MEAN ERGODIC THEOREMS FOR A SEQUENCE OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. Let C be a closed convex subset of a Hilbert space and $\{T_n\}$ a sequence of nonexpansive self-mappings of C. Then we consider the following iterative sequence $\{z_n\}$: $x_1 = x \in C$, $x_{n+1} = T_n x_n$, and $z_n = 1/n \sum_{k=1}^{n} x_k$ for $n \in \mathbb{N}$. In this paper, we obtain a weak convergence theorem for such a sequence $\{z_n\}$. Using our result, we get a nonlinear ergodic theorem which is a generalization of Baillon [2]. Further we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Then a mapping $T\colon C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. In 1975, Baillon [2] proved the first nonlinear ergodic theorem: Define

$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for every $n \in \mathbb{N}$ and $x \in C$ and suppose that $F(T)$ is nonempty. Then the sequence $\{z_n\}$ converges weakly to some element of $F(T)$. It is known that many results concerning the mean ergodic theorem for a nonlinear mapping have been obtained, for example, [2], [11], [12], [5], [7], [8], [19]; see also [6], [3], [18], [1], [10], [9], and the references therein. Reich [13] also proved the following weak convergence theorem; see [16] for a simple proof.

Theorem 1.1 (Reich [13]). *Let C be a nonempty closed convex subset of a real Hilbert space H and T a nonexpansive self-mapping of C. Suppose that $F(T)$ is nonempty. Let $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1)$ satisfies $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to $z \in F(T)$.

Reich [13] really proved such a theorem in a uniformly convex Banach space whose norm is Fréchet differentiable. Motivated by Baillon [2] and Reich [13], we consider the following

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iterative sequence $\{z_n\}$: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = T_n x_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases} \tag{1.1}$$

for $n \in \mathbb{N}$, where $\{T_n\}$ is a sequence of nonexpansive self-mappings of C.

In this paper, we establish a weak convergence theorem for such a sequence $\{z_n\}$ generated by (1.1). Using our result, we obtain a nonlinear ergodic theorem for a nonexpansive mapping which is a generalization of Baillon [2]. Further we apply our theorem to the problem of finding a common fixed point of a countable family of nonexpansive mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, H denotes a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H and $x \in H$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$.

Let C be a nonempty closed convex subset of H and T a mapping of C into H. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. It is known that $F(T)$ is closed and convex if T is nonexpansive. For each $x \in H$, there exists a unique point $z \in C$ such that

$$\|x - z\| = \min\{\|x - y\| : y \in C\}.$$

Such a point z is denoted by Px and P is called the metric projection of H onto C. It is known that

$$\langle x - Px, Px - y\rangle \ge 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$; see [15] for more details.

To prove our results, we need the following lemmas.

Lemma 2.1 (Takahashi-Toyoda [17])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H, P the metric projection of H onto C, and $\{x_n\}$ a sequence in H. If $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $u \in C$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

Lemma 2.2 (Bruck [4])**.** *Let C be a nonempty closed convex subset of a real Hilbert space E. Let $\{S_k\}$ be a sequence of nonexpansive mappings of C into H and $\{\beta_k\}$ a sequence of positive real numbers such that $\sum_{k=1}^{\infty}\beta_k = 1$. If $\bigcap_{k=1}^{\infty} F(S_k)$ is nonempty, then the mapping $T = \sum_{k=1}^{\infty}\beta_k S_k$ is well-defined and $F(T) = \bigcap_{k=1}^{\infty} F(S_k)$.*

Bruck [4] showed this assertion for a strictly convex Banach space.

3. Mean ergodic theorems

Using the technique in [15, p.59], we obtain the following:

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H, $\{z_n\}$ a sequence in H defined by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$, $\{\alpha_n\}$ a sequence of real numbers such that $\alpha_n \to 0$, and T a mapping of C into H. Suppose that there exists $z \in C$ such that

$$\alpha_n \le \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$ and a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to z. Then z is a fixed point of T.

Proof. For all $k \in \mathbb{N}$ we have

$$\begin{aligned}\alpha_k &\leq \|x_k - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz + Tz - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz\|^2 - \|x_{k+1} - Tz\|^2 + 2\langle x_k - Tz, Tz - z\rangle + \|Tz - z\|^2 .\end{aligned}$$

Summing these inequalities from $k = 1$ to n and dividing by n, we get

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\alpha_k &\leq \frac{1}{n}(\|x_1 - Tz\|^2 - \|x_{n+1} - Tz\|^2) + 2\langle z_n - Tz, Tz - z\rangle + \|Tz - z\|^2 \\ &\leq \frac{1}{n}\|x_1 - Tz\|^2 + 2\langle z_n - Tz, Tz - z\rangle + \|Tz - z\|^2 .\end{aligned}$$

Further, replacing n by n_i, we obtain

$$\frac{1}{n_i}\sum_{k=1}^{n_i}\alpha_k \leq \frac{1}{n_i}\|x_1 - Tz\|^2 + 2\langle z_{n_i} - Tz, Tz - z\rangle + \|Tz - z\|^2 .$$

Since $z_{n_i} \rightharpoonup z$ and $1/n_i \sum_{k=1}^{n_i}\alpha_k \to 0$, we obtain

$$0 \leq 2\langle z - Tz, Tz - z\rangle + \|Tz - z\|^2 = -\|Tz - z\|^2$$

and hence $Tz = z$. □

We prove the main result of this paper.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive self-mappings of C. Let $\{x_n\}$ and $\{z_n\}$ be two sequences in C defined by $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = T_n x_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent and T denotes the pointwise limit of $\{T_n\}$, that is, $Ty = \lim_{n\to\infty} T_n y$ for $y \in C$. Then the following hold:

(i) *The mapping T is nonexpansive and $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$.*
(ii) *If $\{x_n\}$ is bounded, then $F(T)$ is nonempty.*
(iii) *If $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$.*

Proof. We first prove (i). Let $x, y \in C$ be fixed. Since each T_n is nonexpansive, we have

$$\begin{aligned}\|Tx - Ty\| &\leq \|Tx - T_n x\| + \|T_n x - T_n y\| + \|T_n y - Ty\| \\ &\leq \|Tx - T_n x\| + \|x - y\| + \|T_n y - Ty\| .\end{aligned}$$

Since $\|T_n y - Ty\| \to 0$ for all $y \in C$, we conclude that $\|Tx - Ty\| \leq \|x - y\|$. Suppose $u \in \bigcap_{n=1}^{\infty} F(T_n)$. It is easy to obtain that

$$\|u - Tu\| \leq \|u - T_n u\| + \|T_n u - Tu\| = \|T_n u - Tu\| \to 0.$$

Therefore $u \in F(T)$.

Let us show (ii). Assume that $\{x_n\}$ is bounded. Then $\{z_n\}$ is also bounded. Thus there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$. Note that $z \in C$. Since T_n is nonexpansive, it is clear that

$$\|x_{n+1} - T_n z\| = \|T_n x_n - T_n z\| \leq \|x_n - z\|$$

for every $n \in \mathbb{N}$. This yields

$$\begin{aligned} \|x_{n+1} - Tz\|^2 &= \|x_{n+1} - T_n z + T_n z - Tz\|^2 \\ &= \|x_{n+1} - T_n z\|^2 + \|T_n z - Tz\|^2 + 2\langle x_{n+1} - T_n z, T_n z - Tz\rangle \\ &\leq \|x_n - z\|^2 + \|T_n z - Tz\| \left(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|\right). \end{aligned}$$

Hence we conclude that

$$\alpha_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$, where $\alpha_n = -\|T_n z - Tz\| \left(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|\right)$. Since $\{T_n\}$ is pointwise convergent and both $\{x_n\}$ and $\{T_n z\}$ are bounded, it follows that $\alpha_n \to 0$. Thus Lemma 3.1 implies that $z \in F(T)$. This means that (ii) holds.

Let us prove (iii). Let $u \in \bigcap_{n=1}^{\infty} F(T_n)$. It is obvious that

$$\|x_{n+1} - u\| = \|T_n x_n - T_n u\| \leq \|x_n - u\| \tag{3.1}$$

for every $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. Then $\{z_n\}$ is also bounded. Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$. As in the proof of (ii), we obtain $z \in F(T)$. On the other hand, Lemma 2.1 and (3.1) imply that $\lim_{n\to\infty} Px_n = w \in \bigcap_{n=1}^{\infty} F(T_n)$. To complete the proof, it is enough to prove $z = w$. From $z \in F(T)$ and (2.1), it holds that

$$\begin{aligned} \langle z - w, x_k - Px_k\rangle &= \langle z - Px_k, x_k - Px_k\rangle + \langle Px_k - w, x_k - Px_k\rangle \\ &\leq \langle Px_k - w, x_k - Px_k\rangle \\ &\leq \|Px_k - w\| \|x_k - Px_k\| \\ &\leq \|Px_k - w\| M \end{aligned}$$

for every $k \in \mathbb{N}$, where $M = \sup\{\|x_k - Px_k\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle z - w, z_{n_i} - \frac{1}{n_i}\sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i}\sum_{k=1}^{n_i} \|Px_k - w\| M.$$

Since $z_{n_i} \rightharpoonup z$ as $i \to \infty$ and $Px_n \to w$ as $n \to \infty$, we obtain $\langle z - w, z - w\rangle \leq 0$. This means $z = w$. This completes the proof. □

Let $T\colon C \to C$ be a nonexpansive mapping. In Theorem 3.2, putting $T_n = T$ for $n \in \mathbb{N}$, we see that $x_{n+1} = T^n x$ and $z_n = 1/n \sum_{k=1}^{n} T^{k-1}x$ for every $n \in \mathbb{N}$, and moreover, it is also clear that $T_n y - Ty = 0$ for all $y \in C$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore Theorem 3.2 (ii) yields a fixed point theorem for a nonexpansive mapping in a Hilbert space.

Theorem 3.3 ([15, Theorem 3.1.6])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and T a nonexpansive self-mapping of C. Then $F(T) \neq \emptyset$ if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

We also obtain a nonlinear ergodic theorem which was proved by Baillon [2]; see also [15, Theorem 3.2.1].

Theorem 3.4 (Baillon [2])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and T a nonexpansive self-mapping of C. Suppose that $F(T)$ is nonempty. Let $x \in C$ and let $\{z_n\}$ be a sequence in C defined by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$.

Further, we obtain the following theorem:

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and T a nonexpansive self-mapping of C. Suppose that $F(T)$ is nonempty. Let $x \in C$ and let $\{x_n\}$ and $\{z_n\}$ be two sequences in C defined by $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$. Then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$.

Proof. Put $T_n = \alpha_n I + (1-\alpha_n)T$ for $n \in \mathbb{N}$, where I is the identity mapping on C. Then T_n is nonexpansive and $F(T_n) = F(T)$ for every $n \in \mathbb{N}$. Therefore $\bigcap_{n=1}^{\infty} F(T_n) = F(T) \neq \emptyset$ and $\|T_n y - Ty\| = \alpha_n \|y - Ty\| \to 0$ for all $y \in C$. So, from Theorem 3.2 (iii), we have the desired result. $\square$

Problem 3.6. *Can we establish a theorem which unifies Theorem 1.1 and Theorem 3.5?*

For the remainder of this paper we discuss the problem of approximating a common fixed point of a given countable family of nonexpansive mappings.

Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive self-mappings of C and $\{\beta_n\}$ a sequence of $(0,1)$ such that $\sum_{n=1}^{\infty} \beta_n = 1$. We define a sequence $\{T_n\}$ of self-mappings of C as follows:

$$\begin{aligned} T_1 &= \beta_1 S_1 + (1-\beta_1)S_2, \\ T_2 &= \beta_1 S_1 + \beta_2 S_2 + (1-\beta_1-\beta_2)S_3, \\ &\vdots \\ T_n &= \sum_{k=1}^{n} \beta_k S_k + (1 - \sum_{k=1}^{n} \beta_k)S_{n+1}, \end{aligned}$$

for $n \in \mathbb{N}$. It is easy to verify that $F(T_n) = \bigcap_{k=1}^{n+1} F(S_k)$, so that we obtain

$$\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k). \tag{3.2}$$

From Lemma 2.2 we may define a nonexpansive self-mapping T of C by

$$T = \sum_{k=1}^{\infty} \beta_k S_k.$$

It also follows from Lemma 2.2 and (3.2) that

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k).$$

Let $u \in \bigcap_{k=1}^{\infty} F(S_k)$ be fixed. Since each S_k is nonexpansive, we see that

$$\|S_k y\| \leq \|S_k y - S_k u\| + \|S_k u\| \leq \|y - u\| + \|u\|$$

for all $y \in C$ and $k \in \mathbb{N}$. Then we obtain

$$\begin{aligned}
\|Ty - T_n y\| &= \left\| \sum_{k=1}^{\infty} \beta_k S_k y - \left(\sum_{k=1}^{n} \beta_k S_k y + (1 - \sum_{k=1}^{n} \beta_k) S_{n+1} y \right) \right\| \\
&= \left\| \sum_{k=n+1}^{\infty} \beta_k S_k y - (1 - \sum_{k=1}^{n} \beta_k) S_{n+1} y \right\| \\
&\leq \sum_{k=n+1}^{\infty} \beta_k \|S_k y\| + (1 - \sum_{k=1}^{n} \beta_k) \|S_{n+1} y\| \\
&\leq M \sum_{k=n+1}^{\infty} \beta_k + M(1 - \sum_{k=1}^{n} \beta_k)
\end{aligned}$$

for all $y \in C$ and $n \in \mathbb{N}$, where $M = \|y - u\| + \|u\|$. From the assumption that $\sum_{k=1}^{\infty} \beta_k = 1$, we conclude that

$$\lim_{n \to 0} \|Ty - T_n y\| = 0$$

for all $y \in C$. So, we obtain the following theorem:

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_k\}$ be a sequence of nonexpansive self-mappings of C such that $\bigcap_{k=1}^{\infty} F(S_k)$ is nonempty and $\{\beta_k\}$ a sequence in $(0,1)$ such that $\sum_{k=1}^{\infty} \beta_k = 1$. Let $\{x_n\}$ and $\{z_n\}$ be two sequences defined by $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \sum_{k=1}^{n} \beta_k S_k x_n + (1 - \sum_{k=1}^{n} \beta_k) S_{n+1} x_n, \\ z_n = \dfrac{1}{n} \sum_{k=1}^{n} x_k \end{cases}$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to $z \in \bigcap_{k=1}^{\infty} F(S_k)$, where $z = \lim_{n \to \infty} P x_n$ and P is the metric projection of H onto $\bigcap_{k=1}^{\infty} F(S_k)$.

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