STRONG HYPERGROUPS OF ORDER FOUR ARISING FROM EXTENSIONS

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Recieved June 26, 2008

Abstract. In this paper we obtain the complete parametrization of commutative hypergroup extensions of a hypergroup of order two by another hypergroup of order two. Among them we characterize all strong hypergroups and splitting extensions. Applying our results, one can determine all strong hypergroups of order four which have non-trivial subhypergroups.

1 Introduction Let $\mathcal{K}$ be a finite commutative hypergroup. If $\mathcal{K}$ has a subhypergroup $\mathcal{H}$ and the quotient hypergroup $\mathcal{K}/\mathcal{H} \cong \mathcal{L}$, then $\mathcal{K}$ is called an *extension* of $\mathcal{L}$ by $\mathcal{H}$ and denoted by the exact sequence $\mathbf{1} \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to \mathbf{1}$. In this case, $\mathrm{Ker}(\varphi) = \mathcal{H}$ and $\varphi(\mathcal{K}) = \mathcal{L}$, where φ is the quotient homomorphism from $\mathcal{K}$ onto $\mathcal{L}$.

Extension problem for the category of commutative hypergroups is to determine all commutative hypergroups $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ when commutative hypergroups $\mathcal{L}$ and $\mathcal{H}$ are given. This will be essential to understand the full structure of hypergroups. We discussed the extension problem of hypergroups in [HKKK], [HK1] and [K] in general situation for splitting extensions. Moreover in the previous papers [HJKK], [HK2], [KST], [IKS], [KKY] and [KM], we have succeeded to determine all extensions in the case that $\mathcal{H}$ or $\mathcal{L}$ is a group. In the present paper we consider the extension problem for the case that both of $\mathcal{H}$ and $\mathcal{L}$ are hypergroups of order two. This is the model case that $\mathcal{H}$ and $\mathcal{L}$ are not necesarily groups. Wildberger[W2] determined all strong hypergroups of order three in 2002. The next task will be expected to determine all strong hypergroups of order four. The present paper is devoted to determine all strong hypergroups of order four which have non-trivial subhypergroups, combining with Wildberger's results for hypergroups of order three.

Our main results are described in Theorem 3.2, Theorem 3.4, Theorem 4.1 and Theorem 4.5.

2 Preliminaries We recall some notions and facts on finite commutative hypergroups from Wildberger's paper [W1] and Bloom-Heyer's book [BH]. $\mathcal{K} := (\mathcal{K}, A)$ is called a finite commutative *signed hypergroup* if the following conditions (1)–(6) are satisfied.

(1) A is a $*$-algebra over $\mathbb{C}$ with the unit c_0,

(2) $\mathcal{K} = \{c_0, c_1, \ldots, c_n\}$ is a linear basis of A,

(3) $\mathcal{K}^* = \mathcal{K}$,

(4) $c_i c_j = \sum_{k=0}^{n} n_{ij}^k c_k$, where n_{ij}^k is a real number such that

(i) $c_i^* = c_j \iff n_{ij}^0 > 0$, (ii) $c_i^* \neq c_j \iff n_{ij}^0 = 0$,

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2000 *Mathematics* ! !! !! ! *Subject Classification.* 43A62, 20N20 .
Key words and phrases. hypergroup, extension .

(5) $\displaystyle\sum_{k=0}^{n} n_{ij}^k = 1$ for any i, j,

(6) $c_i c_j = c_j c_i$ for any i, j.

In the case that $n_{ij}^k \geq 0$ for any i, j, k, $\mathcal{K} = (\mathcal{K}, A)$ is called a finite commutative *hypergroup* with order $|\mathcal{K}| = n + 1$. We often denote $*$-algebra A of $(\mathcal{K}, A)$ by $A(\mathcal{K})$. If $c_i^* = c_i$ for all $i = 1, 2, \ldots, n$, then $\mathcal{K}$ is called a *hermitian* hypergroup.

The *weight* of an element $c_i \in K$ is defined by $w(c_i) := (n_{ij}^0)^{-1}$ where $c_j = c_i^*$, and the *total weight* of $\mathcal{K}$ is given by $w(\mathcal{K}) := \sum_{i=0}^n w(c_i)$.

For a finite commutative signed hypergroup $\mathcal{K}$ a complex valued function χ on $\mathcal{K}$ is called a *character* of $\mathcal{K}$ if

$$(1)\ \chi(c_0) = 1, \quad (2)\ \chi(c_i^*) = \overline{\chi(c_i)}, \quad (3)\ \chi(c_i)\chi(c_j) = \sum_{k=0}^{n} n_{ij}^k \chi(c_k).$$

The set $\widehat{\mathcal{K}}$ of all characters of $\mathcal{K}$ also becomes a finite commutative signed hypergroup with the same order $|\mathcal{K}|$, and the duality $\widehat{\left(\widehat{\mathcal{K}}\right)} \cong \mathcal{K}$ holds in the sense of isomorphisms between signed hypergroups[W1][Z]. A finite hypergroup $\mathcal{K}$ is called a *strong* hypergroup if the dual $\widehat{\mathcal{K}}$ of $\mathcal{K}$ is also a hypergroup.

Let $E_0(\mathcal{K}) \in A(\mathcal{K})$ denote the normalized Haar measure of $\mathcal{K}$ which is given by

$$E_0(\mathcal{K}) = w(\mathcal{K})^{-1} \sum_{i=0}^{n} w(c_i) c_i.$$

For an exact sequence $\mathbf{1} \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to \mathbf{1}$, the total weight $w(\mathcal{K})$ of $\mathcal{K}$ equals to $w(\mathcal{H})w(\mathcal{L})$. Moreover, if $\mathcal{L} = \{\boldsymbol{\ell}_0, \ldots, \boldsymbol{\ell}_l\}$ and $\varphi^{-1}(\boldsymbol{\ell}_j) = \{s_0, \ldots, s_{k(j)}\} \subset \mathcal{K}$ for $\boldsymbol{\ell}_j \in \mathcal{L}$, then $\sum_{i=0}^{k(j)} w(s_i) = w(\mathcal{H})w(\boldsymbol{\ell}_j)$. This is easy to see by the follwing equality:

$$w(\mathcal{K})\varphi(E_0(\mathcal{K})) = w(\mathcal{H})w(\mathcal{L})E_0(\mathcal{L}).$$

We use the notation $\tilde{\rho} = 1 - \rho$ for a real number ρ, a reflection of ρ with respect to a center point $1/2$ throughout this paper. Let $\mathcal{L}(\rho) = \{\boldsymbol{\ell}_0, \boldsymbol{\ell}_1\}$ be a hypergroup of order two which is determined by $\boldsymbol{\ell}_1^2 = \tilde{\rho}\boldsymbol{\ell}_0 + \rho\boldsymbol{\ell}_1$, where $0 \leq \rho < 1$.

3 Extensions and non hermitian type At first, we shall give a general result in the case of $|\mathcal{H}| = |\mathcal{L}| = 2$ and $|\mathcal{K}| = 4$.

Assume that $\mathcal{K}$ is a hypergroup which has a subhypergroup $\mathcal{H} \cong \mathcal{L}(p)$ with $0 \leq p < 1$ and its quotient $\mathcal{K}/\mathcal{H} = \mathcal{L} \cong \mathcal{L}(q)$ with $0 \leq q < 1$. We put $\mathcal{H} = \{h_0, h_1\}$ and $\mathcal{L} = \{\boldsymbol{\ell}_0, \boldsymbol{\ell}_1\}$ with structure equations $h_1^2 = \tilde{p}h_0 + ph_1$ and $\boldsymbol{\ell}_1^2 = \tilde{q}\boldsymbol{\ell}_0 + q\boldsymbol{\ell}_1$.

We write $\mathcal{K} = \{h_0, h_1, s_0, s_1\}$ where $\varphi(s_i) = \boldsymbol{\ell}_1$ $(i = 0, 1)$.

Proposition 3.1. *An extension $\mathcal{K}$ in the above assumption has the following conditions:*

(Ha) $h_1 s_0 = \tau s_0 + \tilde{\tau} s_1$,

(Hb) $h_1 s_1 = (\tilde{p} + \tau)s_0 + (p - \tau)s_1$.

where τ is a parameter such that $0 \leq \tau \leq p$.

Proof. Since $\varphi(h_1 s_i) = \boldsymbol{\ell}_1$ for $i = 0, 1$, we can write $h_1 s_0 = \tilde{x} s_0 + x s_1$ and $h_1 s_1 = \tilde{y} s_0 + y s_1$ for $x, y \in [0, 1]$. Consider a triple product $(s_0 h_1) h_1 = (\tilde{x} s_0 + x s_1) h_1 = (\tilde{x}^2 + x\tilde{y}) s_0 + (\tilde{x} x + xy) s_1$ and $s_0 (h_1)^2 = s_0(\tilde{p} h_0 + p h_1) = \tilde{p} s_0 + p(\tilde{x} s_0 + x s_1) = (\tilde{p} + p\tilde{x}) s_0 + p x s_1$. Compare the coefficients of s_1. Then we have $\tilde{x} x + xy = px$.

Assume that $x = 0$, then $s_0 h_1 = s_0$. Hence $E_0(\mathcal{H}) s_0 = s_0$, where $E_0(\mathcal{H})$ is the normalized Haar measure of $\mathcal{H}$. On the other hand $\mathrm{supp}(E_0(\mathcal{H}) s_0) \ni s_1$. It is a contradiction, so that $x \neq 0$. Therefore, $\tilde{x} + y = p$.

Put $\tau = \tilde{x}$. We note that $y = p - \tau$ and $\tilde{y} = \tilde{p} + \tau$. By the axiom of hypergroups we have $0 \leq \tau \leq p$. □

Now we consider the case that $\mathcal{K}$ is not a hermitian hypergroup, i.e. $s_0^* = s_1$ and $s_1^* = s_0$. Then we have the following theorem about multiplicative structure of $\mathcal{K}$.

Theorem 3.2. *If $\mathcal{K}$ is not a hermitian hypergroup with $|\mathcal{K}| = 4$, then $\mathcal{K} = \mathcal{K}_{nh}^{p,q} = \{h_0, h_1, s_0, s_1\}$ is determined by the following multiplicative structure:*

(a) $h_1 s_0 = \dfrac{p}{2} s_0 + (1 - \dfrac{p}{2}) s_1, \quad h_1 s_1 = (1 - \dfrac{p}{2}) s_0 + \dfrac{p}{2} s_1,$

(b) $s_0^2 = s_1^2 = \tilde{q} h_1 + \dfrac{q}{2} s_0 + \dfrac{q}{2} s_1,$

(c) $s_0 s_1 = \dfrac{2\tilde{p}\tilde{q}}{1 + \tilde{p}} h_0 + \dfrac{p\tilde{q}}{1 + \tilde{p}} h_1 + \dfrac{q}{2} s_0 + \dfrac{q}{2} s_1.$

Proof. The results (Ha)(Hb) in Proposition 3.1 also hold in this case.

By the fact that $\varphi(s_0^2) = \varphi(s_1^2) = \varphi(s_0 s_1) = \boldsymbol{\ell}_1^2$, we put

$$s_0^2 = \tilde{q} h_1 + q\tilde{z} s_0 + q z s_1, \tag{1}$$

$$s_1^2 = \tilde{q} h_1 + q\tilde{w} s_0 + q w s_1, \tag{2}$$

$$s_1 s_0 = \tilde{q}\tilde{x} h_0 + \tilde{q} x h_1 + q\tilde{y} s_0 + q y s_1. \tag{3}$$

where $0 \leq x, y, z, w \leq 1$ but $x \neq 1$. Operate the inverse $*$ on the above (1)–(3). Then we have $q\tilde{w} = qz$ and $q\tilde{y} = qy$. Hence, $q\tilde{y} = qy = q/2$. Moreover it is easy to see that $\tau = p/2$ by $(h_1 s_0)^* = h_1 s_1$, $(h_1 s_1)^* = h_1 s_0$. This shows (a). Next consider a triple product $(s_0 s_1) h_1, (s_0 h_1) s_1$ and $s_0 (h_1 s_1)$. From the coefficients of h_0 of them, it shows that $\tilde{q} x \tilde{p} = \tau \tilde{q} \tilde{x} = (p - \tau) \tilde{q} \tilde{x}$. Since $\tilde{q}\tilde{x} \neq 0$ and $\tau = p/2$, we have $2x\tilde{p} = p\tilde{x} = p(1 - x)$. Hence $x = p/(1 + \tilde{p})$ and $\tilde{x} = 2\tilde{p}/(1 + \tilde{p})$, and this means (b). Compare the coefficients of h_0 on a product $(s_0 s_1) s_0 = (s_0)^2 s_1$. Then $q y \tilde{q} \tilde{x} = q \tilde{z} \tilde{q} \tilde{x}$. Hence $q\tilde{z} = qy = q/2$. Consequently $qz = qw = q/2$. Therefore we have (c).

As the coefficients of s_0 in products $(s_0 s_1) h_1, (s_0 h_1) s_1$ and $s_0 (h_1 s_1)$ equal to $q/2$, we have the associative law: $(s_0 s_1) h_1 = (s_0 h_1) s_1 = s_0 (h_1 s_1)$.

Finally it is easy to see the associativity on $h_1 s_0 s_0, h_1 s_1 s_1, s_0 s_0 s_1, s_0 s_1 s_1$.

□

REMARK. In the previous paper [KST] and [IKS], we studied the cases that $\mathcal{H}$ or $\mathcal{L}$ is a group. When $p = 0$, i.e. $\mathcal{H} \cong \mathbb{Z}_2$, $\mathcal{K}_{nh}^{0,q}$ is parameterized as same as Model 1 in [KST]. When $q = 0$, i.e. $\mathcal{L} \cong \mathbb{Z}_2$, $\mathcal{K}_{nh}^{p,0}$ is parameterized as in [IKS]. When $p = q = 0$, we have $\mathcal{K}_{nh}^{0,0} \cong \mathbb{Z}_4$. This is non splitting exact sequence as in abelian group category.

We remark that all of extensions in Theorem 3.2 are not splitting as hypergroup.

Now we calculate characters of $\mathcal{K}_{nh}^{p,q}$. We denote the dual signed hypergroup $\widehat{\mathcal{K}}_{nh}^{p,q} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. Then we have the exact sequence of dual hypergroups $\mathbf{1} \to \widehat{\mathcal{L}} \to \widehat{\mathcal{K}} \to \widehat{\mathcal{H}} \to \mathbf{1}$, where $\widehat{\mathcal{L}} = \{\chi_0, \chi_1\} \cong \mathcal{L}(q)$.

Proposition 3.3. *A hypergroup $\mathcal{K}_{nh}^{p,q}$ has characters $\{\chi_0, \chi_1, \chi_2, \chi_3\}$ which is determined by the following table with values of weights:*

	h_0	h_1	s_0	s_1	$w(\chi_i)$
χ_0	1	1	1	1	1
χ_1	1	1	$-\tilde{q}$	$-\tilde{q}$	$1/\tilde{q}$
χ_2	1	$-\tilde{p}$	$\sqrt{-\tilde{p}\tilde{q}}$	$-\sqrt{-\tilde{p}\tilde{q}}$	$\dfrac{\tilde{q}+1}{2\tilde{p}\tilde{q}}$
χ_3	1	$-\tilde{p}$	$-\sqrt{-\tilde{p}\tilde{q}}$	$\sqrt{-\tilde{p}\tilde{q}}$	$\dfrac{\tilde{q}+1}{2\tilde{p}\tilde{q}}$
$w(h_i), w(s_i)$	1	$1/\tilde{p}$	$\dfrac{\tilde{p}+1}{2\tilde{p}\tilde{q}}$	$\dfrac{\tilde{p}+1}{2\tilde{p}\tilde{q}}$	$\dfrac{(\tilde{p}+1)(\tilde{q}+1)}{\tilde{p}\tilde{q}}$

Proof. From $\widehat{\mathcal{L}} \subset \widehat{\mathcal{K}}$ we have that $\chi_1(h_j) = 1,\ \chi_1(s_j) = -\tilde{q}$ for $j = 0, 1$. Assume a character $\chi \neq \chi_0, \chi_1$. Since $\chi \notin \widehat{\mathcal{L}}$, it follows that $\chi(h_1) \neq 1$. This means $\chi(h_1) = -\tilde{p}$. We have $\chi(s_0) = -\chi(s_1)$ from (a) in Theorem 3.2. From the conditions (b) and (c), we have the same equation $\chi(s_0)^2 = -\tilde{p}\tilde{q}$. Therefore we write $\chi_2(s_0) = \sqrt{-\tilde{p}\tilde{q}}$ and $\chi_3(s_0) = -\sqrt{-\tilde{p}\tilde{q}}$.

Applying the method of [W2] which calculates the structure coefficients m_{ij}^k of the dual signed hypergroup, we have

$$\begin{aligned} m_{22}^0 = w(\chi_2)^{-1} &= \Big(1 + \chi_2(h_1)^2 w(h_1) + |\chi_2(s_0)|^2 w(s_0) + |\chi_2(s_1)|^2 w(s_1)\Big) \Big/ w(\mathcal{K}_{nh}^{p,q}) \\ &= \Big(1 + \tilde{p} + \frac{1+\tilde{p}}{2} + \frac{1+\tilde{p}}{2}\Big) \cdot \frac{\tilde{p}\tilde{q}}{(\tilde{p}+1)(\tilde{q}+1)} \\ &= \frac{2\tilde{p}\tilde{q}}{\tilde{q}+1}. \end{aligned}$$

Hence $w(\chi_2) = (\tilde{q}+1)/(2\tilde{p}\tilde{q})$. It is shown that $w(\chi_3) = w(\chi_3^*) = w(\chi_2)$.

The value on the right and bottom corner in the table is total weight $w(\mathcal{K}_{nh}^{p,q}) = w(\mathcal{H})w(\mathcal{L})$. □

We determine the dual signed hypergroup of $\mathcal{K}_{nh}^{p,q}$.

Theorem 3.4. *The dual $\widehat{\mathcal{K}}_{nh}^{p,q} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ is determined by the structure equations:*

(A) $\chi_1^2 = \tilde{q}\chi_0 + q\chi_1, \quad \chi_1\chi_2 = \frac{q}{2}\chi_2 + (1 - \frac{q}{2})\chi_3, \quad \chi_1\chi_3 = (1 - \frac{q}{2})\chi_2 + \frac{q}{2}\chi_3,$

(B) $\chi_2^2 = \chi_3^2 = \tilde{p}\chi_1 + \frac{p}{2}\chi_2 + \frac{p}{2}\chi_3,$

(C) $\chi_2\chi_3 = \tilde{p} \cdot \frac{2\tilde{q}}{\tilde{q}+1}\chi_0 + \tilde{p} \cdot \frac{q}{\tilde{q}+1}\chi_1 + \frac{p}{2}\chi_2 + \frac{p}{2}\chi_3.$

The hypergroup $\mathcal{K}_{nh}^{p,q}$ is strong.

Proof. It is easily obtained from the symmetry of the table with respect to p, q in Proposition 3.3. It is shown that all coefficients of the above are non negative. Therefore the dual $\widehat{\mathcal{K}}_{nh}^{p,q}$ is a hypergroup. □

REMARK. It is easy to see that $\widehat{\mathcal{K}}_{nh}^{p,q} \cong \mathcal{K}_{nh}^{q,p}$ from the structure in Theorem 3.2 and Theorem 3.4. In the special case of $p = q$, $\mathcal{K}_{nh}^{p,p}$ is a self dual hypergroup.

4 Hermitian type hypergroups with order four In this section we consider the extension of hermitian type. We have the following theorem.

Theorem 4.1. *Let $\mathcal{K}$ be an extension of $\mathcal{L} \cong \mathcal{L}(q)$ by $\mathcal{H} \cong \mathcal{L}(p)$ whose order is four. If $\mathcal{K}$ is hermitian type, then $\mathcal{K} = \mathcal{K}_h^{p,q}(\tau, \sigma) = \{h_0, h_1, s_0, s_1\}$ is determined by the following structure and condition:*

(a) $h_1 s_0 = \tau s_0 + \tilde{\tau} s_1, \quad h_1 s_1 = (\tilde{p} + \tau) s_0 + (p - \tau) s_1,$

(b) $s_0 s_1 = \tilde{q} h_1 + \sigma s_0 + (q - \sigma) s_1,$

(c) $s_0^2 = \tilde{q} \cdot \dfrac{\tilde{p}}{\tilde{p} + \tau} h_0 + \tilde{q} \cdot \dfrac{\tau}{\tilde{p} + \tau} h_1 + \left(q - \sigma \cdot \dfrac{\tilde{\tau}}{\tilde{p} + \tau} \right) s_0 + \sigma \cdot \dfrac{\tilde{\tau}}{\tilde{p} + \tau} s_1,$

(d) $s_1^2 = \tilde{q} \cdot \dfrac{\tilde{p}}{\tilde{\tau}} h_0 + \tilde{q} \cdot \dfrac{p - \tau}{\tilde{\tau}} h_1 + (q - \sigma) \cdot \dfrac{\tilde{p} + \tau}{\tilde{\tau}} s_0 + \left(q - (q - \sigma) \cdot \dfrac{\tilde{p} + \tau}{\tilde{\tau}} \right) s_1,$

(e) $q \cdot \dfrac{\tilde{p} + \tau}{\tilde{\tau}} \geq \sigma \geq q \cdot (1 - \dfrac{\tilde{\tau}}{\tilde{p} + \tau}),$

where τ, σ are parameters such that $0 \leq \tau \leq p$ and $0 \leq \sigma \leq q$.

Proof. The statement (a) is showed in Proposition 3.1 with a parameter $0 \leq \tau \leq p$. Since $\varphi(s_0^2) = \varphi(s_1^2) = \varphi(s_0 s_1) = \boldsymbol{\ell}_1^2$, we can write

$$s_1 s_0 = \tilde{q} h_1 + \sigma s_0 + (q - \sigma) s_1, \tag{4}$$

$$s_0^2 = \tilde{q} \tilde{x} h_0 + \tilde{q} x h_1 + q \tilde{y} s_0 + q y s_1, \tag{5}$$

$$s_1^2 = \tilde{q} \tilde{z} h_0 + \tilde{q} z h_1 + q \tilde{w} s_0 + q w s_1. \tag{6}$$

where $0 \leq \sigma \leq q$ and $0 \leq x, y, z, w \leq 1$ but x, z are not 1. Then (b) is the first equation (4).

Now consider triple products $(s_0^2) h_1$ and $(s_0 h_1) s_0$. From the coefficients of h_0, it shows that $\tilde{q} x \tilde{p} = \tau \tilde{q} \tilde{x}$. Since $\tilde{q} \neq 0$, we have $x \tilde{p} = \tau \tilde{x}$. Hence $x = \tau / (\tilde{p} + \tau)$ and $\tilde{x} = \tilde{p} / (\tilde{p} + \tau)$. Next compare the coefficients of h_0 on products $(s_1)^2 h_1$ and $(s_1 h_1) s_1$. Then $\tilde{q} z \tilde{p} = (p - \tau) \tilde{q} \tilde{z}$. Hence $z = (p - \tau) / \tilde{\tau}$ and $\tilde{z} = \tilde{p} / \tilde{\tau}$.

Moreover, compare the coefficients of s_0 on the above triple product $(s_0^2) h_1$ and $(s_0 h_1) s_0$. Then we have $\tau q \tilde{y} + \tilde{\tau} \sigma = q \tilde{y} \tau + q y (\tilde{p} + \tau)$. Hence $\tilde{\tau} \sigma = q y (\tilde{p} + \tau)$, so that $q y = \sigma \tilde{\tau} / (\tilde{p} + \tau)$. Compare the coefficients of s_1 on the above triple product $(s_1^2) h_1$, then $q \tilde{w} = (q - \sigma)(\tilde{p} + \tau) / \tilde{\tau}$. Therefore we prove the statements (c) and (d), and get associativity for $(s_0^2) h_1 = (s_0 h_1) s_0$ and similarly $(s_1)^2 h_1 = (s_1 h_1) s_1$.

The condition (e) is immediate from the fact that the coefficients in products on (c) and (d) are non-negative.

It is easy to see that the associativity on $s_0 s_0 s_1, s_0 s_1 s_1, h_1 s_0 s_1$ holds. □

REMARK. Let $\mathfrak{D}_h$ in Figure 1 be a region of two parameter (τ, σ) in which the condition (e) satisfies. A region $\mathfrak{D}_h$ includes the central point $(p/2, q/2)$ of symmetry. The curve (e_b^1) in Figure 1, is the equation of the first inequality of (e) and a curve (e_b^2) is also the one with respect to the second inequality of (e). When a point (τ, σ) is out of $\mathfrak{D}_h$, i.e. in two parts in left top and right bottom corners, $\mathcal{K}_h^{p,q}(\tau, \sigma)$ is a signed hypergroup.

When $p = 0$, i.e. $\mathcal{H} \cong \mathbb{Z}_2$, we have that $\mathcal{K}_h^{0,q}(0, \sigma)$ is parameterized as same as in Model 1 in [KST]. In this case, the condition (e) of the above becomes $0 \leq \sigma \leq q$.

When $q = 0$, i.e. $\mathcal{L} \cong \mathbb{Z}_2$, we have that $\mathcal{K}_h^{p,0}(\tau, 0)$ is parameterized in [IKS]. The condition (e) is always satisfied because all terms of (e) is 0.

Moreover when $p = q = 0$, we have a hypergroup $\mathcal{K}_h^{0,0}(0, 0) = \mathbb{Z}_2 \times \mathbb{Z}_2$, indeed it is a group.

Corollary 4.2. *$\mathcal{K}_h^{p,q}(\tau,\sigma) \cong \mathcal{K}_h^{p,q}(\tau',\sigma')$ if and only if the following (1) or (2) is satisfied.*
(1) $\tau' = p - \tau$ *and* $\sigma' = q - \sigma$
(2) $\tau' = \tau$ *and* $\sigma' = \sigma$

Proof. The isomorphism of $\mathcal{K}_h^{p,q}(\tau,\sigma) \to \mathcal{K}_h^{p,q}(q-\tau, p-\sigma)$ is given by the flip Φ, i.e. $\Phi(h_i) = h_i$ $(i = 0,1)$ and $\Phi(s_0) = s_1$, $\Phi(s_1) = s_0$ from (a),(b) in Theorem 4.1. □

In the figure 1, two points (τ,σ) and (τ',σ') of the condition in Corollary 4.2 are reflection of each other with respect to the central point $(p/2, q/2)$.

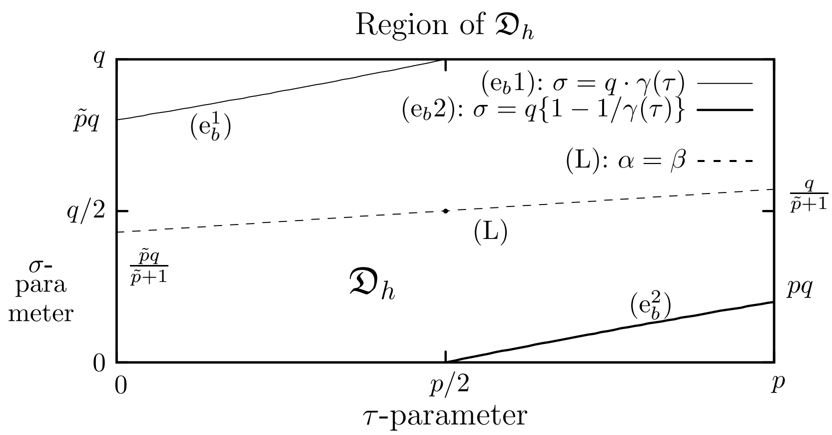

Figure 1: Hypergroup condition (e) of σ, τ

Now we will calculate the characters of hermitian hypergroups $\mathcal{K}_h^{p,q}(\tau,\sigma)$.

Proposition 4.3. *A hypergroup $\mathcal{K}_h^{p,q}(\tau,\sigma)$ has characters $\{\chi_0, \chi_1, \chi_2, \chi_3\}$ which is determined by the following table with values of weights:*

	h_0	h_1	s_0	s_1	$w(\chi_i)$
χ_0	1	1	1	1	1
χ_1	1	1	$-\tilde{q}$	$-\tilde{q}$	$1/\tilde{q}$
χ_2	1	$-\tilde{p}$	α	$-\gamma\alpha$	$\dfrac{(\tilde{q}+1)\beta}{\tilde{p}\tilde{q}(\alpha+\beta)}$
χ_3	1	$-\tilde{p}$	$-\beta$	$\gamma\beta$	$\dfrac{(\tilde{q}+1)\alpha}{\tilde{p}\tilde{q}(\alpha+\beta)}$
$w(h_i), w(s_i)$	1	$1/\tilde{p}$	$\dfrac{\tilde{p}+\tau}{\tilde{p}\tilde{q}}$	$\dfrac{\tilde{\tau}}{\tilde{p}\tilde{q}}$	$\dfrac{(\tilde{p}+1)(\tilde{q}+1)}{\tilde{p}\tilde{q}}$

where two real numbers α, β have the relation:

$$(*) \qquad \alpha - \beta = q - \sigma(1+\gamma^{-1}), \ \ \alpha\beta = \tilde{p}\tilde{q}\gamma^{-1}.$$

Proof. Using the same argument in Proposition 3.3, we suppose that $\widehat{\mathcal{L}} = \{\chi_0, \chi_1\}$. Let $\chi \in \widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma)$ with $\chi \notin \widehat{\mathcal{L}}$. From (a) in Theorem 4.1 under the case of $\chi(h_1) = -\tilde{p}$, it implies that

$$(**) \qquad \chi(s_1) = -\gamma \cdot \chi(s_0),$$

where the ratio of weights $\gamma := (\tilde{p}+\tau)/\tilde{\tau} = w(s_0)/w(s_1)$. Then the equations (b), (c) and (d) become one equation $-\gamma \cdot \chi(s_0)^2 = -\tilde{p}\tilde{q} + \{\sigma - (q-\sigma)\gamma\}\chi(s_0)$. Therefore

$$(***) \qquad \gamma \cdot \chi(s_0)^2 + \{\sigma - (q-\sigma)\gamma\}\chi(s_0) - \tilde{p}\tilde{q} = 0.$$

Two real numbers $\alpha, -\beta$ with $-\beta < 0 < \alpha$ is a pair of solutions of $(***)$ with respect to $w(s_0)$, i.e. the relation $(*)$ in our conlusion holds.

We determine the last two character χ_2, χ_3 by $\chi_2(s_0) = \alpha$, $\chi_2(s_1) = -\gamma\alpha$ and $\chi_3(s_0) = -\beta$, $\chi_3(s_1) = \gamma\beta$.

Using the method of [W2], the weights of these characters are calculated as

$$\begin{aligned}
m_{22}^0 = w(\chi_2)^{-1} &= \Big(1 + \chi_2(h_1)^2 w(h_1) + \chi_2(s_0)^2 w(s_0) + \chi_2(s_1)^2 w(s_1)\Big) \Big/ w(\mathcal{K}_{nh}^{p,q}) \\
&= \Big(1 + \tilde{p} + \frac{\alpha^2(\tilde{p}+\tau)}{\tilde{p}\tilde{q}} + \frac{\gamma^2\alpha^2\tilde{\tau}}{\tilde{p}\tilde{q}}\Big) \cdot \frac{\tilde{p}\tilde{q}}{(\tilde{p}+1)(\tilde{q}+1)} \\
&= \Big(1 + \tilde{p} + \frac{\alpha^2(\tilde{p}+\tau)(\tilde{\tau}+\tilde{p}+\tau)}{\tilde{\tau}\tilde{p}\tilde{q}}\Big) \cdot \frac{\tilde{p}\tilde{q}}{(\tilde{p}+1)(\tilde{q}+1)} \\
&= (1 + \frac{\alpha^2\gamma}{\tilde{p}\tilde{q}}) \cdot \frac{\tilde{p}\tilde{q}}{(\tilde{q}+1)} \\
&= \frac{\alpha+\beta}{\beta} \cdot \frac{\tilde{p}\tilde{q}}{\tilde{q}+1},
\end{aligned}$$

and this leads the value of a weight $w(\chi_2)$. In a similar way we get the value of $w(\chi_3)$.

□

The curve of $\alpha = \beta$, on which the ratio of dual weights $\beta/\alpha = w(\chi_2)/w(\chi_3) = 1$, is a dotted line (L) containing the center point $(p/2, q/2)$ in Figure 1. This is obtained from the equation $\sigma = \frac{q}{\tilde{p}+1} \cdot (\tilde{p}+\tau)$ which comes from $q = \sigma(1 + 1/\gamma)$.

Now we calculate the structure equations of dual hypergroups $\widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma)$.

Theorem 4.4. *The dual* $\widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma) = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ *has the structure equations as a signed hypergroup:*

$$\text{(A)}\ \ \chi_1^2 = \tilde{q}\chi_0 + q\chi_1,\ \ \chi_1\chi_2 = \frac{\beta - \tilde{q}\alpha}{\alpha+\beta}\chi_2 + \frac{\alpha + \tilde{q}\alpha}{\alpha+\beta}\chi_3,\ \ \chi_1\chi_3 = \frac{\beta + \tilde{q}\beta}{\alpha+\beta}\chi_2 + \frac{\alpha - \tilde{q}\beta}{\alpha+\beta}\chi_3,$$

$$\text{(B1)}\ \ \chi_2^2 = \frac{\tilde{p}\tilde{q}(\alpha+\beta)}{(1+\tilde{q})\beta}\chi_0 + \frac{\tilde{p}(\beta - \tilde{q}\alpha)}{(1+\tilde{q})\beta}\chi_1 + \frac{p\beta + \alpha^2(1-\gamma)}{\alpha+\beta}\chi_2 + \frac{p\alpha - \alpha^2(1-\gamma)}{\alpha+\beta}\chi_3,$$

$$\text{(B2)}\ \ \chi_3^2 = \frac{\tilde{p}\tilde{q}(\alpha+\beta)}{(1+\tilde{q})\alpha}\chi_0 + \frac{\tilde{p}(\alpha - \tilde{q}\beta)}{(1+\tilde{q})\alpha}\chi_1 + \frac{p\beta + \beta^2(1-\gamma)}{\alpha+\beta}\chi_2 + \frac{p\alpha - \beta^2(1-\gamma)}{\alpha+\beta}\chi_3,$$

$$\text{(C)}\ \ \chi_2\chi_3 = \tilde{p}\chi_1 + \frac{p\beta - \alpha\beta(1-\gamma)}{\alpha+\beta}\chi_2 + \frac{p\alpha + \alpha\beta(1-\gamma)}{\alpha+\beta}\chi_3.$$

Proof. It is obvious that $\{\chi_0, \chi_1\} = \widehat{\mathcal{L}} \cong \mathcal{L}(q)$ from Proposition 3.3. This shows the first equation of (A). With the relation of the structure and its characters[W2], for example, the coefficient m_{12}^3 of χ_3 in a product of $\chi_1\chi_2$ is

$$\begin{aligned} m_{12}^3 &= \Big(1+\tilde{p}^2/\tilde{p}+\tilde{q}\alpha\beta\cdot\frac{\tilde{p}+\tau}{\tilde{p}\tilde{q}}+\tilde{q}\gamma^2\alpha\beta\cdot\frac{\tilde{\tau}}{\tilde{p}\tilde{q}}\Big)\cdot\frac{w(\chi_3)}{w(\mathcal{K})} \\ &= \Big(\tilde{p}+1+\frac{\alpha\beta}{\tilde{p}}(\tilde{p}+\tau+\gamma^2\tilde{\tau})\Big)\cdot\frac{w(\chi_3)}{w(\mathcal{K})} \\ &= \Big(\tilde{p}+1+\frac{\alpha\beta}{\tilde{p}}\gamma(\tilde{p}+1)\Big)\cdot\frac{\alpha}{(\tilde{p}+1)(\alpha+\beta)} \\ &= \frac{(\tilde{q}+1)\alpha}{\alpha+\beta}. \end{aligned}$$

It is shown that $m_{13}^2=(\tilde{q}+1)\beta/(\alpha+\beta)$ in a similar way. Therefore, we have equations (A).

The coefficient m_{22}^2 of χ_2 in a product of χ_2^2 is

$$\begin{aligned} m_{22}^2 &= \Big(1-\tilde{p}^3/\tilde{p}+\alpha^3\cdot\frac{\tilde{p}+\tau}{\tilde{p}\tilde{q}}-\gamma^3\alpha^3\cdot\frac{\tilde{\tau}}{\tilde{p}\tilde{q}}\Big)\cdot\frac{w(\chi_2)}{w(\mathcal{K})} \\ &= \Big(1-\tilde{p}^2+\frac{\alpha^3}{\tilde{p}\tilde{q}}(\tilde{p}+\tau-\gamma^3\tilde{\tau})\Big)\cdot\frac{w(\chi_2)}{w(\mathcal{K})} \\ &= \Big(1-\tilde{p}^2+\frac{\alpha^3}{\tilde{p}\tilde{q}}(\tilde{p}+\tau)(1-\gamma^2)\Big)\cdot\frac{\beta}{(\tilde{p}+1)(\alpha+\beta)} \\ &= \Big(1-\tilde{p}^2+\frac{\alpha^3\gamma}{\tilde{p}\tilde{q}}(\tilde{p}+1)(1-\gamma)\Big)\cdot\frac{\beta}{(\tilde{p}+1)(\alpha+\beta)} \\ &= \Big(p+\frac{\alpha^3\gamma}{\tilde{p}\tilde{q}}(1-\gamma)\Big)\cdot\frac{\beta}{(\alpha+\beta)} \\ &= \frac{p\beta+\alpha^2(1-\gamma)}{\alpha+\beta}. \end{aligned}$$

The equation (B1) is implied from $m_{22}^0=w(\chi_2)^{-1}$ and $m_{12}^2/w(\chi_2)=m_{22}^1/w(\chi_1)$. We also have (B2). The equation (C) is obtained from $m_{23}^2/w(\chi_2)=m_{22}^3/w(\chi_3)$.

□

Theorem 4.5. *The dual* $\widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma)$ *is a hypergroup if the following conditions* (E) *and* (F) *are satisfied:*

(E) $\beta-\tilde{q}\alpha\geq 0,\ \ \alpha-\tilde{q}\beta\geq 0,$

(F) $p\beta+\alpha^2(1-\gamma)\geq 0,\ \ p\alpha-\beta^2(1-\gamma)\geq 0.$

Proof. We will check that all coefficients are non negative. First we will show that the inequalities (D): $p-\alpha(1-\gamma)\geq 0,\ \ p+\beta(1-\gamma)\geq 0$ hold in the region $\mathfrak{D}_h$ of Firure 1. Notice that $d\gamma/d\tau=1/\tilde{\tau}^2>0$. We can consider that the positive numbers α,β determined by the relation $(*)$ in Proposition 4.3 are functions of two variables γ,σ, where $\tilde{p}\leq\gamma\leq 1/\tilde{p}$.

Since $\partial\alpha/\partial\sigma-\partial\beta/\partial\sigma=-(1+1/\gamma)>0$ and $\beta\cdot\partial\alpha/\partial\sigma+\alpha\cdot\partial\beta/\partial\sigma=0$, we have $\partial\alpha/\partial\sigma<0$ and $\partial\beta/\partial\sigma>0$. Moreover $\partial\alpha/\partial\gamma(\gamma,0)=\partial/\partial\gamma(1/2\cdot\sqrt{q^2+4\tilde{p}\tilde{q}\gamma^{-1}}+q/2)=-\tilde{p}\tilde{q}/(\gamma^2\sqrt{q^2+4\tilde{p}\tilde{q}\gamma^{-1}})<0$.

At first we will prove that $p-\alpha(1-\gamma)\geq 0$. When $\gamma\geq 1$, then the first inequality in (D) holds. If $\gamma<1$, then $p-\alpha(\gamma,\sigma)(1-\gamma)\geq p-\alpha(\gamma,0)(1-\gamma)\geq p-\alpha(\tilde{p},0)(1-\tilde{p})=p-1\cdot p=0$. Hence $p-\alpha(1-\gamma)\geq 0$.

The next it is obtained that $p+\beta(1-\gamma)\geq 0$ from the relations of $\partial\beta/\partial\gamma(\gamma,q)<0$ and $\beta(1/\tilde{p},q)=1/2\cdot\{\sqrt{(q-q(1+\tilde{p}))^2+4\tilde{p}^2\tilde{q}}-q+q(1+\tilde{p})\}=\tilde{p}$.

Therefore inequalities (D) are always satisfied when $\mathcal{K}_h^{p,q}(\tau,\sigma)$ is a hypergroup.

Hence it is shown that the 4th coefficient of (B1) and the 3rd one of (B2) are non-negativite.

The conditions (E) and (F) assure that non-negativity of coefficients in (A), (B1) and (B2) respectively.

□

REMARK. Let $\mathfrak{D}'_h$ be a region in which the conditions (E) and (F) in Theorem 4.5 are satisfied. We give an atlas of $\mathfrak{D}'_h$ and the region $\mathfrak{D}_h$ of Theorem 4.1. In order to view $\mathfrak{D}'_h$, the special values of $\alpha(\gamma,\sigma)$ and $\beta(\gamma,\sigma)$ as functions with respect to γ,σ are the followings:

$\alpha(\tilde{p},0)=1$	$\alpha(1,q/2)=\sqrt{\tilde{p}\tilde{q}}$	$\alpha(1/\tilde{p},q)=\tilde{p}\tilde{q}$	$\alpha(\gamma,0)>\beta(\gamma,0)$
$\beta(\tilde{p},0)=\tilde{q}$	$\beta(1,q/2)=\sqrt{\tilde{p}\tilde{q}}$	$\beta(1/\tilde{p},q)=\tilde{p}$	$\alpha(\gamma,q)<\beta(\gamma,q)$

$\alpha\Big(\tilde{p},\dfrac{2\tilde{p}q}{\tilde{p}+1}\Big)=\tilde{q}$	$\alpha\Big(\dfrac{1}{\tilde{p}},\dfrac{pq}{\tilde{p}+1}\Big)=\tilde{p}$
$\beta\Big(\tilde{p},\dfrac{2\tilde{p}q}{\tilde{p}+1}\Big)=1$	$\beta\Big(\dfrac{1}{\tilde{p}},\dfrac{pq}{\tilde{p}+1}\Big)=\tilde{p}\tilde{q}$

When $p\beta+\alpha^2(1-\gamma)=0$ and $p+\beta(1-\gamma)=0$, it is clear that $\alpha=\beta=p/(\gamma-1)$. The intersection of (F) and (D) are on the line (L) in Figure 1. It is obvious that $\frac{q}{\tilde{p}+1}>pq>\frac{pq}{\tilde{p}+1}$ and $\frac{2\tilde{p}q}{\tilde{p}+1}>\tilde{p}q>\frac{\tilde{p}q}{\tilde{p}+1}$.

There are many varieties of intersections of conditions:(e),(E) and (F), so that we give a typical figure.

Assume $p,q<3/5$. The thick curves (D_b),(E_b) and (F_b) are boundaries of conditions (D)–(F) in the Figure 2 and the boundaries (e_b^1),(e_b^2) of Figure 1 are drown by the thin curves.

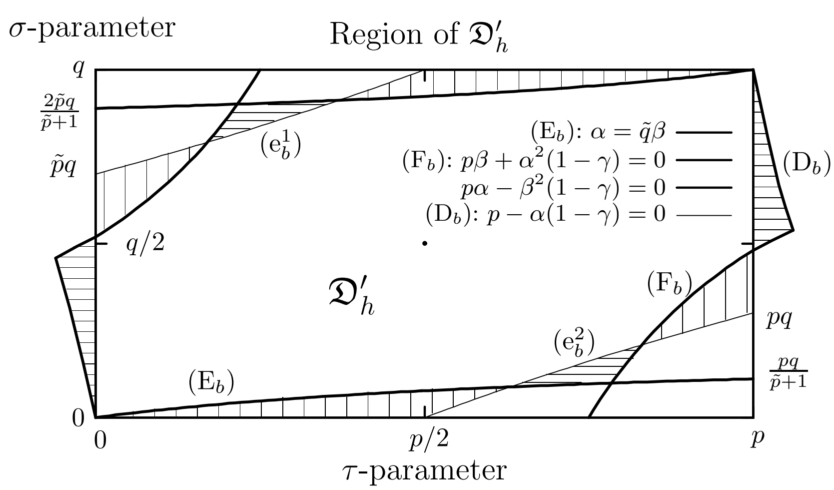

Figure 2: Strong hypergroup condition of σ,τ in the case $p,q<3/5$

Let $\mathfrak{D}'_h$ be the region with the thick curves (D_b),(E_b),(F_b) and etc. including a central point $(p/2,q/2)$ in Figure 2. Using the result in Theorem 4.5, the dual $\widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma)$ is a hypergroup in $\mathfrak{D}'_h$. Since $\mathcal{K}_h^{p,q}(\tau,\sigma)$ is a hypergroup in $\mathfrak{D}_h$, we have that hypergroups $\mathcal{K}_h^{p,q}(\tau,\sigma)$ are strong when (τ,σ) is in the intersection $\mathfrak{D}_h \cap \mathfrak{D}'_h$. In the area of $\mathfrak{D}'_h \setminus \mathfrak{D}_h$ with the horizontal strip, $\widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma)$ is a hypergroup but $\mathcal{K}_h^{p,q}(\tau,\sigma)$ is not.

In the horizontal striped regions out of the rectangle: $0 \le \tau \le p$ and $0 \le \sigma \le q$, which is the extremely right area with the boundary (D_b) and the extremely left one, the dual $\mathfrak{D}'_h$ is a hypergroup but $\mathfrak{D}_h$ is not a hypergroup.

But if p or q is nearly to 1, then there are many varieties of the regions in which the conditions (E),(F) and (e) are satisfied.

We apply our theorems to determine the structure of strong hypergroup of order four which have non-trivial subhypergroups. Given a exact sequence of hypergroups: $\mathbf{1} \to \mathcal{H} \to \mathcal{K} \to \mathcal{L} \to \mathbf{1}$, where $|\mathcal{K}| = 4$. By the order condition : $|\mathcal{H}| + |\mathcal{L}| - 1 \le |\mathcal{K}| \le |\mathcal{H}| \cdot |\mathcal{L}|$, the possible orders of $\mathcal{H}$ and $\mathcal{L}$ are the following cases:

$$|\mathcal{H}| = |\mathcal{L}| = 2, \tag{7}$$

$$|\mathcal{H}| = 3,\ |\mathcal{L}| = 2, \tag{8}$$

$$|\mathcal{H}| = 2,\ |\mathcal{L}| = 3. \tag{9}$$

In the case of (7), $\mathcal{K}$ is already determined by Theorem 3.2 and Theorem 4.1. Using the results of Theorem 3.4 and Theorem 4.5, we can estimate the strong hypergroups.

In the case of (8) or (9), we have $|\mathcal{H}| + |\mathcal{L}| - 1 = |\mathcal{K}| = 4$. When $|\mathcal{H}|$ and $|\mathcal{L}|$ are fixed, $|\mathcal{K}|$ has a minimal order. Hence it is shown that $\mathcal{K}$ is a *join* $\mathcal{H} \vee \mathcal{L}$.

N.J. Wildberger[W2] shows that hypergroups $\mathcal{W}_3$ of order 3 are completely analyzed, which includes Jewett's example[J]. In the case of (8), it is obvious that $\mathcal{K} = \mathcal{W}_3 \vee \mathcal{L}(\rho)$, and $\widehat{\mathcal{K}} = \widehat{\mathcal{L}}(\rho) \vee \widehat{\mathcal{W}_3} \cong \mathcal{L}(\rho) \vee \widehat{\mathcal{W}_3}$. In the case of (9), it is obvious that $\mathcal{K} = \mathcal{L}(\rho) \vee \mathcal{W}_3$ and $\widehat{\mathcal{K}} \cong \widehat{\mathcal{W}_3} \vee \mathcal{L}(\rho)$.

For strong $\mathcal{W}_3$, the joins $\mathcal{W}_3 \vee \mathcal{L}(\rho)$ and $\mathcal{L}(\rho) \vee \mathcal{W}_3$ are also strong.

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