COMPARISON OF WHITTLE TYPE PORTMANTEAU TESTS

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ABSTRACT. For an ARMA adequacy test, Box and Pierce (1970) proposed a portmanteau test T_{BP}. However, because the accuracy of T_{BP} by χ^2-approximation is not good, various modifications of T_{BP} have been introduced by many authors. Taniguchi and Amano (2008) proposed an important portmanteau test T_{WLR} of natural Whittle type which is always asymptotically χ^2 distributed under the null hypothesis that ARMA model is adequate. This paper compares T_{WLR} with another famous portmanteau tests Ljung-Box's T_{LB}, Li-McLeod's T_{LM} and Monti's T_{MN} and proves its accuracy by simulation. Empirical powers of those portmanteau tests are also compared numerically.

1. Introduction One of the most important stages of building a model in time series is to verify the adequacy of a fitted model. In particular, sample residual autocorrelations are usually used. For ARMA adequacy test, Box and Pierce (1970) proposed a test statistic T_{BP} which is the squared sum of m sample autocorrelations of the estimated residual process of ARMA(p,q). Under the null hypothesis that the ARMA(p,q) model is adequate, it is suggested that T_{BP} is approximately distributed as χ^2_{m-p-q}. However, Davies et al. (1977) claimed that the χ^2_{m-p-q} -approximation is not adequate and Ljung and Box (1978) and Li-McLeod (1981) proposed test statistics T_{LB} and T_{LM} as a modification of T_{BP}. Recently Monti (1994) proposed a portmanteau test T_{MN} using the residual partial autocorrelations. Various modified versions of T_{BP} (see Li (2004)) have been proposed. Under the null hypothesis that ARMA(p,q) is adequate, these test statistics are much closer to chi-square distribution than T_{BP}.

The test statistic T_{BP} and modifications of T_{BP} are called the portmanteau test and have been widely used. Taniguchi and Amano (2008) proved that T_{BP} does not converge to χ^2_{m-p-q} distribution for fixed m and for ARMA adequacy test, proposed a portmanteau test of natural Whittle type T_{WLR} and showed that T_{WLR} is always asymptotically chi-square distributed. This paper compares T_{WLR} with another famous portmanteau test statistics T_{BP}, T_{LB} and T_{MN} and we observe that T_{WLR} behaves well numerically.

This paper is organized as follows. Section 2 describes the construction of T_{WLR} and its asymptotics. In Section 3, we compare the means and variances of T_{WLR} with those of other portmanteau tests T_{BP}, T_{LB} and T_{MN} by simulation. Then the empirical significance levels and the empirical powers under contiguous alternatives are compared numerically.

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2. Asymptotics of T_{WLR}

A stationary process $\{X_t\}$ is assumed to satisfy

$$\sum_{j=0}^{p} \alpha_j X_{t-j} = \sum_{j=0}^{q} \beta_j u_{t-j}, \quad (\alpha_0 = \beta_0 = 1,\ \alpha_p \neq 0,\ \beta_q \neq 0), \tag{2·1}$$

where $\{u_t\}$ is an m-dependent sequence with autocovariance $\{\theta_{2,j}\}$ ($\theta_{2,0} \equiv 1$, $\theta_{2,-j} \equiv \theta_{2,j}$) and the innovation process of $\{u_t\}$ is identically distributed with mean 0, variance σ_u^2 and fourth-order cumulant κ_4. Let $\alpha(z) \equiv \sum_{j=0}^{p} \alpha_j z^j$ and $\beta(z) \equiv \sum_{j=0}^{q} \beta_j z^j$, and they are assumed to satisfy $\alpha(z) \neq 0$ and $\beta(z) \neq 0$ on $\mathbf{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ and the equations $\alpha(z) = 0$ and $\beta(z) = 0$ have no common roots. We define $\theta_1 = (\theta_{1,1}, \cdots, \theta_{1,p+q})' \equiv (\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)'$, $\theta_2 = (\theta_{2,1}, \cdots, \theta_{2,m})'$ and $\theta = (\theta_1', \theta_2')'$, then the spectral density of $\{X_t\}$ is

$$f_\theta(\lambda) \equiv f_{(\theta_1,\theta_2)}(\lambda) = \frac{|\sum_{j=0}^{q} \beta_j e^{ij\lambda}|^2}{|\sum_{j=0}^{p} \alpha_j e^{ij\lambda}|^2} \cdot \frac{\sigma_u^2}{2\pi} \left\{ \sum_{j=-m}^{m} \theta_{2,j} e^{-ij\lambda} \right\}$$

For the construction of a portmanteau test, Let $\vec{X}_n = (X_1, \cdots, X_n)'$ be an observed stretch from (1), and write the periodogram as

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{it\lambda} \right|^2, \quad \lambda \in [-\pi, \pi]. \tag{2·2}$$

By use of Whittle likelihood

$$D(f_\theta, I_n) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \{\log f_\theta(\lambda) + \frac{I_n(\lambda)}{f_\theta(\lambda)}\} d\lambda \tag{2·3}$$

estimators for (θ_1', θ_2') are given by

$$\hat{\theta}_1 \equiv \arg\max_{\theta_1} D(f_{(\theta_1,0)}, I_n), \qquad (\tilde{\theta}_1, \tilde{\theta}_2) \equiv \arg\max_{(\theta_1,\theta_2)} D(f_{(\theta_1,\theta_2)}, I_n), \tag{2·4}$$

where 0 in (4) is the m-dimensional zero vector. As an adequacy test for ARMA(p,q) model, a portmanteau test of natural Whittle likelihood type

$$T_{WLR} \equiv 2n[D(f_{(\tilde{\theta}_1,\tilde{\theta}_2)}, I_n) - D(f_{(\hat{\theta}_1,0)}, I_n)] \tag{2·5}$$

was proposed in Taniguchi and Amano (2008).

The following lemmas are due to Taniguchi and Amano (2008).

Lemma 2.1. *Write* $F \equiv \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial\theta} \log f_\theta(\lambda) \frac{\partial}{\partial\theta'} \log f_\theta(\lambda) d\lambda = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$. *Suppose that F is positive definite. If ARMA(p,q) model is adequate, then for any fixed* $m = \dim \theta_2$, *it holds that*

$$T_{WLR} \to \chi_m^2, \qquad \textit{in distribution} \quad \textit{as } n \to \infty. \tag{2·6}$$

Lemma 2.2. *Under* $A_G^{(n)} : \theta_2 = \frac{1}{\sqrt{n}} h$, *where h is a fixed m-dimensional vector, the following holds*

$$T_{WLR} \to \chi_m^2(h' F_{22\cdot1} h) \qquad \textit{in distribution} \quad \textit{as } n \to \infty \tag{2·7}$$

where $F_{22\cdot1} = F_{22} - F_{21} F_{11}^{-1} F_{12}$, *and* $\chi_m^2(h' F_{22\cdot1} h)$ *is a noncentral chi-square random variable with m degrees of freedom and noncentrality parameter* $h' F_{22\cdot1} h$.

3. Numerical study In this section, we give a comparison of the test statistic T_{WLR} with another portmanteau tests

$$T_{LB} = n(n+2)\sum_{k=1}^{m}\frac{\hat{r}_k^2}{n-k}, \tag{3·1}$$

$$T_{LM} = \frac{m(m+1)}{2n} + n\sum_{k=1}^{m}\hat{r}_k^2 \tag{3·2}$$

and

$$T_{MN} = n(n+2)\sum_{k=1}^{m}\frac{\hat{\pi}_k^2}{n-k}, \tag{3·3}$$

by simulation. Here, $\hat{r}_k$ and $\hat{\pi}_k$ are the kth sample autocorrelations and sample partial autocorrelations of the estimated residual process of ARMA(p,q) model, respectively. Under the null hypothesis that ARMA(p,q) is adequate, these portmanteau tests T_{LB}, T_{LM} and T_{MN} are supposed to be approximated by χ^2_{m-p-q}-distribution.

In Example 3.1, the empirical means and variances of T_{WLR} for $m = 1$ are compared with those of T_{LB}, T_{LM} and T_{MN} for $m = 2$ under null hypothesis. In Example 3.2, we compare the significance levels of T_{WLR} for $m = 1$ with those of T_{LB}, T_{LM} and T_{MN} for $m = 2,\ 20$ under null hypothesis. Then we can observe that the test statistic T_{WLR} is more accurate than T_{LB}, T_{LM} and T_{MN}. In Example 3.3, local powers of the test T_{WLR} for $m = 1$ are compared with those of T_{LB}, T_{LM} and T_{MN} for $m = 10,\ 20$ under local alternative and we can see that our test T_{WLR} is more powerful than T_{LB}, T_{LM} and T_{MN}.

Example 3.1. *Let $\{X_t\}$ be the AR(1) process*

$$X_t + \alpha X_{t-1} = u_t \tag{3·4}$$

where u_t's are independent and identically distributed as $N(0,1)$. For (4), we compare the empirical means and variances of T_{WLR} for $m = 1$ with those of T_{LB}, T_{LM} and T_{MN} for $m = 2$, respectively. The parameter values are chosen as $0.85 \le \alpha \le 0.99$. The empirical means and variances are calculated based on length of observations $n = 200$ and 1000 times simulation.

In Figure 1, the empirical means of T_{WLR} for $m = 1$ and T_{LB}, T_{LM} and T_{MN} for $m = 2$ $(0.85 \le \alpha \le 0.99)$ are plotted.

In Figure 2, the empirical variances of T_{WLR} for $m = 1$ and T_{LB}, T_{LM} and T_{MN} for $m = 2$ $(0.85 \le \alpha \le 0.99)$ are plotted.

From Figure 1, the empirical means of T_{WLR} for $m = 1$ are closer to 1 than those of T_{LB}, T_{ML} and T_{MN} for $m = 2$. From Figure 2, the empirical variances of T_{WLR} for $m = 1$ are closer to 2 than those of T_{LB}, T_{ML} and T_{MN} for $m = 2$. Due to Lemma 2.1, T_{WLR} for $m = 1$ is approximated by χ^2_1-distribution and T_{LB}, T_{ML} and T_{MN} for $m = 2$ is supposed to be approximated by χ^2_1-distribution. Hence Figures 1 and 2 imply T_{WLR} is more accurate than another portmanteau tests T_{LB}, T_{ML} and T_{MN}.

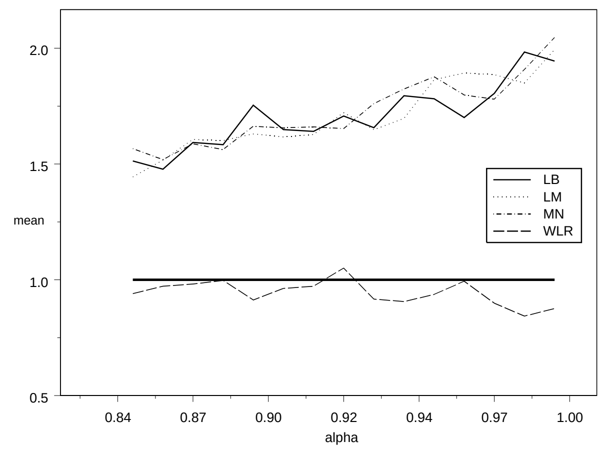

Figure 1: The means of T_{WLR}, T_{LB}, T_{LM} and T_{MN} in Example 3.1 ($0.85 \leq \alpha \leq 0.99$)

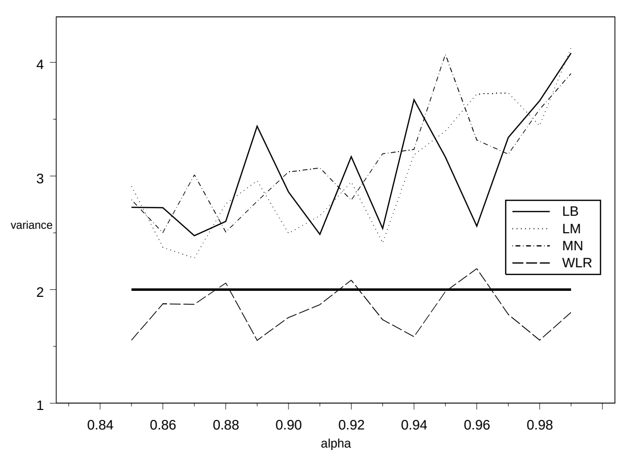

Figure 2: The variances of T_{WLR}, T_{LB}, T_{LM} and T_{MN} in Example 3.1 ($0.85 \leq \alpha \leq 0.99$)

Example 3.2. *For (4), we compare the empirical significance levels of T_{WLR} for $m = 1$ with those of T_{LB}, T_{LM} and T_{MN} for $m = 2,\ 20$, respectively. The parameter values are chosen as $0.85 \le \alpha \le 0.99$. The empirical significance levels are calculated based on length of observations $n = 200$ and 1000 times simulations.*

In Figure 3, the fractions of times that T_{WLR} for $m = 1$ and T_{LB}, T_{LM} and T_{MN} for $m = 2$ $(0.85 \le \alpha \le 0.99)$ exceed the critical values of χ_1^2-distribution for nomial level 5% are plotted.

In Figure 4, the fractions of times that T_{WLR} for $m = 1$ and T_{LB}, T_{LM} and T_{MN} for $m = 20$ $(0.85 \le \alpha \le 0.99)$ exceed the critical values of χ_1^2 and χ_{19}^2-distribution for nomial level 5% are plotted.

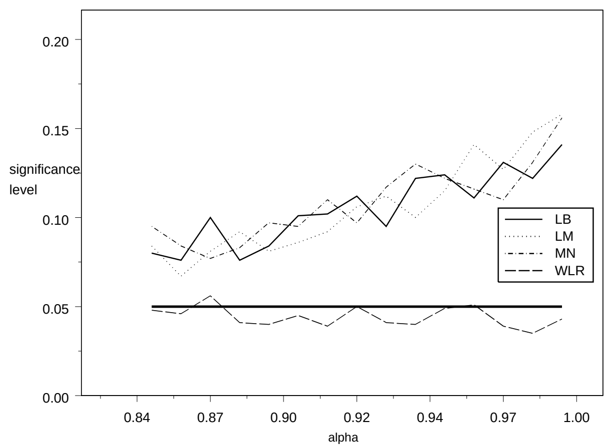

Figure 3: The significance levels with nominal size 5% of T_{WLR}, T_{LB} $(m = 2)$, T_{LM} $(m = 2)$ and T_{MN} $(m = 2)$ in Example 3.2 $(0.85 \le \alpha \le 0.99)$

Due to Lemma 2.1, T_{WLR} for $m = 1$ is approximated by χ_1^2-distribution and T_{LB}, T_{ML} and T_{MN} for m are supposed to be approximated by χ_{m-1}^2-distribution. From Figures 3 and 4, it is seen that T_{WLR} is closer to its asymptotic distribution than T_{LB}, T_{ML} and T_{MN}.

Example 3.3. *Let $\{X_t\}$ be the AR(1) process*

$$X_t + \alpha X_{t-1} = u_t \tag{3·5}$$

where $\{u_t\}$ is the MR(1) with the mean 0, the variance 1 and the autocovariance function $\{\frac{H}{\sqrt{n}}\}$ where $H = \frac{3}{\sqrt{F_{22\cdot 1}}} = \frac{3}{\alpha}$. If T_{WLR} for $m = 1$ exceeds the 95% point of χ_1^2, we reject the null hypothesis. T_{WLR} for $m = 1$ is calculated with length of observations $n = 200$. By use of 1000 times simulation, we give the frequency that the test rejects the hypohtesis. If the T_{LB} for m exceeds the 95% point of χ_{m-1}^2, we reject the null hypothesis. T_{LB} for m is calculated with length of observations $n = 200$. By use of 1000 times simulations, we give

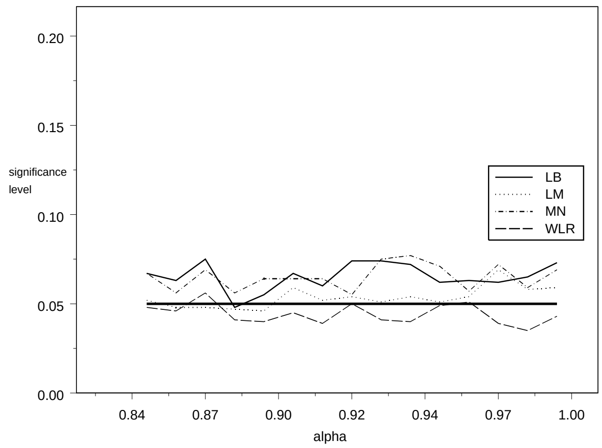

Figure 4: The significance levels with nominal size 5% of T_{WLR}, T_{LB} ($m = 20$), T_{LM} ($m = 20$), T_{MN} ($m = 20$) in Example 3.2 ($0.85 \leq \alpha \leq 0.99$)

the frequency that the test rejects the hypohtesis. Also, we give empirical powers of T_{LM} and T_{MN} for m similary.

In Figure 5, the empirical powers for a 5%-level test of T_{WLR} for $m = 1$ and those of T_{LB}, T_{LM} and T_{MN} for $m = 10$ ($0.45 \leq \alpha \leq 0.99$) are plotted.

In Figure 6, the empirical powers for a 5%-level test of T_{WLR} for $m = 1$ and those of T_{LB}, T_{LM} and T_{MN} for $m = 20$ ($0.45 \leq \alpha \leq 0.99$) are plotted.

From Figures 5 and 6, our test statistic T_{WLR} is more powerful than T_{LB}, T_{LM} and T_{MN}.

Simulation results imply that T_{WLR} is closer to theoretic χ^2-distribution than another famous portmanteau tests T_{LB}, T_{LM} and T_{MN} under null hypothesis that ARMA(p,q) model is adequate. It is implied that under contiguous alternative hypothesis, the ability of T_{WLR} to detect model misspecification is higher than that of another famous portmanteau tests T_{LB}, T_{LM} and T_{MN} by simulation.

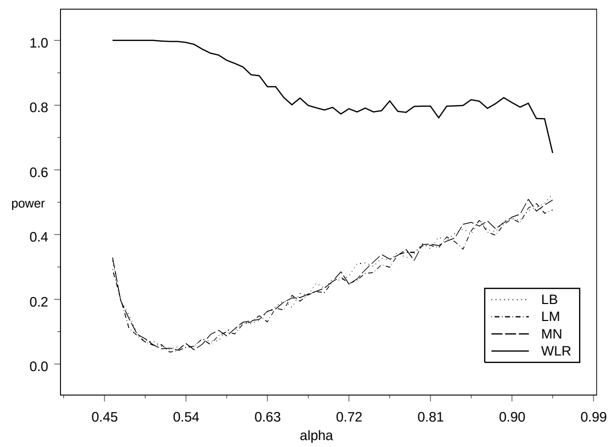

Figure 5: The empirical powers with level test 5% of T_{WLR}, T_{LB} ($m = 10$), T_{LM} ($m = 10$), T_{MN} ($m = 10$) in Example 3.3 ($0.45 \le \alpha \le 0.99$)

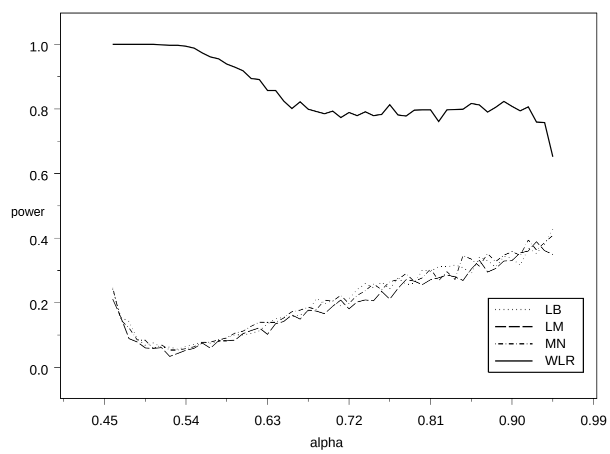

Figure 6: The empirical powers with level test 5% of T_{WLR}, T_{LB} ($m = 20$), T_{LM} ($m = 20$), T_{MN} ($m = 20$) in Example 3.3 ($0.45 \le \alpha \le 0.99$)

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