ON THE BEST CONSTANT FOR L^p SOBOLEV INEQUALITIES

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Abstract. A canonical form of the reproducing kernel for $X \subset W^{m,p}(\Omega)$ is given. (See Theorem 2 as well as Theorem 5.) By its virtue, the best constants for embedding $W^{m,p} \to B^0$ are given for some concrete Sobolev spaces. (See Theorem 8,10 and 14.)

Introduction It was Kametaka et al.[1][2] who clearly pointed out that there exists a close relationship between the Green functions and the reproducing kernels. Using this relationship, they determined the best constants for various Sobolev inequalities, especially in the L^2 framework. In the L^p framework ($p \neq 2$), however, the usual Green functions in themselves are sometimes inappropriate to determine the best constants[3][4]. To deal with the case $p \neq 2$, we modify the notion of the Green functions in the sequel.

1 Notation. We use multi-indices. For

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$$

with nonnegative integers $\alpha_1 \geq 0, \alpha_2 \geq 0, \cdots, \alpha_N \geq 0$, we denote

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_N}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}.$$

In the sequel, p is always a positive constant satisfying $1 < p < \infty$ while $q > 1$ is the conjugate of p which is determined by $1/p + 1/q = 1$. Let $\Omega \subset \mathbf{R}^N$ be an open domain. The norm of $u \in L^p = L^p(\Omega)$ is denoted as

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p}$$

and the notation

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|$$

is also used. For each nonnegative integer $m \geq 0$ and the above $p \in (1, \infty)$, the Sobolev space $W^{m,p}(\Omega)$ is defined as

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega);\, \partial^\alpha u \in L^p(\Omega),\, |\alpha| \leq m\}$$

and we use one of its standard norms

$$\|u\|_{m,p} \equiv \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p^p \right)^{1/p}$$

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in the sequel.

In addition, we use also the notation

$$\mathrm{sgn} z = \begin{cases} z/|z| & (z \neq 0) \\ 0 & (z = 0) \end{cases}$$

for complex $z \in \mathbf{C}$.

2 Results.

Proposition 1. *Let $\Omega \subset \mathbf{R}^N$ be open and X be a closed subspace of $W^{m,p}(\Omega)(1 < p < \infty)$ with the standard norm*

$$\|u\|_{m,p} \equiv \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p^p \right)^{1/p}.$$

Suppose that

$$\|u\|_X \equiv \left(\sum_{|\alpha| \leq m} C_\alpha \|\partial^\alpha u\|_p^p \right)^{1/p}, u \in X$$

with nonnegative constants $C_\alpha \geq 0(|\alpha| \leq m)$ ($C_\alpha > 0$ for some α)determines a norm (possibly $\|u\|_X \equiv \|u\|_{m,p}$) equivalent to $\|u\|_{m,p}$, i.e.,

$$(1/k)\|u\|_{m,p} \leq \|u\|_X \leq k\|u\|_{m,p}, u \in X$$

for some constant $k > 1$. (Notice that the equivalence may fail for the whole $W^{m,p}(\Omega)$.) Then, for an arbitrarily fixed $v \in X$,

$$F(u) = \sum_{|\alpha| \leq m} \int_\Omega C_\alpha \partial^\alpha u(x) |\partial^\alpha v(x)|^{p-1} \overline{\mathrm{sgn} \partial^\alpha v(x)} dx$$

is a bounded linear functional for $u \in X$ and

$$|F(u)| \leq \|v\|_X^{p-1} \|u\|_X, u \in X.$$

Here the equality holds if and only if

$$u(x) \equiv v(x) \quad (x \in \Omega)$$

up to the constant multiplication.

Proof . By the integral version of the Hölder inequality, we have

$$|F(u)| \leq \sum_{|\alpha| \leq m} C_\alpha \|\partial^\alpha u\|_p \|\partial^\alpha v\|_p^{p-1}$$

noticing

$$\| |\partial^\alpha v(x)|^{p-1} \overline{\mathrm{sgn} \partial^\alpha v(x)} \|_q = \|\partial^\alpha v(x)\|_p^{p-1}$$

for each α. Hence, by the finite series version of the Hölder inequality,

$$\begin{aligned} |F(u)| &\leq \sum_{|\alpha|\leq m} C_\alpha^{1/p}\|\partial^\alpha u\|_p C_\alpha^{1/q}\|\partial^\alpha v\|_p^{p-1} \\ &\leq \left(\sum_{|\alpha|\leq m} C_\alpha\|\partial^\alpha u\|_p^p\right)^{1/p}\left(\sum_{|\alpha|\leq m} C_\alpha\|\partial^\alpha v\|_p^p\right)^{1/q} \\ &= \|v\|_X^{p-1}\|u\|_X. \end{aligned}$$

Here the equalities in two "$\leq$" hold at the same time if and only if

$$u(x) \equiv v(x)$$

up to the constant multiplication. Q.E.D.

Theorem 2. *Let the assumption on Ω, X and $\|\cdot\|_X$ be the same as in Proposition 1. Let also $m > N/p$. Suppose there exist $y \in \overline{\Omega}$ and $v_y \in X$ such that*

$$u(y) = \sum_{|\alpha|\leq m}\int_\Omega C_\alpha \partial^\alpha u(x)|\partial^\alpha v_y(x)|^{p-1}\overline{\mathrm{sgn}\partial^\alpha v_y(x)}dx$$

for all $u \in X$. Then

$$v_y(y) = \|v_y\|_X^p$$

and

$$|u(y)| \leq \|v_y\|_X^{p-1}\|u\|_X = v_y(y)^{(p-1)/p}\|u\|_X \text{ for all } u \in X.$$

Here the equality in $\leq$ holds if and only if

$$u(x) \equiv v_y(x) \quad (x \in \Omega)$$

up to the constant multiplication.

Proof. Substituting $u(x) \equiv v_y(x)$ to the integral, we have

$$v_y(y) = \sum_{|\alpha|\leq m}\int_\Omega C_\alpha \partial^\alpha v_y(x)|\partial^\alpha v_y(x)|^{p-1}\overline{\mathrm{sgn}\partial^\alpha v_y(x)}dx = \sum_{|\alpha|\leq m}\int_\Omega C_\alpha|\partial^\alpha v_y(x)|^p dx = \|v_y\|_X^p.$$

Regarding $F(u) = u(y)$ as a functional in $u \in X$, we have only to apply Proposition 1 to obatin the rest of the assertions. Q.E.D.

Corollary 3. *Let Ω, X and $\|\cdot\|_X$ be the same as in Theorem 2, except for v_y . Suppose there exist $y \in \overline{\Omega}$ and $w_\alpha \in L^q(\Omega)$ $(\alpha \in S = \{\alpha; C_\alpha > 0\})$ such that*

$$u(y) = \sum_{\alpha\in S}\int_\Omega C_\alpha(\partial^\alpha u)\overline{w_\alpha(x)}dx$$

for all $u \in X$. Suppose also there exist $v \in X$ such that

$$\partial^\alpha v = |w_\alpha(x)|^{q-1}\mathrm{sgn} w_\alpha(x) \quad (\alpha \in S)$$

Then

$$|u(y)| \leq \|v\|_X^{p-1}\|u\|_X = v(y)^{(p-1)/p}\|u\|_X.$$

Here the equality in $\leq$ holds if and only if

$$u(x) \equiv v(x)(x \in \Omega)$$

up to the constant multiplication.

Proof . Notice $(p-1)(q-1)=1$. Therefore, the condition in the present Corollary is equivalent to

$$w_\alpha(x)=|\partial^\alpha v|^{p-1}\mathrm{sgn}\partial^\alpha v(x) \quad (\alpha\in S).$$

The rest is clear. Q.E.D.

To prove the converse of Theorem 2, we start with a proposition which is itself the converse of Proposition 1.

Proposition 4. *Let the assumption on* Ω, X *and* $\|\cdot\|_X$ *be the same as in Proposition 1. Suppose that* $F(u)$ *is a bounded linear functional on* X. *Then, there exists a unique* $v\in X$ *such that*

$$F(u)=\sum_{|\alpha|\le m}\int_\Omega C_\alpha\partial^\alpha u(x)|\partial^\alpha v(x)|_{p-1}\overline{\mathrm{sgn}\partial^\alpha v(x)}dx$$

for all $u\in X$.

Proof . Let

$$\nu=\sharp\{\alpha;|\alpha|\le m\}.$$

Now

$$Y=\Big\{\{C_\alpha^{1/p}\partial^\alpha u\}_{|\alpha|\le m} \quad ; \quad u\in X\Big\}$$

is a closed subspace of $(L^p(\Omega))^\nu$ with norm

$$\left(\sum_{|\alpha|\le m}\|u_\alpha\|_p^p\right)^{1/p}.$$

Then $F(u)$ can be regarded as a bounded linear functional $G(w)$ for $w\in Y$. By the Hahn Banach theorem, $G(w)$ is extended to $\tilde{G}(w)$ for all $w\in(L^p(\Omega))^\nu$. We note that the norm of $\tilde{G}(w)$ remains the same as $G(w)$. We also know there exist $\{v_\alpha\}\in(L^q(\Omega))^\nu$ $(q=p/(p-1))$such that

$$\tilde{G}(w)=\sum_{|\alpha|\le m}\int_\Omega w_\alpha(x)\overline{v_\alpha(x)}dx, \quad \text{for all } w\in(L^p(\Omega))^\nu$$

hence

$$G(w)=\tilde{G}(w)=\sum_{|\alpha|\le m}\int_\Omega w_\alpha(x)\overline{v_\alpha(x)}dx, \quad \text{for all } w\in Y.$$

Therefore

$$F(u)=\tilde{G}(\{C_\alpha^{1/p}\partial^\alpha u\})=\sum_{|\alpha|\le m}\int_\Omega C_\alpha^{1/p}\partial^\alpha u(x)\overline{v_\alpha(x)}dx, \quad \text{for all } u\in X.$$

Let us now specify the forms of $\{v_\alpha\}$. For this purpose, we consider the norms of the functional $G,\tilde{G},F$. By the Hölder inequality,

$$\|\tilde{G}\|=\left(\sum_{|\alpha|\le m}\|v_\alpha\|_q^q\right)^{1/q}.$$

From the non-increase of the norm, it follows that

$$\|G\| = \left(\sum_{|\alpha|\le m} \|v_\alpha\|_q^q \right)^{1/q}$$

for $G = \tilde{G}|_Y$. By the definition of the norm of the functional G, there exists a sequence $\{w^j\}_{j\ge 0} \subset Y$ such that

$$\|w^j\| = 1 \quad (j = 0, 1, \cdots) \qquad \lim_{j\to\infty} G(w^j) = \|G\|.$$

Since $\{w^j\}_{j\ge 0} \subset Y \subset (L^p)^\nu$ is a bounded sequence, there exists a subsequence with a weak limit $w \in Y$ (recall $Y \subset (L^p)^\nu$ is a closed subspace) and

$$\|w\| \le 1, \quad G(w) = \lim_{j\to\infty} G(w_j) = \|G\| > 0.$$

hence

$$\|G\| = G(w) \le \|G\|\|w\| \le \|G\|.$$

This means $\|w\| = 1$ and

$$G(w) = \|G\| = \sup_{\|\tilde{w}\|_p \le 1} |G(\tilde{w})|$$

Since the supremum $\|G\|$ is attained by w, the Hölder inequality in $(L^p(\Omega))^\nu$ implies

$$\{w_\alpha\} = k\{|v_\alpha|^{q-1}\mathrm{sgn}(v_\alpha)\} \quad (|\alpha| \le m)$$

with some positive constant $k > 0$. On the other hand, the definition of Y implies there exists $v \in X$ such that

$$\{w_\alpha\} = \{C_\alpha^{1/p}\partial^\alpha v\} \quad (|\alpha| \le m).$$

Therefore,

$$\{v_\alpha\} = k^{-(p-1)}\{C_\alpha^{(p-1)/p}|\partial^\alpha v|^{p-1}\mathrm{sgn}(\partial^\alpha v)\}.$$

Redefining $k^{-1}v$ as v, we know

$$\{v_\alpha\} = \{C_\alpha^{(p-1)/p}|\partial^\alpha v|^{p-1}\mathrm{sgn}(\partial^\alpha v)\}, \quad v \in X$$

We have specified the form of $\{v_\alpha\}$. With this $v \in X$, we have

$$\begin{aligned} F(u) &= \sum_{|\alpha|\le m} \int_\Omega C_\alpha^{1/p}\partial^\alpha u(x)\overline{C_\alpha^{(p-1)/p}|\partial^\alpha v|^{p-1}\mathrm{sgn}(\partial^\alpha v)}dx \\ &= \sum_{|\alpha|\le m} \int_\Omega C_\alpha \partial^\alpha u(x)\overline{|\partial^\alpha v|^{p-1}\mathrm{sgn}(\partial^\alpha v)}dx \end{aligned}$$

for all $u \in X$. In addition, the Hölder inequality implies that

$$\sup_{u\in X} |F(u)|/\|u\|_X$$

is attained only by the scalar multiples of the above $v \in X$. This, in turn, implies the uniqueness of $v \in X$ in the expresseion of $F(u)$. Q.E.D.

Theorem 5. *Let Ω, $X, \|\cdot\|_X, m, p$ be the same as in Theorem 2. Suppose that for a prefixed $y \in \bar{\Omega}$, the value $u(y) \in \mathbf{C}$ for each $u \in X$ determines a bounded linear functional on X. Then there exists a unique $v_y \in X$ such that*

$$u(y) = \sum_{|\alpha|\le m} \int_\Omega C_\alpha \partial^\alpha u(x) |\partial^\alpha v_y(x)|^{p-1} \overline{\mathrm{sgn} \partial^\alpha v_y(x)} dx$$

for all $u \in X$. If y is further an interior point of Ω then $v_y \in X$ satisfies

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} C_\alpha \partial^\alpha \left(|\partial^\alpha v_y(x)|^{p-1} \overline{\mathrm{sgn} \partial^\alpha v_y(x)} \right) = \delta(x-y), \qquad x \in \Omega$$

in the distribution sense.

Remark 6. *We may say the above $v_y(x)$ is a kind of Green functions in the L^p case.*

Proof . The assumption ensures that for the fixed y ,

$$u \mapsto u(y)$$

is a bounded linear functional on X. Therefore the previous Proposition 4 implies that there exists $v_y \in X$ such that

$$u(y) = \sum_{|\alpha|\le m} \int_\Omega C_\alpha \partial^\alpha u(x) |\partial^\alpha v_y(x)|^{p-1} \overline{\mathrm{sgn} \partial^\alpha v_y(x)} dx$$

which is the first claim of the present theorem.

Let $y \in \Omega$. Considering only the case of $u \in C_0^\infty(\Omega)$, we have

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} C_\alpha \partial^\alpha \left(|\partial^\alpha v_y(x)|^{p-1} \overline{\mathrm{sgn} \partial^\alpha v_y(x)} \right) = \delta(x-y), \qquad x \in \Omega$$

in the distribution sense. Q.E.D.

3 Examples.

In this section, we give three examples.

Proposition 7. *Let $u \in X = W^{1,p}(-\infty, \infty)$ and $y \in (-\infty, \infty)$ be arbitrarily fixed. Then*

$$|u(y)| \le 2^{-1/p} (p-1)^{(p-1)/p^2} \|u\|_{1,p}.$$

The equality is attained if and only if u equals

$$\phi_y(x) \equiv e^{-(p-1)^{-1/p}|x-y|}$$

up to the constant multiplication.

Proof . Note that

$$\|u\|_X = \|u\|_{1,p} = \{(\|u\|_p)^p + (\|u'\|_p)^p\}^{1/p}$$

is the standard norm for $W^{1,p}(-\infty, \infty)$. Thus Theorem 5 is applicable. The equation

$$-\{|v_y(x)'|^{p-1} \mathrm{sgn}(v_y'(x))\}' + |v_y(x)|^{p-1} \mathrm{sgn}(v_y(x)) = \delta(x-y)$$

has a solution

$$v_y(x) = 2^{-1/(p-1)}(p-1)^{1/p}e^{-(p-1)^{-1/p}|x-y|} \in W^{1,p}(-\infty,\infty).$$

Hence we have

$$u(y) = \int_{-\infty}^{\infty} \frac{du}{dx}\left\{\left|\frac{dv_y}{dx}\right|^{p-1} \operatorname{sgn}\left(\frac{dv_y}{dx}\right)\right\} + u(x)(v_y(x))^{p-1}dx$$

for any $u \in W^{1,p}(-\infty,\infty)$. Therefore this v_y is the very one mentioned in Theorem 2 as well as Theorem 5. Recalling $\|u\|_X = \|u\|_{1,p}$,

$$|u(y)| \le \|v_y\|_{1,p}^{p-1}\|u\|_{1,p}.$$

Here

$$\|v_y\|_{1,p}^{p-1} = (v_y(y))^{(p-1)/p} = 2^{-1/p}(p-1)^{(p-1)/p^2}$$

hence

$$|u(y)| \le 2^{-1/p}(p-1)^{(p-1)/p^2}\|u\|_{1,p}.$$

Here the equality holds if and only if $u(x)$ is a constant multiple of $v_y(x)$, i.e., that of

$$\phi_y(x) \equiv e^{-(p-1)^{-1/p}|x-y|}.$$

Q.E.D.

Theorem 8. *For any* $u \in X = W^{1,p}(-\infty,\infty)$,

$$\|u\|_\infty \le 2^{-1/p}(p-1)^{(p-1)/p^2}\|u\|_{1,p}.$$

The equality is attained if and only if

$$u(x) = \phi_y(x) \equiv e^{-(p-1)^{-1/p}|x-y|} \quad (-\infty < x < \infty)$$

with some $y \in (-\infty,\infty)$ *up to the constant multiplication.*

Proof. Immediate from the previous Proposition 6.

Let us go on to the second example.

Proposition 9. *Let* $u \in X = W_0^{1,p}(-1,1)$ *and* $y \in (-1,1)$ *be arbitrary.*

$$|u(y)| \le \{(1+y)(1-y)\}^{(p-1)/p}\{(1+y)^{p-1} + (1-y)^{p-1}\}^{-1/p}\|u'\|_p$$

The equality is attained only by

$$\phi_y(x) = \begin{cases} (1-y)(x+1) & (-1 \le x \le y \le 1) \\ (1+y)(1-x) & (-1 \le y \le x \le 1) \end{cases}$$

or its scalar multiples.

Proof . By the Poincaré inequality,

$$\|u\|_X = \|u'\|_p$$

is equivalent to the standard norm

$$\|u\|_{1,p} = \{(\|u\|_p)^p + (\|u'\|_p)^p\}^{1/p}$$

for $X = W_0^{1,p}(-1,1)$. The equation we consider is

$$-\{|v_y(x)'|^{p-1}\mathrm{sgn}(v_y'(x))\}' = \delta(x-y), \quad v_y \in X = W_0^{1,p}(-1,1).$$

Its solution

$$v_y(x) = \begin{cases} (1-y)\{(1+y)^{p-1} + (1-y)^{p-1}\}^{-1/(p-1)}(x+1) & (-1 \le x \le y \le 1) \\ (1+y)\{(1+y)^{p-1} + (1-y)^{p-1}\}^{-1/(p-1)}(1-x) & (-1 \le y \le x \le 1) \end{cases}$$

satisfies

$$\left|v_y'(x)\right|^{p-1} \mathrm{sgn}(v_y'(x)) = \begin{cases} \frac{(1-y)^{p-1}}{(1+y)^{p-1}+(1-y)^{p-1}} & (-1 \le x \le y \le 1) \\ -\frac{(1+y)^{p-1}}{(1+y)^{p-1}+(1-y)^{p-1}} & (-1 \le y \le x \le 1). \end{cases}$$

Hence we have

$$u(y) = \int_{-1}^{1} u'(x) \left\{ \left|v_y'(x)\right|^{p-1} \mathrm{sgn}(v_y'(x)) \right\} dx$$

for any $u \in W_0^{1,p}(-1,1)$. Therefore this v_y is the very one mentioned in Theorem 2 as well as Theorem 5. Recalling $\|u\|_X = \|u'\|_p$,

$$|u(y)| \le \|v_y'\|_p^{p-1}\|u'\|_p = (v_y(y))^{(p-1)/p}\|u'\|_p.$$

Here

$$(v_y(y))^{(p-1)/p} = \{(1-y)(1+y)\}^{(p-1)/p}\{(1+y)^{p-1} + (1-y)^{p-1}\}^{-1/p}.$$

Hence

$$|u(y)| \le \{(1-y)(1+y)\}^{(p-1)/p}\{(1+y)^{p-1} + (1-y)^{p-1}\}^{-1/p}\|u'\|_p.$$

Here the equality holds if and only if

$$u(x) \equiv v_y(x)$$

up to the constant multiplication. Q.E.D.

Theorem 10. *For any* $u \in W_0^{1,p}(-1,1)$,

$$\|u\|_\infty \le 2^{-1/p}\|u'\|_p.$$

Here,the equality is attained if and only if

$$u(x) = \phi(x) = 1 - |x| \quad (-1 \le x \le 1)$$

up to the constant multiplication.

Proof . Almost everything is proved in the previous Proposition 8. We have only to notice that

$$\begin{aligned}\|v_y'\|_p^{p-1} = (v_y(y))^{(p-1)/p} &= \{(1-y)(1+y)\}^{(p-1)/p}\{(1+y)^{p-1}+(1-y)^{p-1}\}^{-1/p} \\ &= \{(1+y)^{-p+1}+(1-y)^{-p+1}\}^{-1/p}\end{aligned}$$

attains its maximum $2^{-1/p}$ at $y=0$. And $v_0(x)$ is a constant multiple of

$$\phi(x) = 1-|x|.$$

Q.E.D.

Let us now work on $H(0,1,0,1)$. Different from the above examples, the exact integral expression of $u(y)$ is difficult to obtain except for $y=0$. So $|u(y)|$ $(y \neq 0)$ will be estimated only from above.

Definition.

$$H(0,1,0,1) = \{u \in W^{2,p}(-1,1); u(\pm 1) = u'(\pm 1) = 0\}.$$

Remark 11. *The norm* $\|u\|_X = \|u''\|_p$ *is equivalent to the standard norm* $\|u\|_{2,p}$ *by the Poincaré inequality.*

To calculate the best constant, let us introduce the Green function $G(x,y)$ for $u''(x) = -f(x)(-1<x<1)$ with Dirichlet boundary condition $u(\pm 1)=0$ as follows. ! !

Definition

$$G(x,y) = \begin{cases}(1/2)(1+x)(1-y) & (-1 \le x \le y \le 1) \\ (1/2)(1-x)(1+y) & (-1 \le y \le x \le 1).\end{cases}$$

The next Lemma gives the characterization of $\mathrm{Ran}(d^2/dx^2)$, i.e., the domain of the *Green (resolvent)operator.*

Lemma 12. ! ! *For* $\phi \in L^p(-1,1)$*, the following are equivalent.*

i) $u''(x) = -\phi(x)$ *for some* $u \in H(0,1,0,1)$.

ii) $\int_{-1}^1 \phi(x)dx = \int_{-1}^1 x\phi(x)dx = 0$.

In this case, $u(y)$ $\quad (-1 \le y \le 1)$ *is expressed as*

$$u(y) = \int_{-1}^1 G(x,y)\phi(x)dx.$$

Proof of i) $\to$ ii). Since $u(-1) = u'(-1) = 0$, we find

$$u(y) = -\int_{-1}^y (y-x)\phi(x)dx \quad (-1 \le y \le 1)$$

hence

$$u'(y) = -\int_{-1}^y \phi(x)dx \quad (-1 \le y \le 1).$$

Therefore, $u(1) = u'(1) = 0$ leads

$$\int_{-1}^{1} (1-x)\phi(x)dx = \int_{-1}^{1} \phi(x)dx = 0,$$

i.e., the condition ii).

Proof of ii) → i). Set

$$u(y) = \int_{-1}^{1} G(x,y)\phi(x)dx.$$

The property of the Green function implies

$$u''(y) = -\phi(y) \quad (-1 \le y \le 1)$$

and

$$u(\pm 1) = 0.$$

In addtion, we have

$$\begin{aligned} u'(y) &= \int_{-1}^{1} (\partial G/\partial y)(x,y)\phi(x)dx \\ &= (-1/2)\int_{-1}^{y} (x+1)\phi(x)dx + (1/2)\int_{y}^{1} (1-x)\phi(x)dx \end{aligned}$$

especially

$$\begin{aligned} u'(1) &= (-1/2)\int_{-1}^{1} (x+1)\phi(x)dx \\ u'(-1) &= (1/2)\int_{-1}^{1} (1-x)\phi(x)dx. \end{aligned}$$

From the assumption, we obtain

$$u'(-1) = u'(1) = 0.$$

Q.E.D.

Now we can introduce the reproducing kernel for $H(0,1,0,1)$ which we will use.

Proposition 13. *Let*

$$H_y(x) = -G(x,y) + (1/4)(1-y^2) = \begin{cases} -(1/4)(1-y)(2x-y+1) & (-1 \le x \le y \le 1) \\ -(1/4)(1+y)(-2x+y+1) & (-1 \le y \le x \le 1). \end{cases}$$

Then,for any $u \in H(0,1,0,1)$ *and* $-1 \le y \le 1$, *the following hold:*

$$\begin{aligned} u(y) &= \int_{-1}^{1} H_y(x)u''(x)dx \quad (-1 \le y \le 1) \\ |u(y)| &\le \|H_y\|_q \|u''\|_p = 2^{-(2q-1)/q}(q+1)^{-1/q}(1-y^2)\|u''\|_p \quad (-1 \le y \le 1) \end{aligned}$$

Proof . For any $\in W^{2,p}(-1,1)$,

$$\begin{aligned}\int_{-1}^{1} H_y(x)u''(x)dx &= -\int_{-1}^{1} G(x,y)u''(x)dx + (1/4)(1-y^2)\int_{-1}^{1} u''(x)dx \\ &= -\int_{-1}^{1} G(x,y)u''(x)dx + (1/4)(1-y^2)(u'(1)-u'(-1)).\end{aligned}$$

If $u \in H(0,1,0,1)$, then the previous Lemma 10 is applicable (recall together with $u'(\pm 1) = 0$) ,

$$\int_{-1}^{1} H_y(x)u''(x)dx = -\int_{-1}^{1} G(x,y)u''(x)dx = u(y)$$

Hence by the Hölder inequality,

$$|u(y)| \leq \int_{-1}^{1} |H_y(x)||u''(x)|dx \leq \|H_y\|_q \|u''\|_p \tag{1}$$

Now we evaluate $\|H_y\|_q$

$$\begin{aligned}\|H_y\|_q^q &= \int_{-1}^{1} |H_y(x)|^q dx \\ &= 4^{-q}(1-y)^q \int_{-1}^{y} |2x-y+1|^q dx + 4^{-q}(1+y)^q \int_{y}^{1} |-2x+y+1|^q dx \\ &= 4^{-q}(1-y)^q \int_{-(1+y)/2}^{(1+y)/2} |2x|^q dx + 4^{-q}(1+y)^q \int_{-(1-y)/2}^{(1-y)/2} |2x|^q dx \\ &= 2\cdot 4^{-q}(1-y)^q \cdot 2^{-1}(q+1)^{-1}(1+y)^{q+1} + 2\cdot 4^{-q}(1-y)^q \cdot 2^{-1}(q+1)^{-1}(1+y)^{q+1} \\ &= 2^{-2q+1}(q+1)^{-1}(1-y)^q(1+y)^q.\end{aligned}$$

Thus

$$|u(y)| \leq 2^{-(2q-1)/q}(q+1)^{-1/q}(1-y^2)\|u''\|_p \quad (-1 < y < 1)$$

for all $u \in H(0,1,0,1)$. Q.E.D.

Theorem 14.

$$\|u\|_\infty \leq 2^{-(2q-1)/q}(q+1)^{-1/q}\|u''\|_p$$

for all $u \in H(0,1,0,1)$. Here the equality is attained if and only if

$$u(x) = \int_{-1}^{1} G(x,y)\psi(y)dy \quad (-1 \leq x \leq 1)$$

up to the constant multiplication where

$$\begin{aligned}\psi(x) &= 4^{q-1}|H(x,0)|^{q-1}\mathrm{sgn}(H(x,0)) \\ &= \begin{cases} -(-2x-1)^{q-1} & (-1 \leq x < -1/2) \\ (2x+1)^{q-1} & (-1/2 \leq x < 0) \\ (-2x+1)^{q-1} & (0 \leq x < 1/2) \\ -(2x-1)^{q-1} & (1/2 \leq x \leq 1). \end{cases}\end{aligned}$$

Proof . From the previous Proposition 11, we have

$$|u(y)| \leq 2^{-(2q-1)/q}(q+1)^{-1/q}(1-y^2)\|u''\|_p \leq \|H_0\|_q\|u''\|_p = 2^{-(2q-1)/q}(q+1)^{-1/q}\|u''\|_p$$

for all y and all $u \in H(0,1,0,1) \setminus \{0\}$. Thus the first assertion is clear. And we have only to work on the case $y = 0$ for the second assertion. Putting $y = 0$,

$$u(0) = \int_{-1}^{1} H_0(x)u''(x)dx.$$

Therefore, the equality in $\leq$ of

$$|u(0)| \leq \|H_0\|_q\|u''\|_p = 2^{-(2q-1)/q}(q+1)^{-1/q}\|u''\|_p$$

holds if $u''(x)$ $(u \in H(0,1,0,1))$happens to be

$$\psi(x) = 4^{q-1}|H_0(x)|^{q-1}\mathrm{sgn}(H_0(x))$$

or its scalar multiples (see Corollary 3). This can actually occur since

$$\int_{-1}^{1} \psi(x)dx = 0, \quad \int_{-1}^{1} x\psi(x)dx = 0.$$

The first equality follows from the fact

$$\begin{aligned} \psi(-(1/2)-t) &\equiv \ -\psi(-(1/2)+t) \quad (-1/2 \leq t \leq 1/2), \\ \psi((1/2)-t) &\equiv \ -\psi((1/2)+t) \quad (-1/2 \leq t \leq 1/2) \end{aligned}$$

while the second equality follows from the fact that $\psi(x)$ is an even function hence that $x\psi(x)$ is an odd function. Therefore Lemma 10 is applicable and $u''(x) = -\psi(x)$ has a solution $u \in H(0,1,0,1)$ which is expressed as

$$u(x) = \int_{-1}^{1} G(x,y)\psi(y)dy.$$

Q.E.D.

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