HANS-JÜRGEN HOEHNKE'S WORK ON NORM FORMS AND THE GROUP DETERMINANT

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In 1989, H.-J. Hoehnke and I began to collaborate on a question I had posed on the group determinant of Dedekind and Frobenius. Very few modern accounts of representation theory give a thorough account of Frobenius' early work on group matrices and the group determinant, which led to group representation theory, but there are accounts in [5], [7], [8], [9] , [14], [15].

Our work relied heavily on Hoehnke's great expertise in the theory of norm forms. Frobenius' construction of an irreducible factor of the group determinant from the character χ of an irreducible representation involved the construction of a series of "k-characters" $\chi^{(k)}$: $G^k \to \mathbb{C}$. Each factor of the group determinant is a multiplicative norm form and this was the connection with Hoehnke's work. His approach to constructive algebra could be applied to answer not only the original question of whether the group determinant determines its group (first proved by Formanek and Sibley after Hoehnke's ideas were communicated to them) but that the 1-,2- and 3-characters of any representation which contained at least one copy of each irreducible representation determined a group, see [11], [12]. The result was an answer to one of the questions of Brauer in [1] : which information in addition to the character table determines a group? Other sets of invariants which determine a group were given by Roitman and Gallagher, and in [13] it was possible to describe these in terms of k-characters.

This demonstrates a close connection between group representation theory and the multiplicative norm form work of Hoehnke and the school of A. Bergmann. Hoehnke described the use of norm forms and associated objects as similar to the use by Felix Klein of invariant theory as a substitute for group representations in his work on the solution of equations [16]. While the k-characters have not yet provided new tools within finite group theory, there are indications that in several of the areas in which group representations are applied they have advantages over traditional tools. A few examples are given below.

(1) The problem of how, given a group class function it can be decided whether it is a character of a representation was given a solution by Helling in [10]. Helling's criterion can be restated in terms of k-characters: a class function f is a character if and only if $f^k = 0$ for some positive integer k, where the definition of f^k proceeds exactly as for k-characters (the underling field needs to be algebraically closed). This criterion appeared in work of Wiles and Taylor in papers leading up to the Fermat theorem. For any finite dimensional representation of a group they define a *pseudocharacter* to be a class function f for which $f^k = 0$ for some k and use pseudocharacters instead of genuine representations (for fields which are not algebraically closed) see [22], [18], [17].

(2) Vazirani in [19] has built on the generalisations of k-characters first appearing in [13] to describe connections with Schur functions, combinatorics and other aspects of representation theory.

(3) In [2] Buchstaber and Rees use k-characters to produce a constructive proof of a generalisation of the theorem of Gelfand and Kolmogorov which identifies a compact

Hausdorff space X with space of maximal ideals of the ring of continuous functions on X and later in [3] produce a new proof of the above theorem. They also prove that for a suitable trace function f on a Frobenius algebra $f^{(1)}$,$f^{(2)}$ and $f^{(3)}$ determine the structure constants of the associated Jordan algebra, which leads to a more elementary proof of the result on 3-characters mentioned above. See also [4].

(4) In [6] Cooper and Walsh introduce the group determinant of a group G in geometric work on 3-manifolds.

(5) Wavelets. Waldron and collaborators ([21], [20]), use the gram matrix of a tight frame in their work on wavelets. This is a version of the group matrix of the symmetry group.

I think that the effects of Hoehnke's insight in this area are only beginning to appear.

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