NORMAL HYPER K-ALGEBRAS

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Abstract. In this note first we define the notions of left (right) hyper stabilizers of types 1 and 2 of a nonempty subset of a hyper K-algebra. Also we define the notions of left (right) normal elements of types 1 and 2 of a hyper K-algebra and left (right) normal hyper K-algebras of types 1 and 2. Then we give many examples to show that these notions are different together. Finally we prove some theorems and obtain some related results . In particular we determine the relationships between the proper hyper K-ideals of a left (right) normal hyper K-algebra of types 1 and 2 and the positive implicative hyper K-ideals of types 2, 3, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25 and 26 of a hyper K-algebra of order 3, which satisfies the simple condition. Finally we define some closure operators induced by stabilizers.

1 Introduction

The hyper algebraic structure theory was introduced by F. Marty [6] in 1934. Imai and Iseki [5] in 1966 introduced the notion of BCK-algebra. Boorzooei, Jun and Zahedi et.al. [1,2,8] applied the hyper structure to BCK-algebra and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Yisheng Hung and Zhaomu Chen [4] in 1997 introduced normal BCK-algebra. Roodbari and Zahedi [7] introduced 27 different types of positive implicative hyper$K$-ideals also they introduced 9 different types of commutative hyper K-ideals . In this note we define 4 different types of left (right) stabilizers normal hyper K-algebras and some closure operators induced by stabilizers of a hyper K-algebra. Then we obtain some results as mentioned in the abstract.

2 Preliminaries

Definition 2.1.. [2] Let H be a nonempty set and " $\circ$ " be a *hyper operation* on H, that is " $\circ$ " is a function from $H \times H$ to $P^*(H) = P(H)\backslash\{\emptyset\}$. Then H is called a hyper K- algebra if it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y$, $y < x \Rightarrow x = y$,
(HK5) $0 < x$.
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by $A \circ B$ we mean

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the subset $\bigcup_{\substack{a\in A\\ b\in B}} a \circ b$ of H.

From now on $(H, \circ, 0)$ is a hyper K-algebra.

Theorem 2.2. [2] For all $x, y, z \in H$ and for all non-empty subsets A and B of H the following statements hold:
(i) $x \circ y < x$, (ii) $A \circ B < A$,
(iii) $A \circ A < A$, (iv) $0 \in x \circ (x \circ 0)$,
(v) $x < x \circ 0$, (vi) $A < A \circ 0$,
(vii) $A < A \circ B$, if $0 \in B$.

Lemma 2.3. [1] For all $x, y, z \in H$ the following statements hold:
(i) $x \circ y < z \Leftrightarrow x \circ z < y$, (ii) $x \in x \circ 0$.

Proof. (i) Let $x, y, z \in H$ be such that $x \circ y < z$. Then there exists $t \in x \circ y$ such that $t < z$. Thus $0 \in t \circ z \subseteq (x \circ y) \circ z = (x \circ z) \circ y$, and hence there exists $w \in x \circ z$ such that $0 \in w \circ y$, i.e., $w < y$. Therefore $x \circ z < y$. The proof of the converse is similar.

(ii) By Theorem 2.2.(i) we have $x \circ 0 < x$. So there exists $t \in x \circ 0$ such that $t < x$. Since $t \in x \circ 0$, then $x \circ 0 < t$ and so by (i) , $x \circ t < 0$. Thus there is $h \in x \circ t$ such that $h < 0$. By (HK5) and (HK4) we get that $h = 0$. So $0 \in x \circ t$, that is $x < t$. Since $x < t$ and $t < x$, then by (HK4), $x = t$. Therefore $x \in x \circ 0$.

Definition 2.4. [2] Let I be a nonempty subset of H and $0 \in I$. Then,
(i) I is called a weak hyper K-ideal of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,
(ii) I is called a hyper K-ideal of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.5. [8] Let $H = \{0, 1, 2\}$ be a hyper K-algebra. We say that H satisfies the simple condition if the conditions $1 \not< 2$ and $2 \not< 1$ hold.

Definition 2.6. [7] Let I be a nonempty subset of H such that $0 \in I$. Then I is called a positive implicative hyper K-ideal of
(i) type 1, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(iii) type 3, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$ imply that $x \circ z < I$,

(iv) type 4, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \subseteq I$,

(v) type 5, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \bigcap I \neq \emptyset$

(vi) type 6, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $x \circ z < I$,

(vii) type 7, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $y \circ z < I$ imply that $x \circ z < I$,

(viii) type 8, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $y \circ z < I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(ix) type 9 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $y \circ z < I$ imply that $(x \circ z) \subseteq I$,

(x) type 10, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xi) type 11, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \subseteq I$ imply that$(x \circ z) \subseteq I$,

(xii) type 12, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \subseteq I$ imply that $x \circ z < I$,

(xiii) type 13, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \subseteq I$,

(xiv) type 14, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xv) type 15, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that$x \circ z < I$,

(xvi) type 16, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $y \circ z < I$ imply that $x \circ z < I$,

(xvii) type 17 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $y \circ z < I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xviii) type 18, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $y \circ z < I$ imply that $(x \circ z) \subseteq I$,

(xix) type 19 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $x \circ z < I$,

(xx) type 20, if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \subseteq I$,

(xxi) type 21, if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xxii) type 22, if for all $x, y, z \in H$,$((x \circ y) \circ z) < I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \subseteq I$,

(xxiii) type 23 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \subseteq I$ imply that $x \circ z < I$,

(xxiv) type 24 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xxv) type 25 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $y \circ z < I$ imply that $x \circ z < I$,

(xxvi) type 26 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $y \circ z < I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xxvii) type 27 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $y \circ z < I$ imply that $(x \circ z) \subseteq I$.

For simplicity of notation we use "PIHKI " instead of " Positive Implicative Hyper K-ideal" .

Definition 2.7[7] Let I be a nonempty subset of H such that $0 \in I$.
Then I is called a commutative hyper K-ideal of

(i) type 1 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \subseteq I$,

(ii) type 2 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \bigcap I \neq$

$\emptyset$,

(iii) type 3 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) < I$,

(iv) type 4 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$, $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \subseteq I$,

(v) type 5 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \bigcap I \neq \emptyset$,

(vi) type 6 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) < I$,

(vii) type 7 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$, $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \subseteq I$,

(viii) type 8 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \bigcap I \neq \emptyset$,

(ix) type 9 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) < I$.

For simplicity of notation we use "CHKI " instead of "Commutative hyper K-ideal ".

Definition 2.8. [1] Let H be a hyper K-algebra. An element $a \in H$ is called to be a left(resp. right) scalar if $|a \circ x| = 1$(resp. $|x \circ a| = 1$) for all $x \in H$.

Definition 2.9. [1] Let S be a subset of a hyper K-algebra H . The smallest hyper K-ideal containing S is called the hyper K-ideal generated by S and is denoted by $< S >$.

Definition 2.10 [3] If we are given a set A, a mapping $\varphi : P(A) \rightarrow P(A)$ is called a closure operator on A if for all $X, Y \subseteq A$ it satisfies:

(i) $X \subseteq \varphi(X)$,

(ii) $\varphi^2(X) = \varphi(X)$,

(iii) $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$.

3 Normal hyper K-algebras of types 1(2)

Definition 3.1. Let H be a hyper K-algebra and S be a nonempty subset of H. Then the sets
${}_{l1}S = \{x \in H | a < (a \circ x), \forall\ a \in S\}$, ${}_{l2}S = \{x \in H | a \in (a \circ x), \forall\ a \in S\}$, $S_{r1} = \{x \in H | x < (x \circ a), \forall\ a \in S\}$ and $S_{r2} = \{x \in H | x \in (x \circ a), \forall\ a \in S\}$ are called left hyper stabilizer of type 1 of S, left hyper stabilizer of type 2 of S, right hyper stabilizer of type 1 of S and right hyper stabilizer of type 2 of S, respectively.

Also we let ${}_{l1}S_{r1} = {}_{l1}S \bigcap S_{r1}$ and ${}_{l2}S_{r2} = {}_{l2}S \bigcap S_{r2}$ and they are called hyper stabilizers of types 1 and 2, respectively.

For simplicity of notation , we use ${}_{l1}s({}_{l2}s)$ and $s_{r1}(s_{r2})$ instead of ${}_{l1}\{s\}({}_{l2}\{s\})$ and $\{s\}_{r1}(\{s\}_{r2})$.

Example 3.2. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then ${}_{l1}1 = 1_{r1} = 1_{r2} = H$ and ${}_{l2}1 = \{0,1\}$.

Theorem 3.3. Let H be a hyper K-algebra and C, D ,S and S_j be nonempty subsets of H. Then

(a) If $C \subseteq D$, then ${}_{l1}D \subseteq_{l1} C$, ${}_{l2}D \subseteq_{l2} C$, $D_{r1} \subseteq C_{r1}$ and $D_{r2} \subseteq C_{r2}$,

(b) $D \subseteq_{l1} (D_{r1}), D \subseteq ({}_{l1}D)_{r1}$, $D \subseteq_{l2} (D_{r2})$ and $D \subseteq ({}_{l2}D)_{r2}$,

(c) ${}_{l1}D ={}_{l1}(({}_{l1}D)_{r1})$, ${}_{l2}D ={}_{l2}(({}_{l2}D)_{r2}), D_{r1} = ({}_{l1}(D_{r1}))_{r1}$ and $D_{r2} = ({}_{l2}(D_{r2}))_{r2}$,

(d) ${}_{li}(C \bigcup D) ={}_{li} C \bigcap_{li} D$ and $(C \bigcup D)_{ri} = C_{ri} \bigcap D_{ri}$, for i=1, 2

$(\acute{a})$ $(< S >)_{ri} \subseteq S_{ri}$ and ${}_{li}(< S >) \subseteq_{li} S$, for i=1, 2 and

$(\grave{d})$. $(\bigcup\limits_{j \in J} S_j)_{ri} = \bigcap\limits_{j \in J} (S_j)_{ri}$ and ${}_{li}(\bigcup\limits_{j \in J} (S_j)) = \bigcap\limits_{j \in J} {}_{li}(S_j)$, for any i=1, 2.

Proof. (a). Suppose $x \in_{l1} D$, then $d < d \circ x$ for all $d \in D$. Since $C \subseteq D$, we have $x \in_{l1} C$, therefore ${}_{l1}D \subseteq_{l1} C$. Similarly , ${}_{l2}D \subseteq_{l2} C$, $D_{r1} \subseteq C_{r1}$ and $D_{r2} \subseteq C_{r2}$,

(b). Assume that $d \in D$, then $x \in x \circ d$, for all $x \in D_{r2}$. Hence $x \in_{l2} (D_{r2})$. Therefore $D \subseteq_{l2} (D_{r2})$. By a similar argument $D \subseteq_{l1} (D_{r1})$ and $D \subseteq ({}_{l2}D)_{r2}$,

(c). By (b) we get that ${}_{l1}D \subseteq_{l1} (({}_{l1}D)_{r1})$ and $D \subseteq ({}_{l1}D)_{r1}$, so by (a) ${}_{l1}(({}_{l1}D)_{r1})) \subseteq_{l1} D$. Hence ${}_{l1}(({}_{l1}D)_{r1})) =_{l1} D$. Similarly ${}_{l2}(({}_{l2}D)_{r2})) =_{l2} D$. Also by (b) we get that $D_{r1} \subseteq ({}_{l1}(D_{r1}))_{r1}$ and $D \subseteq_{l1} (D_{r1})$, so by (a) $({}_{l1}(D_{r1}))_{r1} \subseteq D_{r1}$. Hence $({}_{l1}(D_{r1}))_{r1} = D_{r1}$. Similarly $({}_{l2}(D_{r2}))_{r2} = D_{r2}$.

(d). We have $(C \bigcup D)_{r2} \subseteq C_{r2}$ and also D_{r2}. So $(C \bigcup D)_{r2} \subseteq C_{r2} \bigcap D_{r2}$. Assume that $x \in C_{r2} \bigcap D_{r2}$, then $x \in x \circ d$ and $x \in x \circ c$ for all $d \in D$ and $c \in C$. Therefore $x \in (C \bigcup D)_{r2}$. So $(C \bigcup D)_{r2} = C_{r2} \bigcap D_{r2}$. By a similar argument $(C \bigcup D)_{r1} = C_{r1} \bigcap D_{r1}$, ${}_{l1}(C \bigcup D) ={}_{l1} C \bigcap_{l1} D$ and ${}_{l2}(C \bigcup D) ={}_{l2} C \bigcap_{l2} D$.

$(\grave{a})$. Since $S \subseteq < S >$, by (a) we get that $(\grave{a})$.

It is similar to part (d).

Remark 3.4.(i). Note that it may be that S is a hyper K-ideal, while ${}_{l1}S$ and S_{r1} are not hyper K-ideals. To see this consider the following hyper K-algebra structure on $H =$

$\{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0, 2\}$
1	$\{1\}$	$\{0\}$	$\{0, 2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

Assume that $S = \{0, 1\}$. Then S is a hyper K-ideal of H, while ${}_{l1}S = S_{r1} = \{0, 2\}$ are not hyper K-ideals .

(ii). Note that it may be that S is a hyper K-ideal, while ${}_{l2}S$ and S_{r2} are not hyper K-ideals. To see this , consider the following hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 2\}$	$\{0, 1, 2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1, 2\}$

Assume that $S = \{0, 1\}$. Then S is a hyper K-ideal of H, while ${}_{l2}S = S_{r2} = \{0, 2\}$ are not hyper K-ideals.

Example 3.5. The following table shows that a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$

Suppose $S = \{1\}$, then $1_{r1} = 1_{r2} = \{0, 1\}$ and $(< S >)_{r1} = (< S >)_{r2} = \{0\}$. Also $(< S >)_{r1}$ is a proper subset of S_{r1} , $(< S >)_{r2}$ is a proper subset of S_{r2} , ${}_{l1}S =_{l2} S = \{0, 1\}$ and ${}_{l1}(< S >) =_{l2} (< S >) = \{0\}$. Therefore ${}_{l1}(< S >)$ is a proper subset of ${}_{l1}S$ and ${}_{l2}(< S >)$ is a proper subset of $({}_{l2})S$.

These means that in $\grave{a}$ it may be we have proper inclusion.

Theorem 3.6. Let H be a hyper K-algebra, S be a nonempty subset of H and $a \circ a = \{0\}$, for all $a \in S$. Then $S \bigcap_{li} S = \emptyset$ or $S \bigcap_{li} S = \{0\}(S \bigcap S_{ri} = \emptyset$ or $S \bigcap S_{ri} = \{0\})$ for any i=1 or i=2.

Proof. If $S \bigcap_{l1} S \neq \emptyset$, then $x \in S \bigcap_{l1} S$. So $x < x \circ x = \{0\}$. Hence $x = 0$. Therefore $S \bigcap_{l1} S = \{0\}$. The proof of the other case is similar.

Example 3.7. The following table shows that a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 2\}$	$\{0, 1, 2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1, 2\}$

Suppose that $S = \{1,2\}$. We see that ${}_{l1}S = S_{r1} = \{0,1,2\}$, $S\bigcap_{l1} S = S\bigcap S_{r1} = \{1,2\}$,${}_{l2}S = S_{r2} = \{0,2\}$ and $S\bigcap_{l2} S = S\bigcap S_{r2} = \{2\}$. Also $2\circ 2 \neq \{0\}$. Hence the condition "$a\circ a = \{0\}$ for all $a \in S$" in Theorem 3. 6 is necessary.

Theorem 3.8. Let H be a hyper K-algebra and let A and B be nonempty subsets of H such that $a\circ a = \{0\}$, for all $a \in A$. If A and B satisfy any one of $A \subseteq_{l1} B, A \subseteq_{l2} B, A \subseteq B_{r1}$ and $A \subseteq B_{r2}$. Then $A\bigcap B = \{0\}$.

Proof. Let $a \in A\bigcap B$. Then by hypotheses $a < a\circ a = \{0\}$ or $a \in a\circ a = \{0\}$, so $a = 0$.

Example 3.9. Consider the hyper K-algebra H given in Example 3.7 . Suppose $A = \{0,2\}$ and $B = H$. We get that $A \subseteq_{l1} B, A \subseteq_{l2} B, A \subseteq B_{r1}$ and $A \subseteq B_{r2}$. We see that $A\bigcap B = \{0,2\}$ and $2\circ 2 \neq \{0\}$. Thus the condition "$a\circ a = \{0\}$ for all $a \in A$" in Theorem 3.8 is necessary.

Theorem 3.10. Let H be a hyper K-algebra. Then for any $a \in H$ the following conditions are equivalent :

(i). $a_{ri} \subseteq_{li} a$,

(ii). $a_{ri} =_{li} a$,

(iii). ${}_{li}a \subseteq a_{ri}$

for any i=1 or 2, .

Proof. We show that $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$.

$(ii) \Rightarrow (i)$. It is clear.

$(i) \Rightarrow (ii)$. For any $x \in_{l1} a({}_{l2}a)$, we have $a < a\circ x (a \in a\circ x)$. Thus $a \in x_{r1}(x_{r2})$. By (i) we get that $a \in_{l1} x({}_{l2}x)$. Hence $x \in a_{r1}(a_{r2})$. Therefore $a_{ri} =_{li} a$ for any $a \in H$ and for any i=1 or 2 .

$(ii) \Rightarrow (iii)$. It is clear.

$(iii) \Rightarrow (ii)$. For any $x \in a_{r1}(a_{r2})$, we have $x < x\circ a (x \in x\circ a)$. So $a \in_{l1} x({}_{l2}x)$. By (iii) $a \in x_{r1}(x_{r2})$. Therefore $x \in_{l1} a({}_{l2}a)$. Hence $a_{ri} =_{li} a$ for any i=1 or 2 and for any $a \in H$.

Theorem 3.11. Let H be a hyper K-algebra such that $x_{ri} =_{li} x$, for any i=1 or 2 and for any $x \in H$. Then $x < x\circ y (x \in x\circ y)$ implies that $y < y\circ x (y \in y\circ x)$, for $x, y \in H$.

Proof. If $x \in x\circ y$, then $x \in y_{r2} =_{l2} y$ and so $y \in y\circ x$. Similarly $x < x\circ y$ implies $y < y\circ x$.

Definition 3.12. Let H be a hyper K-algebra. If $x\circ(x\circ y) = y\circ(y\circ x)$ for all $x, y \in H$, then H is called a commutative hyper K-algebra.

Example 3.13. The following table shows a commutative hyper K-algebra structure on

$H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Theorem 3.14. Let H be a commutative hyper K-algebra and S be a nonempty subset of H. Then $S_{r1} =_{l1} S$.

Proof. Let $x \in S_{r1}$. Then $x < x \circ y$ for all $y \in S$. Hence $0 \in x \circ (x \circ y) = y \circ (y \circ x)$. Therefore $y < y \circ x$. Thus $S_{r1} \subseteq_{l1} S$. By similarly $_{l1}S \subseteq S_{r1}$. Also $S_{r1} {=}_{l1} S$.

Example 3.15. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1,2\}$
2	$\{2\}$	$\{1\}$	$\{0,1,2\}$

We have $0 \circ (0 \circ 1) \neq 1 \circ (1 \circ 0)$, also $2_{r1} \neq_{l1} 2$. So "the commutativity " in Theorem 3.14 is necessary.

Definition 3.16. Let H be a hyper K-algebra. An element $a \in H$ is called a left (right) hyper normal element of type 1(2) if $_{li}a(a_{ri})$ is a hyper K-ideal of H, for i=1 , 2 .

Definition 3.17. A hyper K-algebra H is called left (right) hyper normal of type 1(2) if $_{li}a(a_{ri})$ of any element $a \in H$ is a hyper K-ideal of H for i=1 or 2 . Also if H is both left and right hyper normal of type 1 (2), then H is called hyper normal K-algebra of type 1 (2).

Example 3.18.(i) Let$(H, *, 0)$ be a normal BCK-algebra [4] and define the hyper operation "o" on H by $x \circ y = \{x * y\}$. Then $(H, *, 0)$ is a normal hyper K-algebra of types 1 and 2.

(ii) Consider Remark 3.4 (i), we have $_{l1}1 = 1_{r1} = 1_{r2} = \{0, 2\}$,$_{l2}1 = \{0\}$, $_{l1}2 = 2_{r1} =_{l2} 2 = \{0, 1\}$ and $2_{r2} = \{0\}$, then 2 is a left (right) hyper normal element of types 1 and 2 , but 1 is not a left (right) hyper normal element of type 1 . In Remark 3.4 (ii) , we get that $_{l2}1 = 1_{r1} = 1_{r2} = \{0, 2\}$,$_{l1}1 = \{0, 1, 2\}$ and $_{l1}2 = 2_{r1} =_{l2} 2 = 2_{r2} = \{0, 1, 2\}$, then 2 is a left (right) hyper normal element of types 1 and 2 , but 1 is not a right hyper normal element of types 1, 2 , also 1 is not a left hyper normal element of type 2 and 1 is a left hyper normal element of type 1 .

(iii) The following table shows a left(right) hyper normal K-algebra of type 1(2) on $H = \{0, 1, 2\}$. Because $_{l1}1 =_{l2} 1 = 1_{r1} = 1_{r2} = \{0, 1, 2\}$,$_{l1}2 =_{l2} 2 = 2_{r1} = 2_{r2} = \{0, 1\}$ and $_{l1}0 =_{l2} 0 = 0_{r1} = 0_{r2} = \{0, 1, 2\}$ are hyper K-ideals.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

Theorem 3.19. Let H be a hyper K-algebra and $a \in H$. If a is a left scalar element of H , then ${}_{l2}a$ is a weak hyper K-ideal.

Proof. Since a is a left scalar element of H , then ${}_{l2}a = \{x \in$H$|a \circ x = \{a\}\}$. We show that ${}_{l2}a$ is a weak hyper K-ideal. By Lemma 2.3 (ii)we have $0 \in_{l2} a$. Now let $x \circ y \subseteq_{l2} a$ and $y \in_{l2} a$. Then $\{a\} = a \circ y$. By Definition 3.1 and $x \circ y \subseteq_{l2} a$ we get that $\{a\} = a \circ (x \circ y)$. Also $a \circ (x \circ y) = (a \circ y) \circ (x \circ y) < a \circ x$, then $a < a \circ x$. Since $a \circ x < a$, we have $\{a\} = a \circ x$. Hence $x \in_{l2} a$ .Therefore ${}_{l2}a$ is a weak hyper K-ideal.

Remark 3.20. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

. We see that ${}_{l2}2 = \{0, 2\}$ is not a weak hyper K-ideal. Also 2 is not a left scalar of H. Therefore the condition "left scalar element in Theorem 3.19" is necessary.

Theorem 3.21 . Let H be a commutative hyper K-algebra . If H is a left(right) normal hyper K-algebra of type 1, then H is a normal hyper K-algebra of type 1.

Proof. It is sufficient to show that H is a right (left) normal of type 1 $\Leftrightarrow$ H is a left(right) normal of type 1. By Theorem 3.14 this can be easily proved.

Note that the following examples show that the notions normal hyper K-algebra of types 1 and 2 are not equivalent.

Example 3.22 .(i) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$

Then H is a normal hyper K-algebra of type 2, but it is not a normal hyper K-algebra of type 1. Because $\{x \in H|x < x \circ 1\} = \{0, 1\}$ is not a hyper K-ideal of H.

(ii) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

H is a normal hyper K-algebra of type 1, but it is not a normal hyper K-algebra of type 2. Because $\{x \in H | x \in x \circ 1\} = \{0, 1\}$ is not a hyper K-ideal of H.

For simplicity of notation we use "RNHKA" (LNHKA) instead of right(left) normal hyper K-algebra and also we use "NHKA" instead of normal hyper K-algebra.

Example 3.23. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2, 3\}$.

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,2,3\}$
1	$\{1\}$	$\{0\}$	$\{1,2,3\}$	$\{1,2,3\}$
2	$\{2\}$	$\{2\}$	$\{0,2,3\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

In this example we see that $(x < x \circ a)$ implies that $(x \in x \circ a)$ for all $a, x \in H$.

Theorem 3.24. Let H be a hyper K-algebra and $(x < x \circ a)$ implies that $(x \in x \circ a)$ for all $a, x \in H$. Then H is a RNHKA(LNHKA) of type 1 if and only if H is a LNHKA(RNHKA) of type 2.

Proof. Let $a \in H$. Then by hypothesis $a_{r1} = a_{r2}$ and ${}_{l1}a =_{l2} a$. Then the proof follows from Definition 3.17.

Remark 3.25.(i) In Example 3.22 (i) H is a RNHKA of type 2, but it is not a LNHKA of type 1. Also in Example 3.22 (ii) H is a RNHKA of type 1, but it is not a LNHKA of type 2.
(ii) Every RNHKA (LNHKA) of type 1 is a RNHKA (LNHKA) of type 2. Because $x \in x \circ a \Rightarrow x < x \circ a$ for all $x, a \in H$.

Theorem 3.26. Let H be a hyper K-algebra.Then H is a RNHKA(LNHKA) of type 1(2) if and only if $S_{ri}({}_{li}S)$ i=1, 2 is a hyper K-ideal of H for any nonempty subset S of H.

Proof.$\Leftarrow$. It is clear.

$\Rightarrow$. We have ${}_{li}S = \bigcap\limits_{a \in S} {}_{li}a$ and $S_{ri} = \bigcap\limits_{a \in S} a_{ri}$. Thus H is a RNHKA(LNHKA) of type 1(2) if and only if $a_{ri}({}_{li}a)$ i=1, 2 is a hyper K-ideal of H for all $a \in S$.

Theorem 3.27. Let H be a hyper K-algebra. H is a RNHKA(LNHKA) of type 1(2) if and only if every subhyper K-algebra N of H is a RNHKA(LNHKA) of type1(2).

Proof. $\Rightarrow$). Assume $a \in N$ and a_{r1} ,$a_{\acute{r}1}$ be the right stabilizers of it with respect to H and N respectively. It is clear that $a_{\acute{r}1} = N \bigcap a_{r1}$. We show that $a_{\acute{r}1}$ is a hyper K-ideal of N. If $x, y \in N$, $x \circ y < a_{\acute{r}1}$ and $y \in a_{\acute{r}1}$, then $x \circ y < a_{r1}$ and $y \in a_{r1}$. Since a_{r1} is a hyper K-ideal of H, so $x \in a_{\acute{r}1}$. Hence N is a RNHKA of type 1. The proof of the other part is the same as a bove.

Remark 3.28. [1] Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be hyper K-algebras. Then $(H_1 \bigotimes H_2, \circ)$ is a hyper K-algebra , where $(x_1, x_2) \circ (y_1, y_2) = (x_1 \circ_1 y_1, x_2 \circ_2 y_2) = \{(a, b)|a \in x_1 \circ_1 y_1, b \in x_2 \circ_1 y_2\}$.

Lemma 3.29.[1] Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be hyper K-algebras. Then

(i) If I_1 and I_2 are hyper K-ideals of H_1 and H_2 , respectively , then $(I_1 \bigotimes I_2, \circ)$ is a hyper K-ideal of $(H_1 \bigotimes H_2, \circ)$,

(ii) If I is a hyper K-ideal of $(H_1 \bigotimes H_2, \circ)$, then there are unique hyper K-ideals I_1 and I_2 of H_1 and H_2, respectively such that $I = I_1 \bigotimes I_2$.

Theorem 3.30. Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be hyper K-algebras. Then H_1 and H_2 are RNHKA(LNHKA) of type1(2) if and only if $H_1 \bigotimes H_2$ is a RNHKA(LNHKA) of type 1(2).

Proof. $\Rightarrow$. Let $(a_1, a_2) \in H_1 \bigotimes H_2$. Then $(a_1, a_2)_{r1} = \{(x_1, x_2) \in H_1 \bigotimes H_2 | (x_1, x_2) < (x_1, x_2) \circ (a_1, a_2)\} = \{(x_1, x_2) \in H_1 \bigotimes H_2 | x_1 < x_1 \circ_1 a_1, x_2 < x_2 \circ_2 a_2\} = (a_1)_{r1} \bigotimes (a_2)_{r2}$. By Lemma 3.29 $(a_1, a_2)_{r1}$ is a hyper K-ideal of $H_1 \bigotimes H_2$. Therefore $H_1 \bigotimes H_2$ is a RNHKA of type 1.

$\Leftarrow$. Let $a \in H_1$. Then $(a, 0) \in H_1 \bigotimes H_2$. We have $(a, 0)_{r1} = a_{r1} \bigotimes 0_{r1}$. Since $(a, 0)_{r1}$ is a hyper K-ideal of $H_1 \bigotimes 0_2$, so a_{r1} is a hyper K-ideal of H_1. Therefore by Lemma 3.29 H_1 is a RNHKA of type 1. The proof of the other parts are the same as above.

Theorem 3.31. Let H be a NHKA of type 1(2) of order 3 , which satisfies the simple condition and let I be a proper hyper K-ideal of H. Then I is PIHKI of types 2, 3, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25 and 26.

Proof. We prove theorem for type 1, the proof of the other cases is similar to it, by same suitable modification.

Since H is simple , then by [8]

(i) $1 \circ 2 \neq \{2\}$ and $2 \circ 1 \neq \{1\}$,

(ii) $1 \circ 0 = \{1\}$ and $2 \circ 0 = \{2\}$.

Since H is a NHKA of type 1 , then $\{x|x < x \circ 1\}$, $\{x|x < x \circ 2\}$, $\{x|1 < 1 \circ x\}$ and $\{x|2 < 2 \circ x\}$ are hyper K-ideals . Also since H is simple , then $\{x \in H|x < x \circ 1\} \supseteq \{0, 2\}, \{x \in H|x < x \circ 2\} \supseteq \{0, 1\}, \{x \in H|1 < 1 \circ x\} \supseteq \{0, 2\}$ and $\{x \in H|2 < 2 \circ x\} \supseteq \{0, 1\}$. Therefore H has only the following possible cases :

case 1: $1 \circ 2 = \{1\}$ and $2 \circ 1 = \{2\}$, case 2: $2 \circ 1 = \{2\}, 1 \circ 2 = \{1, 2\}$ and $1 < 1 \circ 1$,

case 3: $1 \circ 2 = \{1\}, 2 \circ 1 = \{1, 2\}$ and $2 < 2 \circ 2$,

case 4: $1 \circ 2 = 2 \circ 1 = \{1, 2\}, 2 < 2 \circ 2$ and $1 < 1 \circ 1$,

Without loss of generality assume that $I = \{0, 1\}$. Since H is a NHKA of type 1 and I is a hyper K-ideal, then H has only the following possible cases:

case 1: $1 \circ 2 = \{1\}$ and $2 \circ 1 = \{2\}$,

case 2: $2 \circ 1 = \{2\}$, $1 \circ 2 = \{1, 2\}$ and $1 < 1 \circ 1$.

Note that in cases 3 and 4 , $I = \{0, 1\}$ is not a hyper K-ideal .We consider case 1:

On the contrary , let I does not be a PIHKI of type 2. Then there exist $x, y, z \in H$ such that $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$, but $(x \circ z) \bigcap I = \emptyset$. Hence $x \circ z = 2$.

Since H is simple , then by [8] we have the following subcases:

(i) x=2 and z=0,

(ii) x=2 and z=1.

If x=2 and z=0, we show that at least one of the hypothesis $((x \circ y) \circ z) \subseteq I$ or $(y \circ z) \subseteq I$ does not hold.

(a) If y=0, then $2 = ((2 \circ 0) \circ 0) \nsubseteq I$,

(b) if y=1, then $2 = ((2 \circ 1) \circ 0) \nsubseteq I$,

(c) if y=2, then $2 = (2 \circ 0) \nsubseteq I$.

Therefore in this sub case I is a PIHKI of type 2.

We now consider sub case (ii), that is let x=2 and z=1 .

We show that at least one of the hypothesis $((x \circ y) \circ z) \subseteq I$ or $(y \circ z) \subseteq I$ does not hold.

(a) If y=0, then $2 = ((2 \circ 0) \circ 1) \nsubseteq I$,

(b) if y=1, then $2 = ((2 \circ 1) \circ 1) \nsubseteq I$,

(c) if y=2, then $2 = (2 \circ 1) \nsubseteq I$.

Therefore in this sub case I is a PIHKI of type 2.
Thus similarly , by imposing the suitable changes we can prove theorem for other types.

Remark 3.32. The converse of Theorem 3.31 is not true. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

We see that H is simple, moreover $I = \{0, 1\}$ is a PIHKI of types 2, 3, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25 and 26, while H is not a NHKA of type 1(2). Because $1_{ri} = \{0, 2\}$ is not a hyper K-ideal of H for i=1, 2.

Theorem 3.33. Let H be a NHKA of type 1(2) of order 3 , which satisfies the simple condition and let I be a proper hyper K-ideal of H. Then I is a CHKI of types 2, 3, 5, 6, 8 and 9.

Proof. We prove theorem for type 3, the proof of the other cases are similar to it, by same suitable modification.
Without loss of generality assume that $I = \{0, 1\}$. Since H is NHKA of type 1(2) and I is a hyper K-ideal, then by considering the proof of Theorem 3.31 H has only the following possible cases:

case 1: $1 \circ 2 = \{1\}$ and $2 \circ 1 = \{2\}$,

case 2: $2 \circ 1 = \{2\}, 1 \circ 2 = \{1, 2\}$ and $1 < 1 \circ 1$.

On the contrary , let I does not be a CHKI of type 3. Then there exist $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \not< I$. So $(x \circ (y \circ (y \circ x)) = 2$. By simplicity H [8] we have the following subcases:

(i) x=2 and $y \circ (y \circ x) = 0$,

(ii) x=2 and $y \circ (y \circ x) = 2$,

If x=2 and $y \circ (y \circ x) = 0$, then $y \circ (y \circ 2) = 0$.

We show that the hypothesis $((x \circ y) \circ z) \bigcap I \neq \emptyset$ in the following cases does not hold.

(a) If y=0 and z=0, then $((2 \circ 0) \circ 0) \bigcap I = \emptyset$,

(b) if y=0 and z=1, then $((2 \circ 0) \circ 1) \bigcap I = \emptyset$,

(a) if y=1 and z=0, then $((2 \circ 1) \circ 0) \bigcap I = \emptyset$,

(d) if y=1 and z=1, then $((2 \circ 1) \circ 1) \bigcap I = \emptyset$,

Also

(e)if y=2 and z=0 or z=1, then since $0 \in 2 \circ 2$ and $2 \in 2 \circ 0$, we get that $0 \in (2 \circ (2 \circ (2 \circ 2))$. Hence $(2 \circ (2 \circ (2 \circ 2)) < I$, then in this case Theorem 3.33 is true. By a similar argument as above we obtain a contradition. Therefore I is a CHKI of type 3. Thus similarly , by imposing the suitable changes we can prove theorem for the other types.

Remark 3.34. The converse of Theorem 3.33 is not true. Consider hyper K-algebra H in Remark 3.32. We see that $I = \{0, 1\}$ is a CHKI of types 2, 3, 5, 6, 8 and 9, while H is not NHKA of type 1(2). Because $1_{ri} = \{0, 2\}$ is not a hyper K-ideal of H for i=1, 2.

Theorem 3.35. Let H be a hyper K-algebra. Then the functions $\varphi_i : P(H) \to P(H)$ such that $\varphi_i(D) =_{li} (D_{ri})$ i=1 , 2 are closure operators.

Proof. (i) By Theorem 3.3(b), we have $D \subseteq_{li} (D_{ri})$ for all $D \subseteq H$,

(ii) By Theorem 3.3 (c) , we have $\varphi_i(D) =_{li} (D_{ri}) =_{li} ((_{li}(D_{ri}))_{ri}) = \varphi_i(\varphi_i(D))$,

(iii) Let $A \subseteq B$. Then by Theorem 3.3(a), we get that $B_{ri} \subseteq A_{ri}$ and $_{li}(A_{ri}) \subseteq_{li} (Bri), i = 1, 2$. Therefore $\varphi_i(A) \subseteq \varphi_i(B)$. Hence by Definition 2.10 φ_i is a closure operator.

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