H. J. HOEHNKE'S CONTRIBUTION TO RADICAL THEORY

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1 The aim of introducing radicals The purpose of this note is to show how radical theory evolved, and how Hoehnke's profound contribution influenced the developments.
In ring theory the aim was to discard or ignore a "bad ideal" ϱA of a ring A such that the factor ring $A/\varrho A$ becomes a "good ring" which can be represented as a (sub)direct sum of "well behaved" rings. For instance, the *nil radical* $\mathcal{N}(A)$, introduced by Köthe in 1930, is the unique largest nil ideal of a ring A, and the factor ring $B = A\mathcal{N}/\mathcal{N}(A)$ can be represented as a subdirect sum of rings each of which is a prime ring having no nonzero nil ideal. Imposing certain finiteness conditions (such as the descending chain condition on left ideals), *the ring $A/\mathcal{N}(A)$ becomes a finite direct sum of rings of linear transformations on finite dimensional vector spaces over division rings.* Observe that there is no way to represent the nil ring $\mathcal{N}(A)$ as a ring of linear transformations, and the rings $\mathcal{N}(A)$ and $A/\mathcal{N}(A)$ are of very different properties.

Another example is the *Jacobson radical* $\mathcal{J}(A)$, the unique maximal quasi-regular ideal of A, and the factor ring $A/\mathcal{J}(A)$ is a subdirect sum of *primitive rings*, that is, *of dense subrings of linear transformations on vector spaces over division rings.* Notice that the Jacobson radical is the most efficient one in representing rings by dense rings of linear transformations.

Having in mind several concrete radicals for rings, Kurosh and Amitsur independently generalized the notion in the following way. Let ϱ be a property of rings, or equivalently the class of all rings having propery ϱ. The class ϱ is called a *radical class in the sense of Kurosh and Amitsur*, if

- ϱ is closed under taking homomorphic images,
- for every ring A the sum of ideals $\varrho A = \Sigma(I \triangleleft A \mid I \in \varrho)$ is in the class ϱ,
- $\varrho(A/\varrho A) = 0$ for every ring A.

Rings of the class ϱ are called *ϱ-radical rings*, and rings with 0 ϱ-radical are said to be *ϱ-radical free*, or *ϱ-semisimple*. The ideal ϱA is referred to as the *ϱ-radical* of the ring A.

An important feature is that *every ring A is an extension of a ϱ-radical ring ϱA by a ϱ-semisimple ring $A/\varrho A$.*

In the above examples the radical class $\mathcal{N}$ is the class of all nil rings and $\mathcal{J}$ is that of all quasi-regular rings, respectively. Introducing radicals makes sense not only for rings, but also for modules, groups, semigroups, near-rings, Ω-groups, universal algebras, etc. Implementing the above explained aim by defining notions in a suitable way, by developing and using appropriate methods in these categories, decomposition theorems can be proved for semisimple objects. Working on a more general level, however, something will be lost, and therefore it is inevitable to change priorities and the point of view. Hoehnke succeeded in this project by introducing radicals for univeral algebras, which are now referred to as *Hoehnke radicals*. In the special case of rings, imposing further constrains on Hoehnke radicals, one gets the above defined Kurosh–Amitsur radicals.

Hoehnke published, mainly in the 60-s, some 15 papers on radicals. He contributed to this topic roughly in two directions. Firstly, he succeeded carrying over ring theoretic structure theorems to semigroups by introducing radicals, secondly, he generalized the semigroup theoretical results to universal algebras and developed the fundamentals of the radical theory thereof by far-reaching and inseminating ideas. Although he dealt with radical theory only in a short period of his long and successful research career, his ideas and results influenced substantially the further developments of radical and structure theory of universal algebras and special cases thereof.

2 Semigroups and radicals Carrying over radical theory to semigroups, there might be two approaches: a group theoretical or a ring theoretical one. Hoehnke followed the latter way, which turned out to be more efficient, in fact, the natural one. His aim was to represent semigroups by transformation semigroups. Notice that some notions concerning rings are basically multiplicative ones, such as ideal, nilpotency, idempotency, primeness, primitivity, transitivity, etc. Nevertheless, sometimes there may be several semigroup theoretical candidates of a ring theoretic notion (e.g. the corresponding notion of an ideal may be a multiplicatively absorbing subset as well as a congruence of a semigroup; a semigroup ideal may belong to several congruences, the smallest congruence determined by an ideal is the Rees congruence where the ideal is a coset and each element outside the ideal is itself a coset; it may happen that no coset of a congruence is an ideal). Hoehnke developed the structure theory via radicals, at first for semigroups with 0.

An S-act M with 0 is said to be *irreducible*, if $M \neq \{0\}$, and M and $\{0\}$ are the only subacts of M. The 0-*radical* of a semigroup S with zero has been defined as the intersection

$$\text{rad}^0\ S = \bigcap(\text{ann}\ M \mid M \text{ is an irreducible } S\text{-act})$$

where ann $M = \{s \in S \mid Ms = 0\}$ is the *annihilator* of M is S.

A semigroup S with 0 is said to be 0-*primitive* if there exists an irreducible S-act M such that ann $M = \{0\}$. An ideal P of a semigroup S with 0 is called a 0-*primitive ideal* of S if the Rees factor semigroup S/P is a 0-primitive semigroup. In [10] Hoehnke proved that *an ideal P is primitive if and only if P is the annihilator of an irreducible S-act M in S*. Moreover, $\text{rad}^0 S = 0$ *if and only if S is a subdirect product of* 0-*primitive semigroups*, and the subdirect components can be described by transformations acting transitively on irreducible S-acts (cf. Tully [42]).

The assumption of considering only semigroups with 0-element, was removed in the paper [17]. For semigroups not necessarily containing a 0-element, the *radical is a congruence*. Generalizing the 0-radical, *the radical* rad$S$ is defined as the intersection

$$\text{rad} S = \bigcap(\delta_M \mid M = S/\delta_M \text{ is an irreducible S-act}).$$

Then $S/\text{rad}S$ is a subdirect product of the primitive semigroups S/δ_M, and Hoehnke achieved to describe *the structure of primitve semigroups S with an ireducible right ideal generated by an idempotent in terms of dual vector sets* (cf. [17]).

Also other radicals of semigroups were introduced and investigated. A primitive congruence is a prime congruence, therefore the intersection of all prime congruences, that is, *the prime radical is contained in the radical* (cf. [10], [14], [17]). The *nil radical* $\mathcal{N}(S)$ *of a semigroup* S *with* 0 *is defined as the sum of all nil ideals of* S, and $\mathcal{N}(S) = \mathrm{rad}^0\, S$ (see [10], [14], [17], [39]).

The 0*-radical assignment is idempotent*, that is, $\mathrm{rad}^0\ \mathrm{rad}^0 S = \mathrm{rad}^0 S$, as proved in [24]. *All the so far considered semigroup radicals* κ *are congruences for which* $\kappa(S/\kappa)$ *is the identical relation*. Hoehnke investigated and characterized the radicals of semigroups also in the papers [11] and [15]. In [16] Hoehnke interpreted the module theoretic characterizations of ring radicals of Andrunakievich and Ryabukhin to semigroups and went towards universal algebraic aspects (cf. also [22]), so [17] can be regarded also as a prelude of the radical theory of universal algebras. In his later paper [23] he continued the ring theoretical investigations of Andrunakievich and Ryabukhin.
For further developments in the radical theory of semigroups the reader is referred to the extensive study of strict radicals of monoids by Márki, Mlitz and Strecker [25], to the section of special cases and examples in [26], and in the structure theory to Hotzel [26] and Steinfeld [40]. In the semi-expository paper [6] Clifford reformulated, among others, for semigroups some of the material of [16] and [18], following Tully's theory [42]. In [38] Roĭz and Schein surveyed the radical theory of semigroups with emphasis on Hoehnke's contribution.

3 Radicals in universal algebras The investigations of concrete radicals of semigroups led Hoehnke to define and study general radicals for universal algebras. This was done in [18], a paper of fundamental importance. In a variety of universal algebras
• a *Hoehnke radical* ϱ is an assignment which designates to each algebra A a *congruence*

$$\varrho :\ A \mapsto \varrho A,$$

the congruence ϱA such that
• for every surjective homomorphism $f : A \mapsto fA$,

$$f(\varrho A) \subseteq \varrho(fA)$$

where $f(\varrho A)$ means that the homomorphism f is applied componentwise to the subalgebra ϱ of $A \times A$, and
• for every algebra A,

$$\varrho(A/\varrho A) \text{ is the identical relation.}$$

Every Hoehnke radical ϱ determines its *semisimple class*

$$\mathbb{S}_\varrho = \{\ A \mid \varrho A = \text{ is the identical relation }\}$$

which is closed under taking subdirect products. Conversely, every subdirectly closed subclass $\mathbb{S}$ of algebras determines a Hoehnke radical assignment

$$\varrho_{\mathbb{S}} :\ A \mapsto \bigcap(\kappa \mid A/\kappa \in \mathbb{S}).$$

Thus
• *there is a one-to-one correspondence between Hoehnke radical assignments and Hoehnke semisimple classes*, and the primary aim of introducing radicals has been preserved in as much as

• *every Hoehnke semisimple algebra has a subdirect representation*: given a class $\mathbb{C}$ of algebras, its subdirect closure $\overline{\mathbb{C}}$ is the semisimple class of a Hoehnke radical ϱ and

$$A = \prod_{\text{subdirect}} (B \mid B \in \mathbb{C})$$

for every ϱ-semisimple algebra A.
Although every Hoehnke radical ϱ determines its radical class

$$\mathbb{R}_\varrho = \{\ A \mid \varrho A \text{ is the universal relation } \},$$

different Hoehnke radicals may have the same radical class, in contrast to the Kurosh–Amitsur radicals. Equivalently, a Hoehnke radical is a congruence ϱ minimal with respect to the property

$$A/\varrho A \text{ is the identical relation}$$

for all algebras A, and beside this property a Kurosh–Amitsur radical has also a maximal property (for rings, for instance, the nil radical is the unique largest nil ideal, and the Jacobson radical is the unique largest quasi-regular ideal).

In order to get Kurosh–Amitsur radicals, Hoehnke introduced the notion of M-relation to distinguish certain subalgebras (in particular, for rings the relation $B\mathrm{M}A$ may mean that B is a nonzero ideal, right ideal, or subring of A, respectively), and imposed the following additional condition on a Hoehnke radical ϱ:

• ϱA is the identical relation iff ϱB is not the universal relation whenever $B\mathrm{M}A$. These Hoehnke radicals are the Kurosh–Amitsur radicals of universal algebras. In particular, in categories where congruences are determined by distinguished subalgebras (for example, rings, near-rings, or groups, modules) and radicals ϱ assign a distinguished subalgebra, *a Hoehnke radical ϱ is a Kurosh–Amitsur radical iff ϱ satisfies the additional conditions:*
• the radical ϱ is *idempotent*, that is, $\varrho(\varrho A) = \varrho A$,
• the radical ϱ is *complete*, that is, $\varrho B = B$ and $B\mathrm{M}A$ imply $B \subseteq \varrho A$.
Thus, *a Hoehnke radical ϱ is a Kurosh–Amitsur radical iff ϱ is idempotent and complete.* A Hoehnke radical ϱ with an M-relation is said to be *hereditary*, if for every algebra A, and $B\mathrm{M}A$ the restriction of the congruence ϱA to B is the congruence ϱB. Many Hoehnke radicals, in particular, all the classical Kurosh–Amitsur radicals of rings are hereditary.
• In categories where Hoehnke radicals assign distinguished subalgebras, *a hereditary Hoehnke radical is idempotent and complete, whence a Kurosh–Amitsur one*. This is an important fact e.g. in the study of radicals of near-rings. Moreover, in categories of R-modules as well as of S-acts torsion theories can be defined in terms of injectivity. For modules, torsion theories are just the hereditary Kurosh–Amitsur radicals, but not for S-acts because a hereditary Hoehnke radical need not be Kurosh–Amitsur (cf. [44]).

It is amazing how clearly Hoehnke gave the axiomatics of the general radical theory of universal algebras, and foresaw the path of building out this theory for universal algebras and other varieties in which the axioms of Kurosh–Amitsur radicals are too demanding and therefore too restrictive. It took 10 to 20 years till the developments made possible the continuation of Hoehnke's work (cf. [31], [41], and [27]).

In [19] and [21] Hoehnke observed that radicals and also M-radicals can be defined for objects of categories too. In this case it would be clumsy to assign some kind of a kernel of an appropriate morphism from the given object as a radical. For this reason Hoehnke

considered categories endowed with a factorization system (every morphism factors uniquely though a mono- and epimorphism) and
• *a radical ϱ is defined as a covariant factor functor designating to each object A a factor object C* (in terms of algebras this corresponds to $C = A/\varrho A$) which satisfies requirements corresponding to those imposed for algebras.

Finally we give a sample of papers in which the authors followed Hoehnke's profound ideas. It was R. Mlitz who continued Hoehnke's research on radical theory of *universal algebras* in a series of papers [31], [32], [33], and [34]. Strecker [41] investigated M-radicals, that is, Kurosh–Amitsur radicals for universal algebras. Radical theory of universal algebras with idempotent operations was studied by Gardner [8]. Buys and Heidema [3] dealt with Hoehnke's radical theory from a category theoretical aspect. Starting from Hoehnke radicals and using category theoretical tools, Márki, Mlitz and Wiegandt [27] developed a general Kurosh–Amitsur radical theory which includes, as special cases, all the foregoing radical theories, as well as the connectedness–disconnectedness theory of graphs and topological spaces, and the assignment of the greatest semilattice image to semigroups. Radical theory of Ω-*groups* were treated, among others, in the papers of Buys and Gerber [1], [2], Booth, Petersen and Veldsman [5], Mlitz and Veldsman [36]. For *near-rings* the radicals corresponding to the ring theoretic Kurosh–Amitsur ones, are mostly only Hoehnke radicals. Papers dealing with Hoehnke radicals of near-rings are e.g. Groenewald [9], Márki, Mlitz and Wiegandt [28], Mlitz [30], [34] and Veldsman [43]. For Hoehnke radicals of *rings* we refer to Buys and Heidema [4], and de la Rosa, Niekerk and Wiegandt [7].

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