*APPROXIMATING COMMON FIXED POINTS OF NONEXPANSIVE SEMIGROUPS IN BANACH SPACES BY METRIC PROJECTIONS

SACHIKO ATSUSHIBA AND WATARU TAKAHASHI

Received June 25, 2008; revised August 27, 2008

Abstract. In this paper, we prove a strong convergence theorem by the hybrid method for nonexpansive semigroups in Banach spaces. Using this theorem, we obtain some strong convergence theorems in Banach spaces.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then, a mapping $T : C \to C$ is called nonexpansive [5] if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know iteration procedures for finding a fixed point of a nonexpansive mapping; see, for instance, [11, 14]. In 2003, Nakajo and Takahashi [13] studied the following iteration procedure of finding a fixed point of a nonexpansive mapping in a Hilbert space by using the hybrid method in mathematical programming:

$$\begin{aligned}
&x_1 = x \in C,\\
&y_n = \alpha_n x_n + (1-\alpha_n) T x_n,\\
&C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\},\\
&Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, 3, \dots,
\end{aligned}$$

where $0 \le \alpha_n \le 1$ and $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. Xu [26] also introduced another hybrid method. Motivated by Nakajo and Takahashi [13] and Xu [26], Matsushita and Takahashi [12] introduced the following iterative algorithm for finding a fixed point of a nonexpansive mapping in a Banach space:

$$\begin{aligned}
&x_1 = x \in C,\\
&C_n = \overline{\text{co}}\{z \in C : \|z - Tz\| \le t_n \|x_n - Tx_n\|\},\\
&Q_n = \{z \in C : \langle x_n - z, J(x - x_n) \rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, 3, \dots,
\end{aligned}$$

where $0 \le t_n < \infty$ and $P_{C_n \cap Q_n}$ is the metric projection of E onto $C_n \cap Q_n$ (see also [26]). On the other hand, we also know many convergence theorems for finding common fixed points of nonexpansive semigroups in Hilbert spaces or Banach spaces; see, for instance, [1, 2, 3, 4, 15, 16, 17, 18, 19, 21, 22, 23, 24].

In this paper, using the idea of Matsushita and Takahashi [12], we prove a strong convergence theorem for nonexpansive semigroups in Banach spaces by the hybrid method and

*This research was supported by Grant-in-Aid for Young Scientists (B), the Ministry of Education,Culture,Sports,Science and Technology, Japan, and Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

2000 *Mathematics Subject Classification.* Primary 47H09, 47H10.

Key words and phrases. Fixed point, iteration, nonexpansive mapping, hybrid method, nonexpansive semigroup, strong convergence.

metric projections. Using this theorem, we obtain some strong convergence theorems in Banach spaces.

1. Preliminaries

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$. We denote by E^* the topological dual space of E. We denote by $\mathbb{R}$ the set of all real numbers. In addition, we denote by $\mathbb{N}$ and $\mathbb{R}^+$ the sets of all positive integers, and all nonnegative real numbers, respectively.

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^*\rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

Let C be a subset of a Banach space and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$.

A Banach space E is said to be strictly convex if $\dfrac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \le \varepsilon \le 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \;:\; \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \ge \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \le r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x - y\| \ge \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. The multi-valued mapping J from E into E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for every} \quad x \in E$$

is called the duality mapping of E. From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$. A Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

Let C be a closed convex subset of a reflexive, strictly convex and smooth Banach space E. Then, for any $x \in E$, there exists a unique point x_0 in C such that

$$\|x - x_0\| = \min_{y \in C} \|x - y\|.$$

The mapping P_C defined by $P_C x = x_0$ is called the metric projection from E onto C. Let $x \in E$ and $u \in C$. Then, it is known that $u = P_C x$ if and only if

$$\langle u - y, J(x - u)\rangle \geq 0 \tag{1}$$

for all $y \in C$ (see [25]).

The following lemma was proved by Bruck [6].

Lemma 1.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Then, for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\|T\left(\sum_{j=0}^{n} \lambda_j x_j\right) - \sum_{j=0}^{n} \lambda_j T x_j\right\|\right) \leq \max_{0 \leq j \leq k \leq n} \left(\|x_j - x_k\| - \|Tx_j - Tx_k\|\right)$$

for all $n \in \mathbb{N}$, $\{\lambda_i\}_{i=0}^n \in \Delta_n$, $\{x_i\}_{i=0}^n \subset C \cap B_r$ and $T \in Lip(C, 1)$, where $\Delta_n = \{\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\} : 0 \leq \lambda_i (0 \leq i \leq n)$ and $\sum_{i=0}^n \lambda_i = 1\}$, $B_r = \{z \in E : \|z\| \leq r\}$ and $Lip(C, 1)$ is the set of all nonexpansive mappings from C into E.

Let S be a commutative semigroup and let $B(S)$ be the Banach space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$ and $g \in B(S)$, we can define an element $\ell_s g \in B(S)$ by $(\ell_s g)(t) = g(st)$ for all $t \in S$. We also denote by ℓ_s^* the conjugate operator of ℓ_s. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. A linear functional μ on X is called a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ or $\int g(t) d\mu(t)$ instead of $\mu(g)$ for $\mu \in X^*$ and $g \in X$. Further, assume that X is invariant under every $\ell_s, s \in S$, i.e., $\ell_s X \subset X$ for each $s \in S$. Then, a mean μ on X is called invariant if $\mu(\ell_s g) = \mu(g)$ for all $s \in S$ and $g \in X$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean μ on S is also a mean on any subspace X of $B(S)$ containing constants.

The following definition which was introduced by Takahashi [21] is crucial in the nonlinear ergodic theory for abstract semigroups (see also [8]). Let h be a function of S into E such that the weak closure of $\{h(t) : t \in S\}$ is weakly compact. Let X be a subspace of $B(S)$ containing constants and invariant under every ℓ_s, $s \in S$. Assume that for each $x^* \in E^*$, the function $t \mapsto \langle h(t), x^*\rangle$ is an element of X. Then, for any $\mu \in X^*$ there exists a unique element $h_\mu \in E$ such that

$$\langle h_\mu, x^*\rangle = (\mu)_t \langle h(t), x^*\rangle = \int \langle h(t), x^*\rangle \, d\mu(t)$$

for all $x^* \in E^*$. If μ is a mean on X, then h_μ is contained in $\overline{\text{co}}\{h(t) : t \in S\}$ (for example, see [9, 10, 21, 25]). Sometimes, h_μ will be denoted by $\int h(t) d\mu(t)$.

Let C be a closed convex subset of a Banach space E. Then, a family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(a) $T(st) = T(s)T(t)$ for all $s, t \in S$;
(b) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in S$. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that for each $x \in C$ and $x^* \in E^*$, the weak closure of $\{T(t)x : t \in S\}$ is weakly compact and the mapping $t \mapsto \langle T(t)x, x^*\rangle$ is an element of X. Let μ be a mean on X. Following [15], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$ for $x \in C$. We remark that T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathcal{S})$. If μ is a finite mean, i.e.,

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} \ (t_i \in S, a_i \geq 0, \ \sum_{i=1}^{n} a_i = 1),$$

then

$$T_\mu x = \sum_{i=1}^{n} a_i T(t_i) x.$$

The following was proved in [17, 1] (see also [8]).

Lemma 1.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ such that $1 \in X$, it is ℓ_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on X such that $\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$. Then, for each $r > 0$, $w \in C$ and $t \in S$,*

$$\lim_{n\to\infty} \sup_{y \in D_r} \|T_{\mu_n} y - T(t) T_{\mu_n} y\| = 0,$$

where $D_r = \{z \in C : \|z - w\| \leq r\}$.

2. Strong convergence theorem

In this section, we prove a strong convergence theorem by the hybrid method for nonexpansive semigroups in Banach spaces. Before proving it, we obtain the following lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E, let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ such that $1 \in X$, it is ℓ_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on X such that $\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$ for each $s \in S$ and let $\{T_{\mu_n}\}$ be a sequence of nonexpansive mappings of C into itself such that*

$$\langle T_{\mu_n} x, x^* \rangle = (\mu_n)_t \langle T(t)x, x^* \rangle$$

for all $x \in C$ and $x^* \in E^*$. Consider the following iteration scheme:

$$\begin{aligned} x_1 &= x \in C, \\ C_n &= \overline{\text{co}}\{z \in C : \|z - T_{\mu_n} z\| \leq t_n \|x_n - T_{\mu_n} x_n\|\}, \\ D_n &= \{z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap D_n}(x_1) \end{aligned} \tag{2}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap D_n}$ is the metric projection of E onto $C_n \cap D_n$ and $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$. Then, $\{x_n\}$ is well-defined.

Proof. It is easy to check that $C_n \cap D_n$ is closed and convex, and $F(\mathcal{S}) \subset C_n$ for each $n \in \mathbb{N}$. Since $F(\mathcal{S}) \subset C_1$ and $D_1 = C$, we obtain $F(\mathcal{S}) \subset C_1 \cap D_1$. Suppose $F(\mathcal{S}) \subset C_k \cap D_k$ for each $k \in \mathbb{N}$. Then, there exists a unique element $x_{k+1} \in C_k \cap D_k$ such that $x_{k+1} = P_{C_k \cap D_k} x$. It follows from (1) and $F(\mathcal{S}) \subset C_k \cap D_k$ that

$$\langle x_{k+1} - u, J(x - x_{k+1}) \rangle \geq 0$$

for all $u \in F(\mathcal{S})$. This gives us $F(\mathcal{S}) \subset D_{k+1}$. It follows that $F(\mathcal{S}) \subset C_{k+1} \cap D_{k+1}$. By mathematical induction, we obtain that $F(\mathcal{S}) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well-defined. □

Theorem 2.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E. Let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ such that $1 \in X$, it is ℓ_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on X such that $\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$ for each $s \in S$ and let $\{T_{\mu_n}\}$ be a sequence of nonexpansive mappings of C into itself such that*

$$\langle T_{\mu_n} x, x^* \rangle = (\mu_n)_t \langle T(t)x, x^* \rangle$$

for all $x \in C$ and $x^* \in E^*$. Consider the following iteration scheme:

$$\begin{aligned} & x_1 = x \in C, \\ & C_n = \overline{\mathrm{co}}\{z \in C : \|z - T_{\mu_n} z\| \le t_n \|x_n - T_{\mu_n} x_n\|\}, \\ & D_n = \{z \in C : \langle x_n - z, J(x - x_n) \rangle \ge 0\}, \\ & x_{n+1} = P_{C_n \cap D_n}(x_1) \end{aligned} \tag{3}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n \cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection from E onto $F(\mathcal{S})$.

Proof. Since S is commutative, it follows from [7, 20] that $F(\mathcal{S})$ is nonempty. Put $u = P_{F(\mathcal{S})}x$. Since $F(\mathcal{S}) \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n} x$, we have that

$$\|x - x_{n+1}\| \le \|x - u\| \tag{4}$$

for all $n \in \mathbb{N}$. Since $x_{n+1} \in C_n$ and $t_n > 0$, there exists $m \in \mathbb{N}$, $\{\lambda_i\}_{i=0}^m \in \Delta^m$ and $\{y_i\}_{i=0}^m \subset C$ such that

$$\left\| x_{n+1} - \sum_{i=0}^m \lambda_i y_i \right\| < t_n \tag{5}$$

and

$$\|y_i - T_{\mu_n} y_i\| \le t_n \|x_n - T_{\mu_n} x_n\| \tag{6}$$

for all $i = \{0, 1, \dots, m\}$. Put $r_0 = 2\sup_n \|x_n - u\|$. Since C is bounded, it follows from Lemma 1.1, (5) and (6) that

$$\begin{aligned} & \|x_{n+1} - T_{\mu_n} x_{n+1}\| \\ & \le \left\| x_{n+1} - \sum_{i=0}^m \lambda_i y_i \right\| + \left\| \sum_{i=0}^m \lambda_i y_i - \sum_{i=0}^m \lambda_i T_{\mu_n} y_i \right\| \\ & \quad + \left\| \sum_{i=0}^m \lambda_i T_{\mu_n} y_i - T_{\mu_n} \left(\sum_{i=0}^m \lambda_i y_i \right) \right\| + \left\| T_{\mu_n} \left(\sum_{i=0}^m \lambda_i y_i \right) - T_{\mu_n} x_{n+1} \right\| \\ & \le (2 + r_0) t_n + \gamma^{-1} \left(\max_{0 \le i \le j \le m} (\|y_i - y_j\| - \|T_{\mu_n} y_i - T_{\mu_n} y_j\|) \right) \\ & \le (2 + r_0) t_n + \gamma^{-1} \left(\max_{0 \le i \le j \le m} (\|y_i - T_{\mu_n} y_i\| + \|y_j - T_{\mu_n} y_j\|) \right) \\ & \le (2 + r_0) t_n + \gamma^{-1} (2 r_0 t_n). \end{aligned}$$

This implies that

$$\|x_{n+1} - T_{\mu_n} x_{n+1}\| \to 0. \tag{7}$$

Let $t \in S$. We also have

$$\begin{aligned}
&\|T(t)x_{n+1} - x_{n+1}\| \\
&\le \|T(t)x_{n+1} - T(t)T_{\mu_n}x_{n+1}\| + \|T(t)T_{\mu_n}x_{n+1} - T_{\mu_n}x_{n+1}\| + \|T_{\mu_n}x_{n+1} - x_{n+1}\| \\
&\le 2\|T_{\mu_n}x_{n+1} - x_{n+1}\| + \|T(t)T_{\mu_n}x_{n+1} - T_{\mu_n}x_{n+1}\|. \qquad (8)
\end{aligned}$$

We also know from Lemma 1.2 that

$$\lim_{n\to\infty} \sup_{y\in C} \|T_{\mu_n}y - T(t)T_{\mu_n}y\| = 0.$$

So, by (7)and (8) we have

$$\lim_{n\to\infty} \|T(t)x_{n+1} - x_{n+1}\| = 0. \qquad (9)$$

for each $t \in S$.

Since $T(t)$ is nonexpansive, $T(t)$ is demiclosed. So, we have that if $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $w_0 \in C$, then $w_0 \in F(T(t))$ for each $t \in S$.

Finally, we prove that $x_n \to u$. Since $x_{n_i} \rightharpoonup w_0$ and the norm $\|\cdot\|$ is weakly lower semicontinuous, by (4) we also obtain

$$\|x - u\| \le \|x - w_0\| \le \varliminf_{i\to\infty} \|x - x_{n_i}\| \le \varlimsup_{i\to\infty} \|x - x_{n_i}\| \le \|x - u\|. \qquad (10)$$

This implies that $u = w_0$ and hence $x_{n_i} \rightharpoonup u$. Therefore, we have $x_n \rightharpoonup u$. By (10), we also have

$$\lim_{n\to\infty} \|x - x_n\| = \|x - u\|.$$

Since E is uniformly convex, we have $x_n - x \to x - u$ and hence $x_n \to u$. □

3. Applcations

Throughout this section, we assume that C is a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E. Using Theorem 2.2, we can prove some strong convergence theorems as in [25].

Theorem 3.1. *Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x \in C$. Consider the following iteration scheme:*

$$\begin{aligned}
&x_1 = x \in C, \\
&C_n = \overline{\text{co}}\left\{ z \in C : \left\| z - \frac{1}{n}\sum_{i=1}^{n} T^i z \right\| \le t_n \left\| x_n - \frac{1}{n}\sum_{i=1}^{n} T^i x_n \right\| \right\}, \\
&D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\}, \\
&x_{n+1} = P_{C_n \cap D_n}(x_1) \qquad (11)
\end{aligned}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n\cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Theorem 3.2. *Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x \in C$. Let $\{q_{n,m} : n, m \in \mathbb{N}\}$ be a sequence of real numbers such that $q_{n,m} \ge 0$, $\sum_{m=0}^{\infty} q_{n,m} = 1$ for each $n \in \mathbb{N}$ and $\lim_n \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0$. Consider the following* *iteration scheme:*

$$x_1 = x \in C,$$
$$C_n = \overline{\text{co}}\left\{z \in C : \left\|z - \sum_{m=0}^{\infty} q_{n,m} T^m z\right\| \le t_n \left\|x_n - \sum_{m=0}^{\infty} q_{n,m} T^m x_n\right\|\right\},$$
$$D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\},$$
$$x_{n+1} = P_{C_n \cap D_n}(x_1) \tag{12}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n\cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Theorem 3.3. *Let T and U be nonexpansive mappings from C into itself such that $TU = UT$ and $F(T) \cap F(U) \neq \emptyset$ and let $x \in C$. Consider the following iteration scheme:*

$$x_1 = x \in C,$$
$$C_n = \overline{\text{co}}\left\{z \in C : \left\|z - \frac{1}{(n+1)^2} \sum_{i,j=0}^{n} T^i U^j z\right\| \le t_n \left\|x_n - \frac{1}{(n+1)^2} \sum_{i,j=0}^{n} T^i U^j x_n\right\|\right\},$$
$$D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\},$$
$$x_{n+1} = P_{C_n \cap D_n}(x_1) \tag{13}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n\cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(T)\cap F(U)}x$, where $P_{F(T)\cap F(U)}$ is the metric projection from E onto $F(T) \cap F(U)$.

Theorem 3.4. *Let $\mathcal{S} = \{T(t) : t \in [0,\infty)\}$ be a nonexpansive semigroup on C such that the functions $t \mapsto \langle T(t)x, x^*\rangle$, $t \mapsto \|T(t)x - y\|$ are measurable for each $x, y \in C$ and $x^* \in E^*$ and $\bigcap_{t\in\mathbb{R}^+} F(T(t)) \neq \emptyset$. Let $x \in C$ and let $\{s_n\}$ be a sequence of positive real numbers with $s_n \to \infty$. Consider the following iteration scheme:*

$$x_1 = x \in C,$$
$$C_n = \overline{\text{co}}\left\{z \in C : \left\|z - \frac{1}{s_n}\int_0^{s_n} T(t)z\,dt\right\| \le t_n \left\|x_n - \frac{1}{s_n}\int_0^{s_n} T(t)x_n\,dt\right\|\right\},$$
$$D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\},$$
$$x_{n+1} = P_{C_n \cap D_n}(x_1) \tag{14}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n\cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection from E onto $F(\mathcal{S})$.

Theorem 3.5. *Let $\mathcal{S}$ be as in Theorem 3.4 and let $x \in C$. Let $\{r_n\}$ be a sequence of positive real numbers with $r_n \to 0$. Consider the following iteration scheme:*

$$x_1 = x \in C,$$
$$C_n = \overline{\text{co}}\left\{z \in C : \left\|z - r_n\int_0^{\infty} e^{-r_n t} T(t)z\,dt\right\| \le t_n \left\|x_n - r_n\int_0^{\infty} e^{-r_n t} T(t)x_n\,dt\right\|\right\},$$
$$D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\},$$
$$x_{n+1} = P_{C_n \cap D_n}(x_1) \tag{15}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n\cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection from E onto $F(\mathcal{S})$.

Theorem 3.6. *Let $\mathcal{S}$ be as in Theorem 3.4 and let $x \in C$. Let $\{q_n\}$ be a sequence of measurable functions from $[0,\infty)$ into itself such that $\int_0^\infty q_n(t)\,dt = 1$ for each $n \in \mathbb{N}$, $\lim_n q_n(t) = 0$ for almost every $t \geq 0$, $\lim_n \int_0^\infty |q_n(t+s) - q_n(t)|\,dt = 0$ for all $s \geq 0$ and there exists $r \in L^1_{\mathrm{loc}}[0,\infty)$ such that $\sup_n q_n(t) \leq r(t)$ for almost every $t \geq 0$, where $r \in L^1_{\mathrm{loc}}[0,\infty)$ means the restriction of r on $[0,s]$ belongs to $L^1[0,s]$ for each $s > 0$. Consider the following iteration scheme:*

$$\begin{aligned} &x_1 = x \in C, \\ &C_n = \overline{\mathrm{co}}\left\{ z \in C : \left\| z - \int_0^\infty q_n(t)T(t)z\,dt \right\| \leq t_n \left\| x_n - \int_0^\infty q_n(t)T(t)x_n\,dt \right\| \right\}, \\ &D_n = \{ z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0 \}, \\ &x_{n+1} = P_{C_n \cap D_n}(x_1) \end{aligned} \tag{16}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 0$ as $n \to 0$ and $P_{C_n \cap D_n}$ is the metric projection of E onto $C_n \cap D_n$. Then, $\{x_n\}$ converges strongly to the element $P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection from E onto $F(\mathcal{S})$.

References

1. S. Atsushiba, N. Shioji and W. Takahashi, *Approximating common fixed points by the Mann iteration procedure in Banach spaces,* J. Nonlinear Convex Anal. **1** (2000), 351–361.
2. S. Atsushiba and W. Takahashi, *Approximating common fixed points of nonexpansive semigroups by the Mann iteration process*, Ann. Univ. Mariae Curie-Skłodowska **51** (1997), 1–16.
3. S. Atsushiba and W. Takahashi, *Approximating common fixed points of two nonexpansive mappings in Banach Spaces*, Bull. Austral. Math. Soc. **57** (1998), 117-127.
4. S. Atsushiba and W. Takahashi, *A weak convergence theorem for nonexpansive semigroups by the Mann iteration process in Banach spaces,* in Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), pp. 102–109, World Scientific, Singapore, 1999.
5. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.
6. R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1981), 304-314.
7. R. De Marr, *Common fixed pointsfor commuting contraction mappings*, Pacific J. Math. **13** (1963), 1139-1141.
8. N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269-1281.
9. K. Kido and W. Takahashi, *Mean ergodic theorems for semigroups of linear continuous operators in Banach spaces*, J. Math. Anal. Appl. **103** (1984), 387–394.
10. K. Kido and W. Takahashi, *Means on commutative semigroups and nonlinear ergodic theorems*, J. Math. Anal. Appl. **111** (1985), 585–605.
11. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math.Soc. **4** (1953), 506–510.
12. S. Matsushita and W.Takahashi, *Approximating fixed points of nonexpansive mappings in a Banach space by metric projections*, Appl. Math. Comput. **196** (2008), 422-245.
13. K. Nakajo and W.Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372-379.
14. S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.
15. G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172–178.
16. T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings,* J. Math. Anal. Appl. **211** (1997), 71-83.
17. N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Banach spaces*, J. Nonlinear Convex Anal. **1** (2000), 73–87.
18. T. Suzuki, *Strong convergence theorem to common fixed points of two nonexpansive mappings in general Banach spaces*, J. Nonlinear Convex Anal. **3** (2002), 381–391.
19. T. Suzuki and W. Takahashi, *Strong convergence of Mann's type sequences for one-parameter nonexpansive semigroups in general Banach spaces*, J. Nonlinear Convex Anal. **5** (2004), 209–216.

20. W. Takahashi, *Fixed point theorem for amenable semigroups of nonexpansive mappings,* Kodai Math. Sem. Rep. **21** (1969), 383-386.
21. W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space,* Proc. Amer. Math. Soc. **81** (1981), 253-256.
22. W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space,* Proc. Amer. Math. Soc. **96** (1986), 55-58.
23. W. Takahashi, *Fixed point theorem and nonlinear ergodic theorem for nonexpansive semigroups without convexity,* Canad. J. Math. **44** (1992), 880-887.
24. W. Takahashi, *Fixed point theorems and nonlinear ergodic theorems for nonlinear semigroups and their applications,* Nonlinear Anal. **30** (1997), 1287-1293.
25. W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
26. H.K. Xu, *Strong convergence of approximating fixed point sequences for nonexpansive mappings*, Bull. Austral. Math. Soc. **74** (2006), 143–151.

(S. Atsushiba) Department of Mathematics and Physics, Interdisciplinary Sciences Course, Faculty of Education and Human Sciences, University of Yamanashi, 4-4-37 Takeda Kofu, Yamanashi 400-8510, JAPAN

E-mail address: asachiko@yamanashi.ac.jp

(W. Takahashi) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, O-okayama, Meguro-ku, Tokyo 152-8552, Japan

E-mail address: wataru@is.titech.ac.jp