Scientiae Mathematicae Japonicae Online, **e-2008**, 439–451

THE BEST CONSTANT OF SOBOLEV INEQUALITY CORRESPONDING TO DIRICHLET BOUNDARY VALUE PROBLEM FOR $(-1)^M(d/dx)^{2M}$

Yoshinori Kametaka*, Hiroyuki Yamagishi†, Kohtaro Watanabe‡, Atsushi Nagai§ and Kazuo Takemura¶

Received July 17, 2007; revised July 22, 2008

Abstract. For $M = 1, 2, 3, \cdots$, the best constant of Sobolev inequality

$$\left(\sup_{|y|\le 1} |\, u(y)\,|\right)^2 \le C \int_{-1}^{1} \left| u^{(M)}(x) \right|^2 dx$$

for $u(x)$ satisfying $u^{(2i)}(\pm 1) = 0$ $(0 \le i \le [\,(M-1)/2\,])$ is given by

$$C(M) = \left(2^{2M} - 1\right) \pi^{-2M} \zeta(2M)$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (Re $s > 1$) is Riemann zeta function.

1 Conclusion

For $M = 1, 2, 3, \cdots$, we introduce Sobolev space

$$H = H(M) = \left\{ u(x) \,\middle|\, u(x),\ u^{(M)}(x) \in L^2(-1,1), \right.$$
$$\left. u^{(2i)}(\pm 1) = 0 \quad (0 \le i \le [\,(M-1)/2\,]) \right\} \tag{1.1}$$

Sobolev inner product

$$(u, v)_M = \int_{-1}^{1} u^{(M)}(x)\, \overline{v}^{(M)}(x)\, dx \tag{1.2}$$

Sobolev energy

$$\|\, u\,\|_M^2 = \int_{-1}^{1} \left| u^{(M)}(x) \right|^2 dx \tag{1.3}$$

and Sobolev functional

$$S(u) = S(M; u) = \left(\sup_{|y|\le 1} |\, u(y)\,|\right)^2 \Big/ \|\, u\,\|_M^2 \tag{1.4}$$

2000 *Mathematics Subject Classification.* 46E35, 41A44, 34B27.

Key words and phrases. Sobolev inequality, Best constant, Green function, Reproducing kernel, Bernoulli polynomial, Riemann zeta function .

$(\cdot,\cdot)_M$ is proved to be an inner product of H afterwards. H is Hilbert space with inner product $(\cdot,\cdot)_M$.

The purpose of this paper is to find the supremum of Sobolev functional $S(u)$. Our conclusion is as follows.

Theorem 1.1 *$G(x,y)$ is Green function defined later in Theorem 2.1, 2.4.*

(1) $\displaystyle\sup_{u\in H,\ u\not\equiv 0} S(u) = C_0$ *is given by*

$$C_0 \;=\; C(M) \;=\; \max_{|y|\le 1} G(y,y) \;=\; G(0,0) \;=\; \left(2^{2M}-1\right)\pi^{-2M}\zeta(2M) \tag{1.5}$$

$$C(1) = 1/2, \qquad C(2) = 1/6, \qquad C(3) = 1/15,$$
$$C(4) = 17/630, \qquad C(5) = 31/2835, \qquad C(6) = 691/155925,$$
$$C(7) = 10922/6081075, \qquad C(8) = 929569/1277025750, \qquad \cdots$$

(2) $$S(\,G(x,0)\,) \;=\; C_0 \tag{1.6}$$

(3) $$\inf_{u\in H,\ u\not\equiv 0} S(u) \;=\; 0 \tag{1.7}$$

The equality (1.7) in the above theorem is easily proved as follows.

$$S\Big(\sin(\,n\pi x\,)\Big) \;=\; (n\pi)^{-2M} \quad\longrightarrow\quad 0 \qquad (n\to\infty)$$

The above Theorem 1.1 is rewritten equivalently as follows.

Theorem 1.2 *For any function $u(x)\in H$, there exists a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$\left(\sup_{|y|\le 1}|u(y)|\right)^2 \;\le\; C\int_{-1}^{1}\left|u^{(M)}(x)\right|^2 dx \tag{1.8}$$

Among such C the best constant C_0 is the same as that in Theorem 1.1(1).

If we replace C by C_0 in (1.8), the equality holds for

$$u(x) \;=\; c\,G(x,0) \qquad (-1<x<1) \tag{1.9}$$

for every complex number c.

The engineering meaning of Sobolev inequality is that the square of the maximum bending of a string ($M=1$) or a beam ($M=2$) is estimated from above by the constant multiple of the potential energy.

This paper is organized as follows. In section 2, we consider a boundary value problem for $(-1)^M(d/dx)^{2M}$ with Dirichlet boundary condition. In section 3, we show that Green function $G(x,y)$ is expressed in terms of Bernoulli polynomials. In section 4, it is clarified that Green function $G(x,y)$ is a reproducing kernel for H and $(\cdot,\cdot)_M$. Finally, section 5 is devoted to the proof of Theorem 1.2.

2 Dirichlet boundary value problem

We consider the following Dirichlet boundary value problem.

BVP (M)

$$\begin{cases} (-1)^M u^{(2M)} \;=\; f(x) & (-1<x<1) \quad (2.1) \\ u^{(2i)}(\pm 1) \;=\; 0 & (0 \le i \le M-1) \quad (2.2) \end{cases}$$

For later convenience sake, we introduce the following monomials $\{K_j(x)\}$.

$$K_j(x) \;=\; K_j(M;x) \;=\; \begin{cases} x^{2M-1-j} \big/ (2M-1-j)! & (0 \le j \le 2M-1) \\ 0 & (2M \le j) \end{cases} \tag{2.3}$$

Concerning the uniqueness and existence of the solution to BVP(M), we have the following theorem.

Theorem 2.1 *For any bounded continuous function $f(x)$ on an interval $-1<x<1$, BVP(M) has a unique classical solution $u(x)$ expressed as follows.*

$$u(x) \;=\; \int_{-1}^{1} G(x,y)\, f(y)\, dy \qquad (-1<x<1) \tag{2.4}$$

Green function $G(x,y) = G(M;x,y)$ is given by the following two equivalent expressions.

(1) $$G(x,y) \;=\; \frac{(-1)^M}{2}\Big[K_0(\,|x-y|\,) \;-$$
$$\big(\ K_{2j}\ \big)(1+x) \left(\ K_{2(i+j)}\ \right)^{-1}(2) \left(K_{2i} \right)(1-y) \;-$$
$$\big(\ K_{2j}\ \big)(1-x) \left(\ K_{2(i+j)}\ \right)^{-1}(2) \left(K_{2i} \right)(1+y)\ \Big] \qquad (-1<x,y<1) \tag{2.5}$$

$\left(\ K_{2(i+j)}\ \right)^{-1}(2)$ *is the inverse of the $M \times M$ matrix* $\left(\ K_{2(i+j)}\ \right)(2)$ $(0 \le i,j \le M-1)$.

(2) $$G(x,y) \;=\; \frac{(-1)^M}{2}\Big[K_0(\,|x-y|\,) \;+$$
$$\kappa^{-1}\left\{ \begin{vmatrix} K_{2(i+j)}(2) & K_{2i}(1-y) \\ \hline K_{2j}(1+x) & 0 \end{vmatrix} + \begin{vmatrix} K_{2(i+j)}(2) & K_{2i}(1+y) \\ \hline K_{2j}(1-x) & 0 \end{vmatrix} \right\}\Big]$$
$$(-1<x,y<1) \tag{2.6}$$

where $\kappa = (-1)^{M(M-1)/2} 2^M$ *is the determinant of $M \times M$ matrix* $\left(\ K_{2(i+j)}\ \right)(2)$.

Proof of Theorem 2.1 The equivalence between (1) and (2) of Theorem 2.1 is shown from the following well-known lemma.

Lemma 2.1 *For any $N \times N$ regular matrix $\boldsymbol{A}$ and $N \times 1$ matrices $\boldsymbol{a} = {}^t(\cdots, a_j, \cdots)$ and $\boldsymbol{b} = {}^t(\cdots, b_j, \cdots)$ the following relation holds.*

$$
{}^t\boldsymbol{a}\boldsymbol{A}^{-1}\boldsymbol{b} = -\left|\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{b} \\ \hline {}^t\boldsymbol{a} & 0 \end{array}\right| \Big/ \left| \boldsymbol{A} \right| =
$$

$$
-\left|\begin{array}{c|c} a_{ij} & b_i \\ \hline a_j & 0 \end{array}\right| \Big/ \left| a_{ij} \right| \qquad (0 \le i, j \le N-1)
$$

Now we proceed to prove the main part of Theorem 2.1. We suppose that BVP(M) has a classical solution $u(x)$. Introducing new functions

$$
\boldsymbol{u} = {}^t(u_0, \cdots, u_{2M-1}), \qquad u_i = u^{(i)} \quad (0 \le i \le 2M-1) \tag{2.7}
$$

$$
\boldsymbol{N} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \qquad (2M \times 2M \text{ nilpotent matrix}) \tag{2.8}
$$

(2.1) is rewritten as follows

$$
\boldsymbol{u}' = \boldsymbol{N}\,\boldsymbol{u} + {}^t(0, \cdots, 0, 1)\,(-1)^M f(x) \qquad (-1 < x < 1) \tag{2.9}
$$

The fundamental solution $\boldsymbol{E}(x)$ to the above initial value problem is expressed as follows

$$
\boldsymbol{E}(x) = \boldsymbol{K}(x)\,\boldsymbol{K}(0)^{-1} \tag{2.10}
$$

where

$$
\boldsymbol{K}(x) = \Big(K_{i+j} \Big)(x) \qquad (0 \le i, j \le 2M-1) \tag{2.11}
$$

$$
\boldsymbol{K}(0) = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} = \boldsymbol{K}(0)^{-1} \tag{2.12}
$$

Solving (2.9), we have

$$
\boldsymbol{u}(x) = \boldsymbol{E}(x+1)\,\boldsymbol{u}(-1) + \int_{-1}^{x} \boldsymbol{E}(x-y)\,{}^t(0, \cdots, 0, 1)\,(-1)^M f(y)\,dy \tag{2.13}
$$

$$
\boldsymbol{u}(x) = \boldsymbol{E}(x-1)\,\boldsymbol{u}(1) - \int_{x}^{1} \boldsymbol{E}(x-y)\,{}^t(0, \cdots, 0, 1)\,(-1)^M f(y)\,dy \tag{2.14}
$$

or equivalently, for $0 \le i \le 2M-1$,

$$
u_i(x) = \sum_{j=0}^{2M-1} K_{i+j}(x+1)\,u_{2M-1-j}(-1) + \int_{-1}^{x} (-1)^M K_i(x-y)\,f(y)\,dy \tag{2.15}
$$

$$
u_i(x) = \sum_{j=0}^{2M-1} K_{i+j}(x-1)\,u_{2M-1-j}(1) - \int_{x}^{1} (-1)^M K_i(x-y)\,f(y)\,dy \tag{2.16}
$$

Employing the boundary conditions $u_{2i}(\pm 1) = 0$ $(0 \le i \le M-1)$, we have

$$u_i(x) = \sum_{j=0}^{M-1} K_{i+2j}(x+1)\, u_{2(M-1-j)+1}(-1) \,+\, \int_{-1}^{x} (-1)^M K_i(x-y)\, f(y)\, dy \tag{2.17}$$

$$u_i(x) = \sum_{j=0}^{M-1} K_{i+2j}(x-1)\, u_{2(M-1-j)+1}(1) \,-\, \int_{x}^{1} (-1)^M K_i(x-y)\, f(y)\, dy \tag{2.18}$$

for $0 \le i \le 2M-1$. In particular if $i = 0$, we have

$$u_0(x) \,=\, \sum_{j=0}^{M-1} K_{2j}(x+1)\, u_{2(M-1-j)+1}(-1) \,+\, \int_{-1}^{x} (-1)^M K_0(x-y)\, f(y)\, dy \tag{2.19}$$

$$u_0(x) \,=\, \sum_{j=0}^{M-1} K_{2j}(x-1)\, u_{2(M-1-j)+1}(1) \,-\, \int_{x}^{1} (-1)^M K_0(x-y)\, f(y)\, dy \tag{2.20}$$

Using the boundary conditions $u_{2i}(\pm 1) = 0$ $(0 \le i \le M-1)$ again, we have

$$0 \,=\, u_{2i}(1) \,=\, \sum_{j=0}^{M-1} K_{2(i+j)}(2)\, u_{2(M-1-j)+1}(-1) \,+\, \int_{-1}^{1} (-1)^M K_{2i}(1-y)\, f(y)\, dy \tag{2.21}$$

$$0 \,=\, u_{2i}(-1) \,=\, \sum_{j=0}^{M-1} K_{2(i+j)}(-2)\, u_{2(M-1-j)+1}(1) \,-\, \int_{-1}^{1} (-1)^M K_{2i}(-1-y)\, f(y)\, dy \tag{2.22}$$

Solving the above linear system of equations with respect to $u_{2(M-1-i)+1}(-1)$, $u_{2(M-1-i)+1}(1)$ $(0 \le i \le M-1)$, we have

$$\Big(u_{2(M-1-i)+1}\Big)(-1) \,=\, -\int_{-1}^{1} (-1)^M \Big(\; K_{2(i+j)} \;\Big)^{-1}(2) \Big(K_{2i}\Big)(1-y)\, f(y)\, dy \tag{2.23}$$

$$\Big(u_{2(M-1-i)+1}\Big)(1) \,=\, \int_{-1}^{1} (-1)^M \Big(\; K_{2(i+j)} \;\Big)^{-1}(-2) \Big(K_{2i}\Big)(-1-y)\, f(y)\, dy \tag{2.24}$$

Substituting (2.23) and (2.24) into (2.19) and (2.20), we have

$$u_0(x) \,=\, -\int_{-1}^{1} (-1)^M (\;\; K_{2j} \;\;)(x+1) \Big(\; K_{2(i+j)} \;\Big)^{-1}(2) \Big(K_{2i}\Big)(1-y)\, f(y)\, dy \,+$$

$$\int_{-1}^{x} (-1)^M K_0(\,|x-y|\,)\, f(y)\, dy \tag{2.25}$$

$$u_0(x) \;=\; \int_{-1}^{1} (-1)^M \begin{pmatrix} & K_{2j} & \end{pmatrix} (x-1) \begin{pmatrix} & & \\ & K_{2(i+j)} & \\ & & \end{pmatrix}^{-1} (-2) \begin{pmatrix} \\ K_{2i} \\ \\ \end{pmatrix} (-1-y)\, f(y)\, dy \,+$$

$$\int_{x}^{1} (-1)^M K_0(\,|x-y|\,)\, f(y)\, dy \tag{2.26}$$

Taking an average of the above two expressions, we have obtained the expression for a solution $u(x) = u_0(x)$ to BVP(M).

$$u(x) \;=\; \int_{-1}^{1} G(x,y)\, f(y)\, dy \qquad (-1 < x < 1) \tag{2.27}$$

where

$$\begin{aligned} G(x,y) \;=\; & \frac{(-1)^M}{2} \Bigg[K_0(\,|x-y|\,) \;- \\ & \begin{pmatrix} K_0, K_2, \cdots, K_{2M-2} \end{pmatrix} (x+1) \begin{pmatrix} & & \\ & K_{2(i+j)} & \\ & & \end{pmatrix}^{-1} (2) \begin{pmatrix} K_0 \\ K_2 \\ \vdots \\ K_{2M-2} \end{pmatrix} (1-y) \,+ \\ & \begin{pmatrix} K_0, K_2, \cdots, K_{2M-2} \end{pmatrix} (x-1) \begin{pmatrix} & & \\ & K_{2(i+j)} & \\ & & \end{pmatrix}^{-1} (-2) \begin{pmatrix} K_0 \\ K_2 \\ \vdots \\ K_{2M-2} \end{pmatrix} (-1-y) \Bigg] \\ & (-1 < x, y < 1) \end{aligned} \tag{2.28}$$

Theorem 2.1 (1) follows immediately from the relation $K_{2i}(-x) = -K_{2i}(x)$ $(i = 0, 1, 2, \cdots)$.

Since the right-hand side of (2.27) includes only a data function $f(x)$, the solution to BVP(M) is unique. Using the properties (2.29), (2.30) and (2.31) of the following theorem, we can show that $u(x)$ defined by (2.27) satisfies BVP(M), which guarantees the existence of the solution. ■

Theorem 2.2 *Green function $G(x,y) = G(M; x, y)$ satisfies the following conditions.*

(1) $$\partial_x^{2M} G(x,y) \;=\; 0 \qquad (-1 < x, y < 1, \quad x \neq y) \tag{2.29}$$

(2) $$\partial_x^{2i} G(x,y)\Big|_{x=\pm 1} \;=\; 0 \qquad (0 \le i \le M-1, \quad -1 < y < 1) \tag{2.30}$$

(3) $$\partial_x^{i} G(x,y)\Big|_{y=x-0} - \;\partial_x^{i} G(x,y)\Big|_{y=x+0} \;=\; \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \quad (-1 < x < 1) \end{cases} \tag{2.31}$$

(4) $$\partial_x^{i} G(x,y)\Big|_{x=y+0} - \;\partial_x^{i} G(x,y)\Big|_{x=y-0} \;=\; \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \quad (-1 < y < 1) \end{cases} \tag{2.32}$$

(5) $$G(x,y) \;>\; 0 \qquad (-1 < x, y < 1) \tag{2.33}$$

Proof of Theorem 2.2 By rewriting Green function $G(x,y)$ in the form (2.6), it is easy to show that $G(x,y)$ satisfies properties (1) $\sim$ (4) through direct calculation. We only give a proof of (5) by induction with respect to M. If $M=1$, we have

$$G(1;x,y)=\frac{1}{2}(1+x\wedge y)(1-x\vee y)>0 \qquad (-1<x,y<1)$$

For every fixed y $(-1<y<1)$, $u(x)=G(M;x,y)$ $(M=2,3,\cdots)$ satisfies

$$\begin{cases} -u'' = G(M-1;x,y) & (-1<x<1) \\ u(\pm 1) = 0 \end{cases}$$

We can show

$$u(x)=G(M;x,y)=\int_{-1}^{1} G(1;x,z)G(M-1;z,y)dz \;>\; 0 \qquad (-1<x<1)$$

from the inequality $G(M-1;x,y)>0$ $(-1<x,y<1)$. ■

Concerning the uniqueness of Green function, we have the following theorem.

Theorem 2.3 *The smooth function $G(x,y)$ on an open set $-1<x,y<1,\ x\neq y$ satisfying properties* (2.29)*,* (2.30) *and* (2.31) *is unique.*

Proof of Theorem 2.3 Suppose that we have another function $\widetilde{G}(x,y)$ satisfying the same properties (2.29), (2.30) and (2.31). For any function $f(x)$

$$u(x) \;=\; \int_{-1}^{1}\widetilde{G}(x,y)\,f(y)\,dy \qquad (-1<x<1) \tag{2.34}$$

satisfies BVP(M). From Theorem 2.1, we have

$$\int_{-1}^{1}\widetilde{G}(x,y)\,f(y)\,dy \;=\; \int_{-1}^{1}G(x,y)\,f(y)\,dy \qquad (-1<x<1) \tag{2.35}$$

This shows $\widetilde{G}(x,y)=G(x,y)$ $(-1<x,y<1)$. ■

Next we express Green function $G(x,y)$ in terms of Bernoulli polynomials $b_n(x)$ defined by the following relation.

$$b_n'(x) \;=\; b_{n-1}(x), \qquad \int_0^1 b_n(x)\,dx \;=\; 0 \quad (n=1,2,3,\cdots), \qquad b_0(x) \;=\; 1 \tag{2.36}$$

Theorem 2.4 *Green function $G(x,y)=G(M;x,y)$ is expressed as*

(1) $$G(x,y) \;=\; (-1)^{M+1}\,4^{2M-1}\left[\; b_{2M}\left(\frac{|x-y|}{4}\right) \;-\; b_{2M}\left(\frac{2-x-y}{4}\right)\;\right] \qquad (-1<x,y<1) \tag{2.37}$$

where $b_{2M}(x)$ is Bernoulli polynomial of order $2M$.

(2) $$G(x,y) \;=\; 2^{-1}\sum_{j=1}^{\infty}(\pi j/2)^{-2M}\left[\;\cos(\,\pi j(x-y)/2\,) \;-\; \cos(\,\pi j(2-x-y)/2\,)\;\right] \;=\; \sum_{j=1}^{\infty}(\pi j/2)^{-2M}\,\sin(\,\pi j(1+x)/2\,)\,\sin(\,\pi j(1+y)/2\,) \qquad (-1<x,y<1) \tag{2.38}$$

Proof of Theorem 2.4 (2) follows from (1) by Fourier series expansion of Bernoulli polynomial

$$(-1)^{M+1}\, b_{2M}(x) \;=\; 2\sum_{j=1}^{\infty}(2\pi j)^{-2M}\ \cos(2\pi jx) \qquad (0<x<1) \tag{2.39}$$

In order to prove (1), it is enough to show that $G(x,y)$ defined by (2.37) satisfies the properties (2.29), (2.30) and (2.31).

Differentiating $G(x,y)$ with respect to x, we have

$$\begin{aligned}&\partial_x^i G(x,y) \;=\\ &(-1)^{M+1}4^{2M-1-i}\left[\Big(\mathrm{sgn}(x-y)\Big)^i b_{2M-i}\left(\frac{|x-y|}{4}\right)-(-1)^i b_{2M-i}\left(\frac{2-x-y}{4}\right)\right]\\ &(-1<x,y<1,\quad x\neq y,\quad 0\le i\le 2M)\end{aligned} \tag{2.40}$$

Putting $i=2M$ we have (2.29).

For $0\le i\le 2M-1$ we put $x=\pm 1$ in (2.40) and have

$$\partial_x^i G(x,y)\Big|_{x=1} \;=\; (-1)^{M+1}4^{2M-1-i}\Big(1-(-1)^i\Big)\, b_{2M-i}\left(\frac{1-y}{4}\right) \tag{2.41}$$

$$\begin{aligned}&\partial_x^i G(x,y)\Big|_{x=-1} \;=\; (-1)^{M+1+i}4^{2M-1-i}\left[b_{2M-i}\left(\frac{1+y}{4}\right)-b_{2M-i}\left(\frac{3-y}{4}\right)\right] \;=\\ &(-1)^{M+1+i}4^{2M-1-i}\Big(1-(-1)^i\Big)\, b_{2M-i}\left(\frac{1+y}{4}\right)\end{aligned} \tag{2.42}$$

where we used the fact

$$b_n(x) \;=\; (-1)^n b_n(1-x) \qquad (0\le x\le 1/2,\quad n=0,1,2,\cdots)$$

Hence we have (2.30).

Putting $y=x\mp 0$ in (2.40) and taking their difference, we have

$$\begin{aligned}&\partial_x^i G(x,y)\Big|_{y=x-0} \;-\; \partial_x^i G(x,y)\Big|_{y=x+0} \;=\; (-1)^{M+1}4^{2M-1-i}\Big(1-(-1)^i\Big)\, b_{2M-i}(0) \;=\\ &\begin{cases} 0 & (0\le i\le 2M-2)\\ (-1)^M & (i=2M-1) \qquad (-1<x<1)\end{cases}\end{aligned} \tag{2.43}$$

where we have employed the following facts.

$$b_{2n+1}(0) \;=\; -\,1/2 \quad (n=0), \qquad 0 \quad (n=1,2,3,\cdots)$$

This completes the proof of Theorem 2.4. ■

We will list up the concrete form of Green function $G(x,y)=G(M;x,y)$ and related functions for $M=1,2,3$.

$$\begin{aligned}&G(1;x,y) \;=\; -\frac{1}{2}\,|x-y| \;+\; \frac{1}{2}\,(1-xy)\\ &G(1;x,x) \;=\; \frac{1}{2}\,\Big(1-x^2\Big)\,, \qquad G(1;0,0) \;=\; \frac{1}{2}\\ &G(1;x,0) \;=\; \frac{1}{2}\,(1-x) \qquad (0<x<1)\end{aligned}$$

$$G(2;x,y) \;=\; \frac{1}{12}\,|x-y|^3 \;+\; \frac{1}{12}\left[2+2xy-3\left(x^2+y^2\right)+\left(x^3y+xy^3\right)\right]$$

$$G(2;x,x) \;=\; \frac{1}{6}\left(1-x^2\right)^2, \qquad G(2;0,0) \;=\; \frac{1}{6}$$

$$G(2;x,0) \;=\; \frac{1}{12}\,(1-x)\,(2+2x-x^2) \;=$$

$$\frac{1}{12}\,(2-3x^2+x^3) \;=\; \frac{1}{12}\left[\;3(1-x)-(1-x)^3\;\right] \qquad (0<x<1)$$

$$G(3;x,y) \;=\; -\,\frac{1}{240}\,|x-y|^5 \;+\; \frac{1}{720}\Big[48+8xy-60(x^2+y^2)+$$

$$90x^2y^2-20(x^3y+xy^3)+15(x^4+y^4)-10x^3y^3-3(x^5y+xy^5)\Big]$$

$$G(3;x,x) \;=\; \frac{1}{45}\,(1-x^2)^2\,(3-x^2), \qquad G(3;0,0) \;=\; \frac{1}{15}$$

$$G(3;x,0) \;=\; \frac{1}{240}\,(1-x)\,(4+2x-x^2)^2 \;=\; \frac{1}{240}\,(16-20x^2+5x^4-x^5) \;=$$

$$\frac{1}{240}\left[\;25(1-x)-10(1-x)^3+(1-x)^5\;\right] \qquad (0<x<1)$$

3 The method of reflection

In this section, we derive the expression (2.37) by the so-called method of reflection.

In the previous work [1], we proved the following theorem.

Theorem 3.1 *For any bounded continuous function $f(x)$ on an interval $-1<x<3$ which satisfies the solvability condition*

$$\int_{-1}^{3} f(y)\,dy \;=\; 0 \tag{3.1}$$

periodic boundary value problem

BVP (M,P)

$$\begin{cases} (-1)^M u^{(2M)} \;=\; f(x) & (-1<x<3) \quad (3.2)\\ u^{(i)}(3) \;-\; u^{(i)}(-1) \;=\; 0 & (0\le i\le 2M-1) \quad (3.3)\\ \displaystyle\int_{-1}^{3} u(x)\,dx \;=\; 0 & \quad (3.4) \end{cases}$$

has a unique classical solution $u(x)$ which is given by

$$u(x) \;=\; \int_{-1}^{3} (-1)^{M+1}4^{2M-1}b_{2M}\!\left(\frac{|x-y|}{4}\right) f(y)\,dy \qquad (-1<x<3) \tag{3.5}$$

For any bounded continuous function $f(x)$ on an interval $-1<x<1$, we extend the domain of definition to $-1<x<3$ by the symmetry

$$f(x) \;=\; -\,f(2-x) \qquad (1<x<3) \tag{3.6}$$

This extended function $f(x)$ satisfies

$$\int_{-1}^{3} f(y)\,dy \;=\; 0 \tag{3.7}$$

The solution $u(x)$ of BVP(M,P) for this extended $f(x)$ is given by

$$u(x) \;=\; \int_{-1}^{3} (-1)^{M+1} 4^{2M-1} b_{2M}\left(\frac{|x-y|}{4}\right) f(y)\,dy \qquad (-1<x<3) \tag{3.8}$$

For $-1<x<1$, then we have

$$\begin{aligned}
&(-1)^{M+1} 4^{-(2M-1)} u(x) \;=\; I_1 \;+\; I_2 \\
&I_1 \;=\; \int_{-1}^{1} b_{2M}\left(\frac{|x-y|}{4}\right) f(y)\,dy, \qquad I_2 \;=\; \int_{1}^{3} b_{2M}\left(\frac{y-x}{4}\right) f(y)\,dy
\end{aligned} \tag{3.9}$$

Using the symmetry of $f(x)$ (3.6), we have

$$I_2 \;=\; -\int_{1}^{3} b_{2M}\left(\frac{y-x}{4}\right) f(2-y)\,dy \;=\; -\int_{-1}^{1} b_{2M}\left(\frac{2-x-y}{4}\right) f(y)\,dy \tag{3.10}$$

Finally we have

$$u(x) \;=\; \int_{-1}^{1} G(x,y)\, f(y)\,dy \qquad (-1<x<1) \tag{3.11}$$

where $G(x,y)$ is given by (2.37). We have already shown that the above function satisfies (2.29), (2.30) and (2.31) of Theorem 2.2. ■

4 Reproducing kernel

In this section, it is shown that Green function $G(x,y)$ is a reproducing kernel for a set of Hilbert space H and its inner product $(\cdot,\cdot)_M$ introduced in section 1.

Theorem 4.1 (1) *For any $u(x)\in H$, we have the following reproducing relation.*

$$u(y) \;=\; (\,u(x), G(x,y)\,)_M \;=\; \int_{-1}^{1} u^{(M)}(x)\, \partial_x^M G(x,y)\,dx \qquad (-1\le y\le 1) \tag{4.1}$$

(2) $$G(y,y) \;=\; \int_{-1}^{1} \left|\partial_x^M G(x,y)\right|^2 dx \qquad (-1\le y\le 1) \tag{4.2}$$

Proof of Theorem 4.1 For functions $u=u(x)$ and $v=v(x)=G(x,y)$ with y arbitrarily fixed in $-1\le y\le 1$, we have

$$u^{(M)}\, v^{(M)} \;-\; u\,(-1)^M v^{(2M)} \;=\; \left(\sum_{j=0}^{M-1} (-1)^{M-1-j}\, u^{(j)}\, v^{(2M-1-j)}\right)' \tag{4.3}$$

Integrating this with respect to x on intervals $-1 < x < y$ and $y < x < 1$, we have

$$\begin{aligned}
&\int_{-1}^{1} u^{(M)}(x)\, v^{(M)}(x)\, dx \;-\; \int_{-1}^{1} u(x)\, (-1)^M v^{(2M)}(x)\, dx \;= \\
&\left[\, \sum_{j=0}^{M-1} (-1)^{M-1-j}\, u^{(j)}(x)\, v^{(2M-1-j)}(x) \right] \left\{ \Big|_{x=-1}^{x=y-0} + \Big|_{x=y+0}^{x=1} \right\} = \\
&\sum_{j=0}^{M-1} (-1)^{M-1-j} \left[u^{(j)}(1)\, v^{(2M-1-j)}(1) \,-\, u^{(j)}(-1)\, v^{(2M-1-j)}(-1) \right] + \\
&\sum_{j=0}^{M-1} (-1)^{M-1-j}\, u^{(j)}(y) \left[v^{(2M-1-j)}(y-0) \,-\, v^{(2M-1-j)}(y+0) \right]
\end{aligned} \tag{4.4}$$

The first term on the right-hand side is rewritten as follows.

$$\begin{aligned}
&\sum_{j=0}^{M-1} (-1)^{M-1-j} \left[u^{(j)}(1)\, v^{(2M-1-j)}(1) \,-\, u^{(j)}(-1)\, v^{(2M-1-j)}(-1) \right] = \\
&\sum_{j=0}^{[\frac{M-1}{2}]} (-1)^{M-1} \left[u^{(2j)}(1)\, v^{(2(M-1-j)+1)}(1) \,-\, u^{(2j)}(-1)\, v^{(2(M-1-j)+1)}(-1) \right] + \\
&\sum_{j=0}^{[\frac{M-2}{2}]} (-1)^{M} \left[u^{(2j+1)}(1)\, v^{(2(M-1-j))}(1) \,-\, u^{(2j+1)}(-1)\, v^{(2(M-1-j))}(-1) \right]
\end{aligned} \tag{4.5}$$

Using (2.29), (2.30) and (2.32) in Theorem 2.2, we have (1). (2) follows from (1) by putting $u(x) = G(x,y)$ in (4.1). We have proved Theorem 4.1. ■

In order to observe the behavior of the diagonal value

$$G(y,y) \;=\; (-1)^{M+1}\, 4^{2M-1} \left[\, b_{2M}(0) \,-\, b_{2M}\!\left(\frac{1-y}{2}\right) \right] \qquad (-1 < y < 1) \tag{4.6}$$

we prepare the following lemma concerning Bernoulli polynomials (see [1]).

Lemma 4.1 *For $n = 1, 2, 3, \cdots$, $u(x) = (-1)^{n+1} b_{2n}(x)$ satisfy the following properties.*

$$u(x) \;=\; u(1-x) \qquad (0 \le x \le 1/2) \tag{4.7}$$

$$\max_{0 \le x \le 1} u(x) \;=\; u(0) \;=\; u(1) \;=\; (-1)^{n+1} b_{2n}(0) \;>\; 0 \tag{4.8}$$

$$\begin{aligned}
&\min_{0 \le x \le 1} u(x) \;=\; u(1/2) \;=\; (-1)^{n+1} b_{2n}(1/2) \;= \\
&-\left(1 - 2^{-(2n-1)}\right) (-1)^{n+1} b_{2n}(0) \;<\; 0
\end{aligned} \tag{4.9}$$

$$\max_{0 \le x \le 1} |\, u(x)\, | \;=\; u(0) \;=\; u(1) \tag{4.10}$$

$$(-1)^{n+1}\, b_{2n}(0) \;=\; 2 \sum_{j=1}^{\infty} (2\pi j)^{-2n} \;=\; \frac{2}{(2\pi)^{2n}}\, \zeta(2n) \tag{4.11}$$

From this lemma, it is shown that $G(y,y)$ attains its maximum at $y = 0$. As a conclusion, we have obtained the following theorem.

Theorem 4.2

$$\max_{|y|\le 1} G(y,y) \;=\; G(0,0) \;=\; (-1)^{M+1}\,4^{2M-1}\Big[\; b_{2M}(0) - b_{2M}(1/2) \;\Big] \;=$$
$$2^{2M-1}\left(2^{2M}-1\right)(-1)^{M+1}\,b_{2M}(0) \;=\; \left(2^{2M}-1\right)\pi^{-2M}\,\zeta(2M) \tag{4.12}$$

5 Sobolev inequality

In this section, we give a proof of Theorem 1.2, from which Theorem 1.1 is derived simultaneously.

Applying Schwarz inequality to (4.1) and using (4.2), we have

$$|\,u(y)\,|^2 \;\le\; \int_{-1}^{1} \big|\,\partial_x^M\, G(x,y)\,\big|^2\, dx \int_{-1}^{1} \Big|\,u^{(M)}(x)\Big|^2 dx \;=\; G(y,y)\int_{-1}^{1} \Big|\,u^{(M)}(x)\Big|^2 dx$$

Noting that

$$C_0 \;=\; \max_{|y|\le 1} G(y,y) \;=\; G(0,0)$$

we have following Sobolev inequality.

$$\left(\sup_{|y|\le 1}|\,u(y)\,|\right)^2 \;\le\; C_0\int_{-1}^{1}\Big|\,u^{(M)}(x)\Big|^2 dx \tag{5.1}$$

This inequality shows that $(\cdot,\cdot)_M$ is positive definite. It should be noted that it requires Schwarz inequality but does not require "positive definiteness" of the inner product in order to prove (5.1).

In the second place, we apply this inequality to $u(x) = G(x,0) \in H$ and have

$$\left(\sup_{|y|\le 1}|\,G(y,0)\,|\right)^2 \;\le\; C_0\int_{-1}^{1}\big|\,\partial_x^M G(x,0)\,\big|^2 dx \;=\; C_0^2$$

Combining this and trivial inequality

$$C_0^2 \;=\; G^2(0,0) \;\le\; \left(\sup_{|y|\le 1}|\,G(y,0)\,|\right)^2$$

we have

$$C_0^2 \;\le\; \left(\sup_{|y|\le 1}|\,G(y,0)\,|\right)^2 \;\le\; C_0\int_{-1}^{1}\big|\,\partial_x^M G(x,0)\,\big|^2 dx \;=\; C_0^2$$

That is to say

$$\left(\sup_{|y|\le 1}|\,G(y,0)\,|\right)^2 \;=\; C_0\int_{-1}^{1}\big|\,\partial_x^M G(x,0)\,\big|^2 dx \tag{5.2}$$

which completes the proof of Theorem 1.2. ■

Acknowledgement One of the authors A. N. is supported by J. S. P. S. Grant-in-Aid for Scientific Research for Young Scientists No.16740092, K. T. is supported by J. S. P. S. Grant-in-Aid for Scientific Research (C) No.17540175 and H. Y. is supported by the 21st Century COE Program named "Towards a new basic science : depth and synthesis".

References

[1] Y.Kametaka, H.Yamagishi, K.Watanabe, A.Nagai and K.Takemura, *Riemann zeta function, Bernoulli polynomials and the best constant of Sobolev inequality*, Sci. Math. Jpn. **e-2007** (2007), 63–89.

[2] A.Nagai, K.Takemura, Y.Kametaka, K.Watanabe and H.Yamagishi, *Green function for boundary value problem of $2M$-th order linear ordinary equations with free boundary condition*, Far East J. Appl. Math. **26** (2007), 393–406.

[3] Y. Kametaka, K. Watanabe, A. Nagai and S. Pyatkov, *The best constant of Sobolev inequality in an n dimensional Euclidean space*, Sci. Math. Jpn. **e-2004** (2004), 295–303.

[4] Y. Kametaka, K. Watanabe and A. Nagai, *The best constant of Sobolev inequality in an n dimensional Euclidean space*, Proc. Japan Acad. **81** Ser. A (2005), 1–4.

[5] K. Watanabe, T. Yamada and W. Takahashi, *Reproducing Kernels of $H^m(a,b)$ $(m = 1, 2, 3)$ and Least Constants in Sobolev's Inequalities*, Appl. Anal. **82** (2003), 809–820.

* He has retired at March 2004, and now he is an emeritus professor of Osaka University.
Faculty of Engineering Science, Osaka University
1-3 Matikaneyamatyo, Toyonaka 560-8531, Japan
E-mail address: kametaka@sigmath.es.osaka-u.ac.jp

† Faculty of Engineering Science, Osaka University
1-3 Matikaneyamatyo, Toyonaka 560-8531, Japan
E-mail address: yamagisi@sigmath.es.osaka-u.ac.jp

‡ Department of Computer Science, National Defense Academy
1-10-20 Yokosuka 239-8686, Japan
E-mail address: wata@nda.ac.jp

§ Liberal Arts and Basic Sciences, College of Industrial Technology
Nihon University, 2-11-1 Shinei, Narashino 275-8576, Japan
E-mail address: a8nagai@cit.nihon-u.ac.jp

¶ Liberal Arts and Basic Sciences, College of Industrial Technology
Nihon University, 2-11-1 Shinei, Narashino 275-8576, Japan
E-mail address: k8takemu@cit.nihon-u.ac.jp