(\mu)$ may be written in the form $\alpha(\mu) = a\mu + \mu^2\sigma(\mu)$, $a > 0$ and $\sigma \in C^2$, $\sigma(0) = 0$. By a convenient change of variables, and using polar coordinates, system $(4.3)_{y\mu}$ may be written as

$$(4.4)_{y\mu} \qquad \begin{aligned} \dot{r} &= a\mu r - br^3 + r^4\gamma(\theta, r) + \mu r^2\delta(t, \theta, r, \mu) + \mu^2 r\eta(\mu), \\ \dot{\theta} &= \beta(\mu) + r^2\varphi(\theta, r) + \mu r\psi(t, \theta, r, \mu), \end{aligned}$$

where $b > 0$ is a constant and δ, ψ are ω–periodic in t. Underthe hypotheses of Theorem 4.3 the following statement holds: (1) there exists $\mu^\star > 0$, $r^\star > 0$ such that for every $\mu \in (0, \mu^\star)$ we can determine $r_1, r_2 \in (0, r^\star)$, $r_1 < r_2$, with the condition that the annulus $r_1 \leq r \leq r_2$ (say $\mathcal{A}_\mu$) is asymptotically stable, positively invariant, and its region of attraction contains $B^2(r^\star) - \{0\}$; (2) it may be assumed $r_1 = k(\mu^p - \mu^q)$, $r_2 = k(\mu^p + \mu^q)$,$p = 1/2$, $k = (a/b)^p$, $q \in (p, 2p)$. (For the proof see Lemma 3.1 in [1]).

Consider the change of variables

(4.5) $$\rho = (r - k\mu^p)(\mu^q)^{-1}.$$

The annulus $\mathcal{A}_\mu$ becomes the set $\{(\theta,\rho)\,:\,|r| \leq 1\}$ and system $(4.4)_{y\mu}$ assumes the form

$$(4.4)'_{y\mu} \qquad \begin{aligned} \dot{\rho} &= -2a\mu\rho + \mu^\lambda d(t,\theta,\rho,\mu), \\ \dot{\theta} &= \beta_1(\mu) + \mu^\lambda j(t,\theta,\rho,\mu). \end{aligned}$$

where $\lambda \in (1,2)$. By integrating over $[t_0, t_0+\omega]$ and expanding in μ near $\mu = 0$ we obtain the θ, ρ coordinates of $\pi_{t_0\mu}(1,(\theta_0,\rho_0))$. Here $\pi = \pi_{t_0\mu}$ is the dynamical system relative to $(4.4)'_{y\mu}$:

$$(4.6)_{y\mu} \qquad \begin{aligned} \rho &= [1 - 2a\omega\mu]\rho_0 + \mu^\lambda D(t_0,\theta_0,\rho_0,\mu), \\ \theta &= \theta_0 + \beta_1(\mu)\omega + \mu^\lambda J(t_0,\theta_0,\rho_0,\mu). \end{aligned}$$

Let E be the set of all functions $e \in C^3(\mathbf{R},\mathbf{R})$ such that (1) $e(\theta+2\pi) = e(\theta)$ and $|e(\theta)| \leq 1$ for all $\theta \in \mathbf{R}$, (2) $|e(\theta_1) - e(\theta_2)| \leq |\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in \mathbf{R}$. By using the same arguments as in [4] it is easy to prove that:

(a) for every $\theta \in [0,2\pi]$ there exists a unique $\tilde{\theta} \in [0,2\pi]$ such that

$$\theta = \tilde{\theta} + \beta_1(\mu)\omega + \mu^\lambda J(t_0,\tilde{\theta},\rho_0,\mu) \text{ ((mod}2\pi);$$

(b) the map $\mathcal{F} : E \to E$ such that

$$\mathcal{F}(e(\theta)) = [1 - 2a\omega\mu]e(\tilde{\theta}) + \mu^\lambda D(t_0,\tilde{\theta},e(\tilde{\theta}),\mu), \text{ for every } \theta \in \mathbf{R},$$

is a contraction.

Then the manifold Γ_μ corresponding to the unique fixed point of $\mathcal{F}$ is invariant under the discrete dynamical system π. By applying again the contraction principle we have that Γ_μ is an attracting set under π and its region of attraction contains $\mathcal{A}_\mu$. Hence the properties of $\mathcal{A}_\mu$ imply $\Gamma_\mu = m_\mu(t_0)$. The proof is clearly complete. ■

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