# ON "PARTIAL ALGEBRAS AND THEIR THEORIES" BY HANS-JÜRGEN HOEHNKE AND JÜRGEN SCHRECKENBERGER

K. Denecke

Received August 29, 2008

The aim of the book is to develop an appropriate category-theoretical background to characterize the category $\mathbf{P}ar$ of all sets and all partial mappings between sets and to use this characterization to study theories for partial algebras. For a better understanding we first will give the most important basic concepts from Category Theory which are used throughout the book.

**1 Basic Concepts from Category Theory** First we provide a short, self-contained introduction to some elementary topics in Category Theory, including categories, functors and natural transformations to make the material presented in "Partial Algebras and Their Theories" more understandable. For more information on Category Theory we refer the reader to [11] and [9].

A category $\mathbf{C}$ consists of a class $|C|$ of *objects* and a class $mor_{\mathbf{C}}C$ (or $morC$, for short) of *morphisms* between these objects. Let $C(A,B)$ be the set of all morphisms between the objects $A$ and $B$. Each morphism $f$ has exactly one object $A$ as its *source* and exactly one object $B$ as its *target*. We will write $f : A \to B$ or also $A \xrightarrow{f} B$ for a morphism from source $A$ to target $B$. The objects of a category are classes which are not necessarily sets. For a set-theoretical foundation one can use for instance the Set Theory of *von Neumann-Bernays-Gödel*. The concept of a class is then taken as a basic concept, and sets are those classes which are elements of a class. There exists a universal class which contains all sets as elements. For details see for instance [7] or [10].

A category in which the objects are sets with additional structure (such as operations, partial operations or relations) is called a *concrete category*. A category is called *small* if $morC$ is a set. The following properties have to be satisfied by the objects and morphisms of such a category:

(i) If $A$ is the base set of an object, then $1_A : A \to A$ defined by $1_A(x) = x$ for all $x \in A$ is a morphism.

(ii) The class of morphisms is closed under composition: if $A$, $B$ and $D$ are objects and $f : A \to B$ and $g : B \to D$ are morphisms, then $g \circ f : A \to D$ is a morphism.

(iii) For any objects $A$ and $B$ and any morphism $f : A \to B$ the equations $f \circ 1_A = f$ and $1_B \circ f = f$ are satisfied.

(iv) The composition of morphisms is associative.

**Example 1.1** 1. The objects of the category $\mathbf{S}et$ are sets and the morphisms are the usual mappings between sets.

2. The objects of the category $\mathbf{P}ar$ are sets and the morphisms are the partial mappings between sets.

3. The objects of the category $\mathbf{T}op$ are the topological spaces and the morphisms are the continuous mappings between them.

4. The objects of the category $\mathbf{G}roup$ are groups and morphisms are homomorphisms between them.

5. The objects of the category $\mathbf{A}lg(\tau)$ are the algebras of type $\tau$ and the morphisms are (algebra) homomorphisms between them.

6. More generally, each variety of algebras can be regarded as a category, where the objects are the algebras of the variety and the morphisms are the homomorphisms between them.

7. The product $\mathbf{A} \times \mathbf{B}$ of two categories $\mathbf{A}$ and $\mathbf{B}$ is a category which has as its objects the ordered pairs $(A, B), A \in |A|, B \in |B|$. The sets of morphisms are defined by $mor_{\mathbf{A}\times\mathbf{B}}((A, B), (A', B')) := mor_{\mathbf{A}}(A, A') \times mor_{\mathbf{B}}(B, B')$. The composition of morphisms is defined componentwise by $(f', g') \cdot (f, g) := (f' \cdot f, g' \cdot g)$ . (Instead of $(f, g)$ we will also write $f \otimes g$.)

Morphisms can have some special properties.

**Definition 1.2** A morphism $f : A \to B$ is called
a) left-invertible or a coretraction if there exists a morphism $h : B \to A$ such that $h \circ f = 1_A$. In this case the morphism $h$ is called a left inverse of $f$.

b) right-invertible or a retraction if there exists a morphism $g : B \to A$ such that $f \circ g = 1_B$. In this case $g$ is called a right inverse of $f$.

c) an isomorphism if there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Such a morphism $g$ is called an inverse of $f$.

**Remark 1.3** 1. If the morphism $f : A \to B$ is both left-invertible and right-invertible, then it is an isomorphism. It is a simple exercise to prove that if $f$ has both a left and a right inverse, then these two inverses must be equal.

2. If $f : A \to B$ is an isomorphism, then there exists exactly one morphism $g$ with $g \circ f = 1_A$ and $f \circ g = 1_B$. This mapping $g$ is denoted by $f^{-1}$, and is called the inverse of $f$.

3. If there is an isomorphism between the objects $A$ and $B$, then $A$ and $B$ are called isomorphic and we write $A \cong B$.

**Definition 1.4** A morphism $f : A \to B$ is said to be a *monomorphism* (or a mono for short) if

$$\forall g_1, g_2 : C \to A \qquad (f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2),$$

and an *epimorphism* (or an epi) if

$$\forall h_1, h_2 : B \to C \qquad (h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2).$$

Clearly, any left-invertible morphism is a monomorphism, and dually any right-invertible morphism is an epimorphism. The converse of this proposition is not true.

Terminal and initial objects of a category are objects with certain special properties.

**Definition 1.5** An object $T$ of a category $\mathbf{C}$ is called *terminal* if for every object $A$ of $\mathbf{C}$ there is exactly one morphism $\tau_A : A \to T$.

If there are terminal objects in the category **C**, then such objects are uniquely determined up to isomorphism.

**Example 1.6** In the category **S**$et$ every one-element set is terminal. In the category **G**$roup$ every one-element group is terminal, and in **A**$lg(\tau)$ the one-element algebras of type $\tau$ are terminal.

**Definition 1.7** An object $I$ of a category **C** is called *initial* if for every object $A$ of **C** there is exactly one morphism $I \to A$.

The initial objects of a category are also uniquely determined up to isomorphism.

**Example 1.8** In the category **S**$et$ the empty set $\emptyset$ is initial. In **G**$roup$ every one-element group is initial.

Now we look at two constructions defined on a family $(A_i)_{i\in I}$ of objects in a category.

**Definition 1.9** Let $(A_i)_{i\in I}$ be a family of objects in a category **C**. An object $S$ together with a family $(e_i : A_i \to S)_{i\in I}$ of morphisms is called the *sum* of the $A_i$'s if for every object $Q$ from **C** with morphisms $(q_i : A_i \to Q)_{i\in I}$ there is exactly one morphism $s : S \to Q$ with $q_i = s \circ e_i$ for all $i \in I$. If such a sum exists, we write $S := \sum\limits_{i\in I} A_i$ and call the morphisms $(e_i)_{i\in I}$ *injections* of the sum.

The diagram in Figure 1 illustrates the sum construction. The sum of a family of objects is also called their *coproduct*.

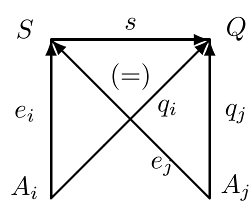


Figure 1: Coproduct

**Example 1.10** Consider the category **S**$et$ of sets and set mappings. We take $S$ to be the disjoint sum

$$A_1 \ + \ A_2 \ = \ \{(1,a) \mid a \in A_1\} \ \cup \ \{(2,b) \mid \ b \in A_2\},$$

and as morphisms the mappings $e_1 : A_1 \to A_1 + A_2$ and $e_2 : A_2 \to A_1 + A_2$ defined by $e_1(a) = (1,a)$ and $e_2(a) = (2,b)$. To see that this fits our categorical definition of the sum, suppose that $Q$ is any set with mappings $q_i : A_i \to Q$ for $i = 1,2$. Then there exists precisely one mapping $s : A_1 + A_2 \to Q$ with $q_i = s \circ e_i$, defined by $s(i,a) := q_i(a)$. For any $a \in A_1 \cup A_2$, we have $q_i(a) = (s \circ e_i)(a) = s(e_i(a)) = s((i,a))$, so $q_i = s \circ e_i$ for $i = 1,2$.

The sum of a family $(A_i)_{i\in I}$ of objects of a category **C** is uniquely determined up to isomorphism. The injections $(e_i)_{i\in I}$ associated with a sum must be epimorphisms. That is, for all morphisms $h_1, h_2 : S \to C$ and all $i \in I$ we have

$$h_1 \circ e_i = h_2 \circ e_i \quad \Rightarrow \quad h_1 = h_2.$$

The product of a family of objects is defined in the following dual way.

**Definition 1.11** Let $(A_i)_{i\in I}$ be a family of objects in a category **C**. An object $P$ together with a family $(p_i : P \to A_i)_{i\in I}$ of morphisms is called the *product* of the $A_i$'s if for every object $Q$ from **C** with morphisms $(q_i : P \to A_i)_{i\in I}$ there is exactly one morphism $s : Q \to P$ with $q_i = p_i \circ s$ for all $i \in I$. If the product exists, we write $P := \prod_{i\in I} A_i$, and the morphisms $(p_i)_{i\in I}$ are called *projections*.

The diagram in Figure 2 illustrates the product construction.

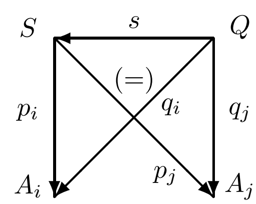


Figure 2: Product

In the category **S**$et$ of sets, the usual cartesian product of a family of sets is their product, with the usual projection mappings as morphisms. In the category **A**$lg(\tau)$ the direct product of a family $(\mathcal{A})_{i\in I}$ is a product.

Pushouts and Pullbacks are defined as follows:

**Definition 1.12** Let $(f_i : A \to B_i)_{i\in I}$ be a family of morphisms in a category **C**. An object $P$ together with a family $(p_i : B_i \to P)_{i\in I}$ of morphisms is called a *pushout* of the $f_i$'s if

(i) for all $i, j \in I$, $\quad p_i \circ f_i \;=\; p_j \circ f_j$.

(ii) for all objects $Q$ and all morphisms $(q_i : B_i \to Q_i)_{i\in I}$ with $q_i \circ f_i \;=\; q_j \circ f_j$ for all $i, j \in I$, there exists a unique morphism $h : P \to Q$ satisfying $h \circ p_i = q_i$ for all $i \in I$.

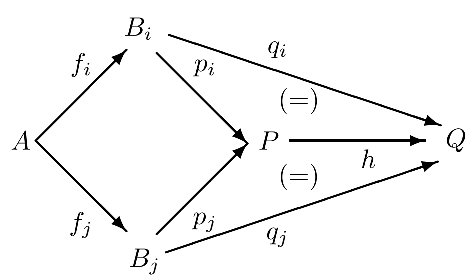


Figure 3: Pushout

**Definition 1.13** Let $(f_i : A_i \to B)_{i \in I}$ be a family of morphisms. An object $P$ together with a family $(p_i : P \to A_i)_{i \in I}$ of morphisms is called the *pullback* of the $f_i$'s if

(i) for all $i, j \in I$, $\quad f_i \circ p_i = f_j \circ p_j$;

(ii) for all objects $Q$ and all morphisms $q_i : Q \to A_i$ with $f_i \circ q_i = f_j \circ q_j$ for all $i, j \in I$, there exists a unique morphism $h : Q \to P$ such that $p_i \circ h = q_i$ for all $i \in I$.

So far we have considered morphisms between objects in one category. Now we turn to mappings between objects and morphisms of two different categories. A functor from a category **C** to a category **D** maps the objects of **C** to objects of **D** and the morphisms of **C** to morphisms of **D**, in a way that is compatible with the composition of morphisms and preserves the identity morphism. The next definition makes this more precise.

**Definition 1.14** Let **C** and **D** be categories. A *(covariant) functor* $F : \mathbf{C} \to \mathbf{D}$ defines a mapping $A \mapsto F(A)$ of objects $A$ from the object class $|C|$ of the category **C** to objects $F(A)$ from the object class $|C'|$ of the category **D**; and also a mapping for morphisms which maps each morphism $f : A \to B$ from **C** to a morphism $F(f) : F(A) \to F(B)$ from the category **D**, such that for all morphisms $f, g$ and all objects $A$ from **C** the following two conditions are satisfied:

(i) $F(g \circ f) = F(g) \circ F(f)$,

(ii) $F(1_A) = id_{F(A)}$.

(The functor is called *contravariant* if instead of (i), (ii) the following conditions are satisfied:

(i') $F(g \circ f) = F(f) \circ F(g)$,

(ii') $F(1_A) = id_{F(A)}$.)

We are interested here only in covariant functors, and shall refer to them merely as functors.

A bifunctor is a functor $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$. If $F(A, B)$ and $F(f, g)$ are the images of $(A, B)$ and of $(f, g)$, respectively, then $F(f' \cdot f, g' \cdot g) = F(f', g') \cdot F(f, g)$.

We can use the following constructions to combine functors into new functors. Let $F_1$ and $F_2$ be functors from $\mathbf{S}et$ to $\mathbf{S}et$.

(i) $F_1 \circ F_2$ defined by $A \mapsto F_1(F_2(A))$ for objects $A$ and $f \mapsto F_1(F_2(f))$ for morphisms $f$ is a functor. This construction is called composition of functors.

(ii) $F_1 \times F_2$ defined by $(F_1 \times F_2)(A) := F_1(A) \times F_2(A)$ for objects and $(F_1 \times F_2)(f)(u, v) := (F_1(f)(u), F_2(f)(v))$ for morphisms is a functor. This construction is called the cartesian product of functors. The cartesian product can also be defined for arbitrary families of endofunctors of $\mathbf{S}et$.

(iii) $F_1 + F_2$ is defined on objects by $(F_1 + F_2)(A) := F_1(X) + F_2(X)$, where $F_1(X) + F_2(X)$ is the disjoint union of the sets $F_1(X), F_2(X)$. For each morphism $f$ we define

$$(F_1 + F_2)(f)(u) := \begin{cases} F_1(f)(u), & \text{if } u \in F_1(X) \\ F_2(f)(u), & \text{if } u \in F_2(X) \end{cases} .$$

Then $F_1 + F_2$ is a functor.

**Remark 1.15** 1. If $F : \mathbf{S}et \to \mathbf{S}et$ is a functor and $f$ is injective (surjective) then $F(f)$ is also injective (respectively surjective).

2. Let $\mathbf{C}$ be a category and let $C(A, B)$ be the set of all morphisms $f : A \to B$ in $\mathbf{C}$. Every functor $F : \mathbf{C} \to \mathbf{D}$ defines a mapping $F_{A,B} : C(A, B) \to D(F(A), F(B))$.

3. A functor $F : \mathbf{C} \to \mathbf{D}$ is called *full* if for any two objects $A$ and $B$ of $\mathbf{C}$ the mapping $F_{A,B}$ is surjective. $F$ is called *faithful* if $F_{A,B}$ is injective for any two objects $A$ and $B$ of $\mathbf{C}$.

4. The functor $F : \mathbf{C} \to \mathbf{D}$ is called an isomorphism if there is a functor $G : \mathbf{D} \to \mathbf{C}$ such that $F \circ G = I_{\mathbf{D}}$ and $G \circ F = I_{\mathbf{C}}$.

Natural transformations connect functors with functors.

**Definition 1.16** Let $F_1, F_2 : \mathbf{C} \to \mathbf{D}$ be functors. A natural transformation $\eta$ from $F_1$ to $F_2$ maps each object $X$ from $\mathbf{C}$ to a morphism $\eta_X : F_1(X) \to F_2(X)$ such that for every morphism $f : X \to Y$ in $\mathbf{C}$ the equation

$$F_2(f) \circ \eta_X = \eta_Y \circ F_1(f)$$

is satisfied. If $\eta$ is a natural transformation from $F_1$ to $F_2$ we write $\eta : F_1 \to F_2$. If $\eta_X$ is an isomorphisms for each object $X$, the natural transformation $\eta$ is called a natural isomorphism.

**2 Symmetric Monoidal Categories** The book is based on an important class of categories, the *symmetric monoidal categories*. This concept was introduced by Eilenberg and Kelly (see [6])

**Definition 2.1** $\mathbf{C} := (\mathbf{C_0}, \otimes, I, a, c, r, l)$ is said to be a *symmetric monoidal category* if the following conditions hold:

(i) $\mathbf{C_0}$ is a category.

(ii) $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor, the monoidal product of $\mathbf{C}$ (also called the tensor product or the multiplication).

(iii) $I$ is an object of $\mathbf{C}$.

(iv) $a = (a_{ABC})_{A,B,C \in |C|}$ with $a_{ABC} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$,

(v) $c = (c_{AB})_{A,B \in |C|}$ with $c_{AB} : A \otimes B \to B \otimes A$,

(vi) $r = (r_A)_{A \in |C|}$ with $r_A : A \otimes I \to A$,

(vii) $l = (l_A)_{A \in |C|}$ with $l_A : I \otimes A \to A$ are families of isomorphisms of $\mathbf{C}$ such that for arbitrary morphisms $f : A \to A', g : B \to B', h : C \to C'$ the equations:

$$\begin{array}{lcl} a_{ABC} \cdot ((f \otimes g) \otimes h) & = & (f \otimes (g \otimes h)) \cdot a_{A'B'C'}, \\ c_{AB} \cdot (g \otimes f) & = & (f \otimes g) \cdot c_{A'B'}, \\ r_A \cdot f & = & (f \otimes 1_I) \cdot r_{A'}, \\ l_A \cdot f & = & (1_I \otimes f) \cdot l_{A'} \end{array}$$

and the coherence conditions

$$\begin{array}{lcl}(1_A \otimes a_{ABC}) \cdot a_{AB\otimes CD} \otimes (a_{ABC} \otimes 1_D) & = & a_{ABC\otimes D} \cdot a_{A\otimes BCD}, \\ a_{AIB} \cdot (r_A \otimes 1_B) & = & 1_A \otimes l_B), \\ c_{AB} \cdot c_{B,A} & = & 1_{A\otimes B},\ i.e. c_{AB}^{-1} = c_{BA}, \\ a_{ABC} \cdot c_{A\otimes BC} \cdot a_{CAB} & = & (1_A \otimes c_{BC}) \cdot a_{ACB} \cdot (c_{AC} \otimes 1_B), \\ c_{AI} \cdot l_A & = & r_A\end{array}$$

are satisfied.

This means that $a, c, r, l$ are natural isomorphisms.

**3 dht-Symmetric Categories** With the aim to give an axiomatic description of the category $\mathbf{P}ar$ in [8] H.-J. Hoehnke defined the concept of a *dht-symmetric category*.

**Definition 3.1** $\mathbf{K} := (\mathbf{K_0}, \otimes, I, a, c, r, l, d, t, o)$ is said to be a *dht*-symmetric category if the following conditions hold:

(i) $(\mathbf{K_0}, \otimes, I, a, c, r, l)$ is a symmetric monoidal category.

(ii) $O$ is an object of $\mathbf{K}$, $o : I \to O$ is a morphism of $\mathbf{K}$ and $A \otimes O = O \otimes A = O$ for each $A \in |K|$.

(iii) $d = (d_A)_{A\in|K|}$ with $d_A : A \to A \otimes A$, $A \in |K|$

(iv) $t = (t_A), A \in |K|$ with $t_A : A \to I, A \in |K|$, are families of morphisms of $\mathbf{K}$ satisfying the following conditions:

(1) $d_A \cdot (f \otimes f) = f \cdot d_{A'}$ for any $f : A \to A'$;

(2) if $f \cdot g = 1_A$, then $f \cdot t_B = t_A$ for any $f : A \to B, g : B \to A$;

(3) $f = t_A \cdot o$, $g = (1_B \otimes t_o) \cdot r_B$ for any $f : A \to O, g : O \to B$.

(4) Let $G := \{a_{ABC} \mid A, B, C \in |K|\} \cup \{c_{AB} \mid A, B \in |K|\} \cup \{r_A \mid A \in |K|\} \cup \{l_A \mid A \in |K|\} \cup \{t_A \mid A \in |K|\}$. Then $f \cdot t_B = t_A$ for each $f : A \to B \in \mathbf{M_K}$ where $\mathbf{M_K}$ is the $\otimes$-closed subcategory of $\mathbf{K}$ such that $|M_K| = |K|$ which is generated by $G$.

(5) $d_A \cdot (1_A \otimes t_A) \cdot r_A = 1_A, d_A \cdot (t_A \otimes 1_A) \cdot l_A = 1_A$ for each $A \in |K|$.

(6) $d_{A\otimes B} \cdot ((1_A \otimes t_B) \otimes (t_A \otimes 1_B)) \cdot (r_A \otimes l_B) = 1_{A\otimes B}$ for any $A, B \in |K|$.

Because of (2) *dht*-symmetric categories are not defined by equations, but the authors can give an equational description of *dht*-symmetric categories. The basic example of a *dht*-symmetric category is the category $\mathbf{P}ar$ of all sets and all partial mappings between sets. Here $d$ and $t$ are the families of morphisms with $d_A : A \to A \times A$ defined by $x \mapsto (x, x)$ and $t$ with $t_A : A \to I$ and $I = \{\emptyset\}$ defined by $x \mapsto \emptyset$ for all $x \in A$. The morphism $o$ is defined by $o : I \to O$ where $O = \emptyset$. Let $Db(f)$ be the definition domain of the partial operation $f$. Then for partial mappings $f, g : A \to B$ a partial order $\leq$ is defined by

$$f \leq g :\Leftrightarrow Db(f) \subseteq Db(g) \text{ and } \forall a \in A\ (f(a) = g(a)).$$

This means, $f$ is the restriction of $g$ onto the definition domain of $f$. It is well-known, that $\leq$ is a partial order on $\mathrm{Par}(A, B)$. The partial order $\leq$ can be characterized as follows:

$$(\forall f, g : A \to B \in \mathrm{Par}) \quad f \leq g \Leftrightarrow d_A \cdot (f \otimes g) = d_A \cdot (f \otimes f).$$

This observation motivates the following definition:

**Definition 3.2** Let $\mathbf{K}$ be any dht-symmetric category. Then we denote by "$\leq_{\mathbf{K}}$" or simply by "$\leq$" if no confusion is possible, the binary relation in mor$\mathbf{K}$ defined by:

$$f \leq g \Longleftrightarrow A = A', B = B' \text{ and } d_A \cdot (f \otimes g) = d_A \cdot (f \otimes f)$$

for all $f : A \to B, g : A' \to B'$ of $\mathbf{K}$.
The binary relation $\leq$ is said to be the *canonical partial order of* $\mathbf{K}$.

Based on this canonical partial order the authors define the important concept of a *subidentity.*

**Definition 3.3** For any dht-symmetric category $\mathbf{K}$, a morphism $e$ of $\mathbf{K}$ is called a *subidentity of* $\mathbf{K}$, if there is an object $A$ of $\mathbf{K}$ such that $e \leq 1_A$.

In $\mathbf{P}ar$ subidentities can be used to describe the definition domain of a partial function.

## 4 Monoidal *dht*-symmetric functors

**Definition 4.1** For any two dht-symmetric categories $\mathbf{K}$, $\mathbf{K}'$ $F : \mathbf{K} \to \mathbf{K}'$ is said to be a *monoidal dht-symmetric functor* ( of $\mathbf{K}$ to $\mathbf{K}'$), if $F$ is a functor and there are a family $\tilde{F} = (\tilde{F}_{AB})_{A,B \in |\mathbf{K}|}$ of morphisms of $\mathbf{K}'$ and morphisms $F^1, F^0$ of $\mathbf{K}'$, such that:

(i) $\tilde{F} = \tilde{F}_{AB} : F(A) \otimes F(B) \to F(A \otimes B) \qquad (A, B \in |\mathbf{K}|)$ is a natural isomorphism, i. e., for arbitrary objects $A, B$ of $\mathbf{K}$, $\tilde{F}_{AB}$ is an isomorphism of $\mathbf{K}'$, and for arbitrary morphisms $F : A \to A', g : B \to B'$ of $\mathbf{K}$ one has

$$\tilde{F}_{AB} \cdot F(f \otimes g) = (F(f) \otimes F(g)) \cdot \tilde{F}_{A'B'},$$

(ii) $F^1 : I' \to F(I)$ is an isomorphism of $\mathbf{K}'$,

(iii) $F^0 : O' \to F(O)$ is an isomorphism of $\mathbf{K}'$.

For arbitrary objects $A, B, C$ of $\mathbf{K}$ the following equations hold:

(1) $(1_{F(A)} \otimes \tilde{F}_{BC}) \cdot \tilde{F}_{A,B\otimes C} \cdot F(a_{ABC}) = a'_{F(A)F(B)F(C)} \cdot (\tilde{F}_{AB} \otimes 1_{F(C)}) \cdot \tilde{F}_{A\otimes B,C}$,

(2) $\tilde{F}_{AB} \cdot F(c_{AB}) = c'_{F(A)F(B)} \cdot \tilde{F}_{BA}$,

(3) $(1_{F(A)} \otimes F^1) \cdot \tilde{F}_{AI} \cdot F(r_A) = r'_{F(A)}$,

(4) $(F^1 \otimes 1_{F(A)}) \cdot \tilde{F}_{AI} \cdot F(l_A) = l'_{F(A)}$,

(5) $F(d_A) = d'_{F(A)} \cdot \tilde{F}_{AA}$,

(6) $F(t_A) = t'_{F(A)} \cdot F^1$.

The concept of a monoidal *dht*-symmetric functor was introduced by H.-J. Hoehnke in [8]. Monoidal *dht*-symmetric functors are characterized as follows:

**Proposition 4.2** *For any two dht-symmetric categories* $\mathbf{K}, \mathbf{K}'$, *a functor* $F : \mathbf{K} \to \mathbf{K}'$ *is a monoidal dht-symmetric functor iff there hold:*

(1) $\hat{F} = \hat{F}_{AB} : F(A \otimes B) \to F(A) \otimes F(B) \quad (A, B \in |\mathbf{K}|)$
*is a natural isomorphism,*
(2) $t'_{F(I)} \in \mathrm{iso}\mathbf{K}'$ *and* $(\forall A \in |\mathbf{K}|): \; F(t_A) \cdot t'_{F(I)} = t'_{F(A)}$,
(3) $F(O) \cong O'$.

For any two dht-symmetric categories $\mathbf{K}, \mathbf{K}'$ and any functor $F : \mathbf{K} \to \mathbf{K}'$ the following conditions are pairwise equivalent:

(1) $F(e)$ is a subidentity of $\mathbf{K}'$ for all subidentities $e \in \mathbf{K}$.
(2) For arbitrary morphisms $f : A \to A', g : B \to B'$ of $\mathbf{K}$ one has:

$$\hat{F}_{AB} \cdot (F(f) \otimes F(g)) = F(f \otimes g) \cdot \hat{F}_{A'B'}.$$

(3) For arbitrary morphisms $f, g$ of $\mathbf{K}$ there holds:
If $f \leq g$, *then* $F(f) \leq F(g)$.

For a *dht*-symmetric category $\mathbf{K}$ the mapping

$$\alpha_{\mathbf{K}} : mor\mathbf{K} \to \{e \mid e \text{ is a subidentity of } \mathbf{K}\},$$

(for short $\alpha$) is the mapping which maps each morphisnm $f : A \to B$ to the subidentity $\alpha(f) := d_A \cdot (1_A \otimes f) \cdot p_1^{A,B}$.

**Definition 4.3** For any two *dht*-symmetric functors $\mathbf{K}$ and $\mathbf{K}'$ a functor $F : \mathbf{K} \to \mathbf{K}'$ is called an *$\alpha$-compatible functor*, if $F : \mathbf{K} \to \mathbf{K}'$ is a functor and the following identity is valid:

$$(\forall f \in \mathbf{K}) : F(\alpha(f)) = \alpha(F(f)).$$

Then $\alpha$-compatible functors were characterized as follows:

**Proposition 4.4** *For arbitrary dht-symmetric categories* $\mathbf{K}$ *and* $\mathbf{K}'$*, a functor* $F : \mathbf{K} \to \mathbf{K}'$ *is $\alpha$-compatible, iff there hold:*

(1) $\hat{F} = \hat{F}_{AB} : F(A \otimes B) \to F(A) \otimes F(B) \quad (A, B \in |\mathbf{K}'\mathbf{K}|)$ *is a natural transformation,*
(2) $F(t_A) \cdot t'_{F(I)} = t'_{F(A)}$ *for every object $A$ of* $\mathbf{K}$.

**Corollary 4.5** *For arbitrary dht-symmetric categories* $\mathbf{K}, \mathbf{K}'$ *and every functor* $F : \mathbf{K} \to \mathbf{K}'$ *one has:*

(1) *If $F$ is a monoidal dht-symmetric functor, then $F$ is an $\alpha$-compatible functor.*
(2) *If $F$ is an $\alpha$-compatible functor, then one has:*
$(\forall f, g \in \mathbf{K}) : f \leq g \Longrightarrow F(f) \leq F(g)$.

The monoidal dht-symmetric functors can be characterized in the following way as special $\alpha$-compatible functors:

**Theorem 4.6** *For arbitrary dht-symmetric categories* $\mathbf{K}$ *and* $\mathbf{K}'$, $F : \mathbf{K} \to \mathbf{K}'$ *is a monoidal dht-symmetric functor, iff there hold:*

(1) *$F$ is an $\alpha$-compatible functor,*
(2) $\hat{F}_{AB} \in \mathrm{iso}\mathbf{K}'$ *for arbitrary objects $A, B$ of* $\mathbf{K}$,
(3) $F(I) \cong I'$,
(4) $F(O) \cong O'$.

Strict monoidal *dht*-symmetric functors are defined as follows:

**Definition 4.7** Let $\mathbf{K}, \mathbf{K}'$ be dht-symmetric categories. $F : \mathbf{K} \to \mathbf{K}'$ is called a *strictly monoidal dht-symmetric functor*, if $F$ is a monoidal dht-symmetric functor and for arbitrary objects $A, B$ of $\mathbf{K}$ one has:

$$\tilde{F}_{AB} = 1_{F(A)\otimes F(B)},$$

$$F^1 = 1_{I'},$$

$$F^0 = 1_{O'}.$$

$F : \mathbf{K} \to \mathbf{K}'$ is called an *embedding* (of $\mathbf{K}$ into $\mathbf{K}'$), if $F$ is a faithful, monoidal *dht*-symmetric functor; and then $\mathbf{K}$ is said to be embeddable in $\mathbf{K}'$ , if there exists an embedding of $\mathbf{K}$ into $\mathbf{K}'$.
$F : \mathbf{K} \to \mathbf{K}'$ is called an *isomorphism*, if $F$ is a strictly monoidal dht-symmetric funtor, the mappings of objects and of morphisms are bijective; and then $\mathbf{K}$ and $\mathbf{K}'$ are called isomorphic (in signs: "$\mathbf{K} \cong \mathbf{K}'$"), if there is an isomorphism $F : \mathbf{K} \to \mathbf{K}'$.

Now the the following two categories can be considered:
*dht*-**C**at is the category with all small *dht*-symmetric categories as objects and with the class of all $\alpha$-compatible functors $F : \mathbf{K} \to \mathbf{K}'$, where $\mathbf{K}, \mathbf{K}'$ are objects of *dht*-**C**at, as morphisms. The composition of morphisms of *dht*-**C**at is induced by consecutively carrying out the functors to be composed. The category *dht*-**C**at will be called the *category of $\alpha$-compatible functors* (between small dht-symmetric categories).
*dht*-**M**on is the subcategory of *dht*-**C**at with the class of all monoidal *dht*-symmetric functors $F : \mathbf{K} \to \mathbf{K}'$, where $\mathbf{K}, \mathbf{K}$ are objects of *dht*-**C**at, as morphisms. This category is called the *category of monoidal dht-symmetric functors* (between small *dht*-symmetric categories).

**5 The Category of all Partial Arrows of a Category** Any morphism (arrow) $f : A \to B$ of the category $\mathbf{P}ar$, i. e. any mapping of a subset $X$ of $A$ (the definition domain of $f$) into the set $B$ can be considered as an ordered pair $(m, x)$ of morphisms of $\mathbf{S}et$, where $m = id_X : X \rightarrowtail A$ and $x = f : X \to B$:

$$A \overset{m}{\leftarrowtail} X \overset{x}{\to} B.$$

Here $m = id_X : X \rightarrowtail A$ is a monomorphism of $\mathbf{S}et$. The composition of morphisms of $\mathbf{P}ar$ can be expressed by means of pullbacks in the category $\mathbf{S}et$ of sets. This connection between the categories $\mathbf{S}et$ and $\mathbf{P}ar$ can be used as a model for gaining a category of partial arrows by a suitable locally small category gaining a category of partial arrows by a suitable locally small category $\mathbf{C}$ admitting pullbacks. A category $\mathbf{C}$ is called *locally small*, if for every object $A$ of $\mathbf{C}$ there is a set of monomorphisms $R$ of $\mathbf{C}$ having the source $A$, such that for each subobject $U$ of $A$, that is an equivalence class of isomorphic monomorphisms of $\mathbf{C}$, there exists exactly one $m$ in $R$ belonging to $U$.

Now the authors apply a construction of categories of partial arrows for an arbitrary category which is based on an $M$-class of a category.

**Definition 5.1** For any category $\mathbf{C}$, $\mathfrak{M}$ is an $M$-class of $\mathbf{C}$, if $\mathfrak{M}$ is a class of morphisms of $\mathbf{C}$, which is closed with respect to composition of morphisms, such that

(1) $iso\mathbf{C} \subseteq \mathfrak{M} \subseteq mono\mathbf{C}$,
(2) $\mathbf{C}$ is $\mathfrak{M}$-locally small, i. e., each class of pairwise non-isomorphc monomorphisms of $\mathbf{C}$ from $\mathfrak{M}$ having the same target, forms a set.
(3) $\mathbf{C}$ has inverse $\mathfrak{M}$-images, i.e., for arbitrary morphisms of $\mathbf{C}$, $X \xrightarrow{x} B \xleftarrow{n} Y$ where $n \in \mathfrak{M}$, there exists a cartesian square in $\mathbf{C}$ with $n' \in \mathfrak{M}$.

Examples of $M$-classes are the following ones:

(a) The class of all injective mappings between sets is an $M$-class of $\mathbf{S}et$.

(b) For every category $\mathbf{C}$ the class of all isomorphisms of $\mathbf{C}$ is an $M$-class of $\mathbf{C}$.

(c) For every locally small category having pullbacks, the class of all monomorphisms of $\mathbf{C}$, $mono\mathbf{C}$, is an $M$-class of $\mathbf{C}$.

Let $Part_{\mathfrak{M}}\mathbf{C}$ be the category of all partial arrows of $\mathbf{C}$ as defined in 2.1.2 of the book.

Let $\mathbf{C}$, $\mathbf{C}'$ be cartesian categories and let $\mathfrak{M}$, $\mathfrak{M}'$ be $M$-classes of $\mathbf{C}$ and $\mathbf{C}'$ respectively, such that all conditions are satisfied, which enable the construction of the *dht*-symmetric categories of partial morphisms of $\mathbf{C}$ and $\mathbf{C}'$ respectively.

Let $F : (\mathbf{C}, \mathfrak{M}) \longrightarrow (\mathbf{C}', \mathfrak{M}')$ be an M-functor. Then

$$Part\,F : \mathbf{P}art_{\mathfrak{M}}\mathbf{C} \longrightarrow \mathbf{P}art_{\mathfrak{M}}\mathbf{C}'$$

is an $\alpha$-compatible functor.

## 6 The Category of Subidentities of a *dht*-symmetric Category

For any category $\mathbf{K}$, let $Sub\mathbf{K}$ be the category of subidentities of $\mathbf{K}$, which is constructed as follows:

The class of objects of $Sub\mathbf{K}$ is the class of all subidentities of $\mathbf{K}$.
For subidentities $e \leq A, e' \leq A'$ of $\mathbf{K}$, $Sub\mathbf{K}\,(e, e')$ designates the corresponding set of morphisms, i. e. of all triples $(e, f, e')$, for which $f \in \mathbf{K}(A, A')$ and

$$\alpha(f) = e, f \cdot e' = f.$$

Then a composition of morphisms will be defined.

The next important result is the Embedding Theorem.

**Theorem 6.1** *For any dht-symmetric category* $\mathbf{K}$

$$H_{\mathbf{K}} : \mathbf{K} \rightarrow \mathbf{P}art(Sub\mathbf{K})$$

*is a full and faithful strict monoidal dht-symmetric functor.*

## 7 Embedding of Small *dht*-symmetric Categories into $\mathbf{P}ar$

The authors are able to characterize the category $\mathbf{P}ar$ up to categorical equivalence as a *dht*-symmetric category. To do so they give necessary and sufficient conditions for any *dht*-symmetric category $\mathbf{K}$ to admit a fully faithful representative monoidal *dht*-symmetric functor from $\mathbf{K}$ to $\mathbf{P}ar$. These conditions are given in Proposition 4.2.2 of the book.

**8 Partial Theories** In the second part of the book the theory of *dht*-symmetric categories is used to characterize partial theories, i.e. theories for many-sorted partial algebras.

For any set $\mathcal{J}$ let $\underline{H} = (H, \otimes, I)$ be a free algebra of type $(2,0)$, freely generated by $\mathcal{J}$, let $O \notin H$ be an arbitrary element, and $\underline{H}^O = (H^O, \otimes, I, O)$ for $H^O = H \cup \{O\}$ be the algebra of type $(2,0,0)$, which arises from $\underline{H}$ by adjoining $O$ as the zero element. Thus for every element $X$ of $H^O$ there holds $O \otimes X = X \otimes O = O$.

**Definition 8.1** Any dht-symmetric category $\mathbf{T}$ such that $(|\mathbf{T}|, \otimes, I, O) = \underline{H}^O$ is called a *partial theory* (*relative* $\underline{H}^O$).

Then every partial theory necessarily is a small *dht*-symmetric category. Since every small category can be defined as a multi-based algebra, also any partial theory $\mathbf{T}$ can be considered as a multi-based algebra. Its carrier family is given by $(\mathbf{T}(A,B))_{A,B \in H^O}$. The composition of morphisms $\cdot$ of $\mathbf{T}$ and the bifunctor $\otimes : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ induce families $(\cdot_{ABC})_{A,B,C \in H^O}$ and $(\otimes_{ABCD})_{A,B,C,D \in H^O}$ as defining operations:

$$\cdot_{ABC} : \mathbf{T}(A,B) \times \mathbf{T}(B,C) \to \mathbf{T}(A,C),$$

$$\otimes_{ABCD} : \mathbf{T}(A,C) \times \mathbf{T}(B,D) \to \mathbf{T}(A \otimes B, C \otimes D)$$

$(A,B,C,D \in H^O)$.
Morerover we have the following families of 0-ary defining operations:
$(1_A)_{A \in H^O}, (a_{ABC})_{A,B,C \in H^O}, (a^{-1}_{ABC})_{A,B,C \in H^O}, (c_{AB})_{A,B \in H^O}, (r_A)_{A \in H^O}, (r^{-1}_A)_{A \in H^O},$
$(l_A)_{A \in H^O}, (l^{-1}_A)_{A \in H^O}, (d_A)_{A \in H^O}, (t_A)_{A \in H^O}$, and $o$.

The authors prove that *dht*-symmetric categories can be characterized by a system of axioms, which exclusively consists of equations. Hence the class $\mathcal{T}h$ of all partial theories with respect to $H^O$, considered as multi-based algebras, forms a variety (equationally defined class) of multi-based algebras. Thus for any set $J$ and a family $(\mathbf{T}_j)_{j \in J}$ of partial theories of $\mathcal{T}h$, the direct product $(\times)_{j \in J}\, \mathbf{T}_j$ of the family $(\mathbf{T}_j)_{j \in J}$ (in the sense of the theory of multi-based algebras) again is a partial theory. Moreover each subalgebra of a partial theory of $\mathcal{T}h$ is again a partial theory, and $\mathcal{T}h$ is closed with respect to homomorphic images of partial theories of $\mathcal{T}h$.

The first main result gives a representation of partial theories.

**Theorem 8.2** *Every partial theory* $\mathbf{T}$ *is isomophic to a subdirect product of partial theories which are embeddable into* $\mathbf{P}ar$.

Let $\mathcal{T}h_0$ be the subclass of the variety $\mathcal{T}h$ consisting of those partial theories of $\mathcal{T}h$ which are embeddable into the *dht*-symmetric category $\mathbf{P}ar$, i. e. $\mathcal{T}h_0$ is the class of all partial theories $\mathbf{T}'$ relative to $H^O$, for which there is a faithful monoidal *dht*-symmetric functor from $\mathbf{T}'$ to Par. Then we have:

**Theorem 8.3** *The variety* $\mathcal{T}h$ *is generated by* $\mathcal{T}h_0$.

In chapter 5 the authors describe how free binary partial theories can be obtained by generators and relations. The main tools for this presentation are taken from braid theory and the Daile monoid introduced in [5]. In her thesis from 1956 (advisor: B. H. Neumann) E. C. Dale characterized varieties of algebras by a certain monoid and by braids. Hoehnke generalized this to varieties of many-sorted partial algebras.

**9 Partial Theories and Mal’cev Clones** In the case of total algebras, i.e. algebras with everywhere defined fundamental operations F. W. Lawvere described varieties by “Lawvere-Theories”. These theories are dually isomorphic to the clones of those varieties.

Clones (of total operations) are sets of operations defined on the same set $A$ which are closed under composition and contain all projection operations. But clones are understood also in a more general sense. It turns out that clones of operations form many-sorted algebras. They belong to a variety of many-sorted algebras whose members are called abstract clones. In this sense not only operations, but also partial operations, sets of operations, terms, sets of terms, cooperations and relations form clones. The notion of a clone was first used in P. M. Cohn’s book “Universal Algebra”, but probably it goes back to Ph. Hall. The concept of a clone is one of the basic concepts of General Algebra and has applications in the theory of data bases and in other fields of Theoretical Computer Science.

Composition of partial operations can be described by operations $\bar{S}^n_m$ as follows:

$$\bar{S}^n_m(f, g_1, \dots, g_n) := f(g_1(a_1, \dots, a_m), \dots, g_n(a_1, \dots, a_m))$$

for all $(a_1, \dots, a_m)$ for which $g_1, \dots, g_n$ are defined and for which the values $b_1 = g_1(a_1, \dots, a_m), \dots, b_n = g_n(a_1, \dots, a_m)$ form an $n$-tuple $(b_1, \dots, b_n)$ belonging to the domain of $f$. Then one can consider the many-sorted algebra $((P^n(A))_{n\geq 1}; (\bar{S}^n_m)_{m,n\geq 1})$. If we add a family of nullary operation symbols, corresponding to the total projections, then we obtain a many-sorted algebra

$$P - cloneA := ((P^n(A))_{n\geq 1}; (\bar{S}^n_m)_{m,n\geq 1}, (e^n_i)_{n\geq 1, 1\leq i\leq n}).$$

This algebra satisfies the following axioms (C1) and (C3).

(C1) $\tilde{S}^p_m(\tilde{Z}, \tilde{S}^n_m(\tilde{Y}_1, \tilde{X}_1, \dots, \tilde{X}_n), \dots, \tilde{S}^n_m(\tilde{Y}_p, \tilde{X}_1, \dots, \tilde{X}_n))$
$\approx \tilde{S}^n_m(\tilde{S}^p_n(\tilde{Z}, \tilde{Y}_1, \dots, \tilde{Y}_p), \tilde{X}_1, \dots, \tilde{X}_n),\ (m, n, p = 1, 2, \dots),$

(C3) $\tilde{S}^n_n(\tilde{Y}, \tilde{e^n_1}, \dots, \tilde{e^n_n}) \approx \tilde{Y},\ (n = 1, 2, \dots).$

Here $\tilde{Z}, \tilde{Y}_1, \dots, \tilde{Y}_p, \tilde{X}_1, \dots, \tilde{X}_n$, are variables for terms, $\tilde{S}^n_m, \tilde{S}^p_m, \tilde{S}^p_n, \tilde{S}^n_n$ are operation symbols and $\tilde{e^n_i}$ are symbols for variables.

The axiom (C1) is called the *superassociative law*. It generalizes the associative law which we obtain from (C1) if we set for $m, n$ and $p$ the integer 1.

One can check that the following identities are also satisfied:

(R) $\bar{S}^n_n(f, \bar{S}^2_n(e^2_1, e^n_1, g), \dots, \bar{S}^2_n(e^2_1, e^n_n, g)) \approx \bar{S}^2_n(e^2_1, f, g),$

(L1) $\bar{S}^1_n(e^1_1, f) = f,$

(L2) $\bar{S}^2_n(e^2_1, f, e^n_i) = f, 1 \leq i \leq n,$

(L3) $\bar{S}^2_n(e^2_1, \bar{S}^2_n(e^2_1, f, g), h) \approx \bar{S}^2_n(e^2_1, f, \bar{S}^2_n(e^2_1, g, h)),$

(L4) $\bar{S}^n_m(e^n_{i+1}, g_n, g_0, \dots, g_{n-1}) \approx \bar{S}^n_m(e^n_i, g_1, \dots, g_n), 1 \leq i \leq n,$

(L5) $\bar{S}^2_m(e^2_1, g_1, \bar{S}^n_m(e^n_1, g_1, \dots, g_n)).$

In [1] a many-sorted algebra of the corresponding type was called an abstract $P$-clone algebra if (C1), (R), (L1), (L2), (L3), (L4), (L5) are satisfied. The algebra $P - cloneA$ is an abstract $P$-clone algebra. $P$-clone algebras consisting of partial operations are called concrete $P$-clone algebras. Answering to a question of H. J. Hoehnke (posed on occasion of the 4-th Conference for Young Algebraists, Potsdam 1988), F. Börner proved in [1] that any abstract $P$-clone algebra is isomorphic to a subdirect product of concrete ones. This result is a universal-algebraic version of the representation theorem (Theorem 8.2).

Another important example of a clone is the clone of all terms of a given type $\tau$.

We define *terms of type* $\tau$ using an indexed sequence $(f_i)_{i\in I}$ of operation symbols and individual variables from an alphabet $X$. Let $X_n := \{x_1, \ldots, x_n\}$ be a finite alphabet and let $X := \{x_1, \ldots, x_n, \ldots\}$ be countably infinite. To every operation symbol $f_i$ there belongs an integer $n_i$ as its *arity*. The type of the formal language which we want to define is the indexed set $(n_i)_{i\in I}$ of the arities. We define $n$-ary terms of type $\tau$ as follows:

**Definition 9.1** Let $n \geq 1$. The *n-ary terms* of type $\tau$ are defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an $n$-ary term.

(ii) If $t_1, \ldots, t_{n_i}$ are $n$-ary terms and $f_i$ is an $n_i$-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an $n$-ary term.

*(iii)* The set $W_\tau(X_n) = W_\tau(x_1, \ldots, x_n)$ of all $n$-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

We denote by $W_\tau(X)$ the set of all terms of type $\tau$ over the countably infinite alphabet $X$:

$$W_\tau(X) \quad = \quad \bigcup_{n\geq 1} W_\tau(X_n).$$

**Definition 9.2** Let $W_\tau(X_n)$ be the set of all $n$-ary terms of type $\tau$. Then the composition operations $\bar{S}^n_m$ (for terms) are inductively defined by the following steps:

(i) If $x_j \in X_n$ is a variable and $t_1, \ldots, t_n \in W_\tau(X_m)$, then
$\bar{S}^n_m(x_j, t_1, \ldots, t_n) := t_j$, for $1 \leq j \leq n$;

(ii) If $f_i(s_1, \ldots, s_{n_i})$ is a composite term, then
$\bar{S}^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(\bar{S}^n_m(s_1, t_1, \ldots, t_n), \ldots, \bar{S}^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then we obtain a $\mathbb{N}^+$-sorted algebras

$$\mathcal{C}lone\tau := ((W_\tau(X_n))_{n\in\mathbb{N}^+}; (\bar{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\leq i\leq n, n\in\mathbb{N}^+}),$$

the term clone of type $\tau$. This algebra satisfies (C1) and (C3).

It turns out that terms for partial algebras $\underline{A} = (A; (f_i^{\underline{A}})_{i\in I})$, where $f_i^{\underline{A}}$ is an $n_i$-ary partial operation defined on $A$ should be defined by adding one more set of symbols (for projection mappings). This gives the set $W^C_\tau(X)$ of $C$-terms. These terms were introduced by W. Craig (see [4]). Welke defined in [12] many-sorted superposition operations for $C$-terms which satisfy the superassociative law (C1).

In the next chapter the authors study "clone theory" in the language of *dht*-symmetric categories.

**10 The Clone-part of a Theory** For any dht-symmetric category $\mathbf{C}$ and any subset $G \subseteq \mathbf{C}$, the authors introduce the notion of a "$\mathbf{C}$-clone generated by $G$":
First one chooses a set $\mathcal{J}$ of objects of $\mathbf{C}$ such that any $g \in G$ is of the form $g: A \to B$, where $A$ and $B$ belong to the submonoid with zero $K^o_{\mathcal{J}}$ of $(|\mathbf{C}|, \otimes, I, O)$, which is generated by $\mathcal{J}$. Second one forms the full subcategory $\mathbf{C}(K^o_{\mathcal{J}})$ of $\mathbf{C}$ such that $|\mathbf{C}(K^o_{\mathcal{J}})| = K^o_{\mathcal{J}}$; this is a *dht*-symmetric category. Third one considers the *dht*-symmetric subcategory $G_{\mathbf{C}}(K^o_{\mathcal{J}})$ of $\mathbf{C}(K^o_{\mathcal{J}})$ which is generated by $G$. The $\mathbf{C}$-*clone generated by* $G$ is the set $\langle G\rangle^{\mathbf{C}}_{\mathcal{J}}$ of all $f: A \to B \in G_{\mathbf{C}}(K^o_{\mathcal{J}})$ such that $B \in \mathcal{J} \cup \{I, O\}$. For $\langle G\rangle^{\mathbf{C}}_{\mathcal{J}} = G$, the set $G$ itself is called a $\mathbf{C}$-*clone*. The only essential case is that one where the monoid with zero $K^o_{\mathcal{J}}$ is freely generated by $\mathcal{J}$. This is assumed henceforth.

These notions apply to $\mathbf{C} = \underline{T} \in |\underline{\mathcal{T}h_{\mathcal{J}}}|$, and then one also speaks of an *abstract Mal'cev clone* $G$. Further, these notions apply to $\mathbf{C} = \mathbf{P}ar$. Instead of a $\mathbf{P}ar$-clone $G$, one speaks of a *concrete Mal'cev clone* $G$.

Abstract and concrete Mal'cev clones are commonly denoted as *Mal'cev clones*. (Of course, each concrete Mal'cev clone is an abstract one.)

Let $\underline{T}$ be any partial theory and $\underline{K}$ be an arbitrary *dht*-symmetric category. Then the monoidal *dht*-symmetric functors $\underline{F}: \underline{T} \to \mathbf{K}$ can be considered as the natural functorial description of the notion of a $\underline{T}$-*algebra over* $\mathbf{K}$. Especially for $\mathbf{K} = \mathbf{P}ar$ one obtains thus the natural functorial description of a *partial* $\underline{T}$-*algebra*.

For any functorial $\underline{T}$-algebra over $\mathbf{K}$, $\underline{F}: \underline{T} \to \mathbf{K}$, the authors introduce the arity-interpretation ("typization") of $\underline{T}\,F \subseteq \mathbf{K}$, which is again a partial theory, Ary $\underline{\text{Im}}$ $\underline{F}$.

One observes, that each $\mathbf{C}$-clone $G$ by assumption generates a partial theory $G_{\mathbf{C}(K^o_{\mathcal{J}})} = \underline{T} \in |\underline{\mathcal{T}h_{\mathcal{J}}}|$, and then $G$ is also called the *(Mal'cev) clone-part of* $\underline{T}$. One may consider the problem to characterize the clone-part of a partial theory $\underline{T}$ independently of its connection to $\underline{T}$.

If $\underline{T} \in |\mathcal{T}h_{\mathcal{J}}|$ and $\underline{F}: \underline{T} \to \underline{K}$ is a $\underline{T}$-algebra over $\underline{K}$, then we define

$$\text{Mcl } \underline{F} := \{(A, fF, B);\ f: A \to B \in \underline{T},\ B \in \mathcal{J} \cup \{I, O\}\}.$$

This is the clone-part of the partial theory Ary$\underline{\text{Im}}\,\underline{F}$. It is also called the *Mal'cev clone of the* $\underline{T}$-*algebra* $\underline{F}$. In particular this notion applies to the case $\underline{K} = \mathbf{P}ar$ . In this case, Mcl $\underline{F}$ is obviously a concrete Mal'cev clone. Observe, that $\underline{F}$ maps the clone-part Mcl $Id_{\underline{T}}$ of $\underline{T}$ onto Mcl $\underline{F}$. The connection between the structures of partial theories $\underline{T}$ and their clone-parts Mcl $\underline{T}$ can be derived from the clone-part functor $\underline{Cp} = (|Cp|, Cp): \underline{\mathcal{T}h_{\mathcal{J}}} \to \underline{Mcl_{\mathcal{J}}}$ , which is defined by:

$$\begin{aligned} \underline{T}|Cp| &= \text{Mcl } \underline{T}\ (\underline{T} \in |\underline{\mathcal{T}h_{\mathcal{J}}}|), \\ \underline{F}\ Cp &= F\big|_{\text{Mcl } \underline{T}}\ (\underline{F} = (|F|, F): \underline{T} \to \underline{T}' \in \underline{\mathcal{T}h_{\mathcal{J}}}). \end{aligned}$$

The clone-part functor is fully faithful. This has several consequences which are described in section 7.2 of the book. In section 8 the authors study clones which are derived from free algebras.

**11 Strong Varieties, Regular Hyperidentities and Solid Strong Varieties** Strong varieties of partial algebras of the same type are model classes of partial algebras defined by so-called strong identities. A strong identity in a partial algebra $\underline{A}$ is a pair $(s, t) \in$

$(W_\tau^C(X))^2$ such that the realization of $s$ on $\underline{A}$ is defined if and only if the realization of $t$ on $\underline{A}$ is defined and both are equal. An identity is said to be a strong hyperidentity if $(s,t)$ is satisfied as strong identity after each replacement of $n_i$-ary operation symbols $f_i$ occurring in $s$ and in $t$ by terms containg all variables $x_1, \ldots, x_{n_i}$. Such replacements can be performed by so-called regular hypersubstitutions. If every identity satisfied in $\underline{A}$ is satisfied as a strong hyperidentity, the strong variety is called solid. The theory of strong hyperidentities and solid varieties is studied on an universal-algebraic level in [**?**] and [3] (see also [2]). The authors can formulate this theory in terms of *dht*-symmetric theories. Other highlights of the book are the formulation of the one-point extension of partial algebras to total ones in 9.6 and the main theorem on strong hypervarieties in 9.9.

## References


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Universität Potsdam,
Institut für Mathematik,
Potsdam, Germany,
e-mail:kdenecke@rz.uni-potsdam.de