TWO PERSON GAMES ON SALE IN WHICH THE PRICE FLUCTUATES WITH TIME

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Abstract. We consider a class of games which is suggested from a timing problem for putting some kind of farm products on the market. Two players, Player I and II, take possession of the right to put some kind of farm products on the market with even ratio. Each of the players can put the farm products at any time in $[0, 1]$. The price of them increases over $[0, m] \subset [0, 1]$ and decreases over $(m, 1]$ with pass time t so long as both of the players do not sell them, however if one of the two players puts his farm products on the market, the price falls discontinuously and then fluctuates analogously as before. Both players have to put their farm products on the market within the unit interval $[0, 1]$. In such a situation, each player wishes to put at the optimal time which gives him the highest price, considering opponents action time with each other. This model yields us a certain class of two person non-zero sum infinite games on the unit square.

1 Introduction We consider a class of games which is suggested from the correlative phenomena between the price fluctuations and supply in a market on farm products. Two players, Player I and II, take possession of the right to put some kind of farm products on the market with even ratio. We call such kind of products product A in this paper. We can harvest product A at a specific season every year periodically. Each of the two players wants to decide the optimal time to put his product A on the market until the next harvest season. We consider one time period where the harvest time in each year is the beginning and the next harvest time is the end. The price of product A increases smoothly until some point and then decreases with time as long as the both players don't put on the market and keep their own products. But, when one of the players puts his product A on the market, the price of product A possessed by his opponent falls discontinuously and then fluctuates with time analogously as before until his opponent puts the rest on the market. In such a situation, each player has to decide the optimal action time considering the current price and his opponent's action time, with each other.

This problem is applicable to the correlation phenomena between the price and supply on land, not only to the problem of farm products. As well as the usual games of timing [1,2], we have to introduce two patterns of information available to the players. If a player is informed of his opponent's action time as soon as his opponent put product A on the market, we say they are in a noisy version. If neither player learns when nor whether his opponent has put product A on the market, we say both players are in a silent version. We shall discuss three cases according to the information patterns mentioned above, as follows:

1. Both players are in a noisy version. We call this case noisy game.

2. Both players are in a silent version. We call this case silent game.

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3. Player Iis informed of II's action time whereas II does not learn when or whether I has already put his product A on the market, that is, I is the silent player and II is the noisy player. We call this case silent-noisy game.

We consider the silent game and the noisy game in this paper. Related to our models, there are two works. Teraoka and Yamada considered two person games on rivalry over territory [3] and Teraoka and Hohjo extended it to n person games [4]. Also Teraoka and Hohjo proposed and analyzed the games on sale in which the price is an increasing function with respect to pass time [5,6].

2 Notations and Assumptions Since we consider one period game, we express the period as the unit interval $[0, 1]$. Throughout of this paper, we use the following notations: $v(t)$ is the price of product A at time $t \in [0, 1]$, when both players don't put their product A. We assume that $v(t)$ is differentiable and

$$v'(t) \left\{ \begin{array}{c} \geq \\ < \end{array} \right\} 0 \quad \text{for} \quad \left\{ \begin{array}{c} 0 \leq t \leq m \\ m < t \leq 1 \end{array} \right\},$$

that is, $v(t)$ is a unimodal function with respect to t. And we assume that $v(t)$ is known to both players.

r is the discount factor after one of the players already puts product A on the market and is assumed $0 < r < 1$. That is, if one of the players sells his product A at time $t \in [0, 1]$, the price of his opponent's falls down from $v(t)$ to $rv(t)$ immediately. It is natural to assume $0 < v(0) < \infty$.

Here we also assume the following. If both of the players put their product A at a same time $t \in [0, 1]$, each of the both players has to sell his product A at the price after fall $rv(t)$.

Throughout this paper we use notations on the expectation for real valued function $M_i(x, y)$ defined on the unit square when Player I and II employ mixed strategies (*cdfs*) on $[0, 1]$ $F(x)$ and $G(y)$, respectively, as follows:

$$M_i(F, G) = \int_0^1 \int_0^1 M_i(x, y) dF(x) dG(y)$$

and

$$M_i(x, G) = \int_0^1 M_i(x, y) dG(y); \quad M_i(F, y) = \int_0^1 M_i(x, y) dF(x).$$

3 Silent Game Here, we deal with the case where both players are in a silent version. Since both players can't learn when nor whether his opponent has acted and each player is informed of the current price $\bar{v}(t)$ immediately after he has sold, we establish the pure strategies for Player I and II as $x \in [0, 1]$ and $y \in [0, 1]$, respectively. Then the expected payoff kernels $M_1(x, y)$ for I and $M_2(x, y)$ for II are given as follows:

$$M_1(x, y) = \left\{ \begin{array}{ll} v(x), & 0 \leq x < y \\ rv(x), & y \leq x \leq 1 \end{array} \right. ; \tag{1}$$

$$M_2(x, y) = \left\{ \begin{array}{ll} v(y), & 0 \leq y < x \\ rv(y), & x \leq y \leq 1 \end{array} \right. . \tag{2}$$

Observing the above payoff kernels, we can't find any Nash equilibrium points in the class of pure strategies. Hence we try to find them from a certain class of mixed strategies. Since $v(x)$ is a unimodal function which has the maximal value at point $x = m \in [0, 1]$, we suppose that Player I and II use same mixed strategy (*cdf*) $F(x)$ which consists of density part $f(x) > 0$ over an interval (a, m), where $0 \leq a < m$, that is,

$$F(x) = \begin{cases} 0, & 0 \leq x < a \\ \int_a^x f(t)dt, & a \leq x < m \\ 1, & m \leq x \leq 1 \end{cases} . \tag{3}$$

Supposing that Player I uses pure strategy x and Player II employs mixed strategy given by (3), we have the expected payoff kernel to Player I $M_1(x, F)$ as follows:

$$M_1(x, F) = \begin{cases} v(x), & 0 \leq x < a \\ v(x)[1 - (1-r)F(x)], & a \leq x < m \\ rv(x), & m \leq x \leq 1 \end{cases} .$$

In a similar way, we get

$$M_2(F, y) = \begin{cases} v(y), & 0 \leq y < a \\ v(y)[1 - (1-r)F(y)], & a \leq y < m \\ rv(y), & m \leq y \leq 1 \end{cases} .$$

Putting

$$M_1(x, F) = \text{const} \quad \text{for } x \in (a, m),$$

we have

$$v'(x)[1 - (1-r)F(x)] = (1-r)f(x)v(x) > 0, \quad a < x < m.$$

Then we get

$$F(x) = \{1/(1-r)\}[1 - \{c/v(x)\}], \quad a < x < m,$$

where c is an integration constant. Since $F(x)$ has to satisfy the boundary value conditions

$$F(a) = 0 \quad \text{and} \quad F(m) = 1,$$

both of the following equalities have to hold:

$$c = rv(m); \quad v(a) = rv(m).$$

The latter equation has its solution, provided that the next inequality holds.

$$v(0) \leq rv(m).$$

Hence, we consider the case where $v(0) \leq rv(m)$ first. Since the equation $v(a) = rv(m)$ has the unique root we denote it by a^0. Then the following relations hold:

$$M_1(x, F) = \begin{cases} v(x) < v(a^0) = rv(m), & 0 \leq x < a^0 \\ v(a^0) = rv(m), & a^0 \leq x \leq m \\ rv(x) < rv(m), & m < x \leq 1 \end{cases} .$$

After all, we obtain Theorem 1.

Theorem 1. Assume that $v(0) \le rv(m)$, and let a^0 be the unique root of the equation in the interval $[0, m]$. Also consider the following mixed strategy (*cdf*) given by

$$F^0(x) = \begin{cases} 0, & 0 \le x < a^0 \\ \{1/(1-r)\}[1 - \{v(a^0)/v(x)\}], & a^0 \le x < m \\ 1, & m \le x \le 1 \end{cases} .$$

Then the pair of mixed strategy (F^0, F^0) constitutes a Nash equilibrium point of two person non-zero sum game given by (1) and (2). And the corresponding equilibrium values v_1 to Player I and v_2 to II are given as

$$v_1 = M_1(F^0, F^0) = rv(m) \ ; \quad v_2 = M_2(F^0, F^0) = rv(m).$$

According to Theorem 1, each player is forced to concentrate his probability to take his action over the interval where the price of product A increases under the equilibrium, irrespective of the concrete shape of function $v(t)$.

Now we consider the case where $v(0) > rv(m)$. If we consider

$$F(x) = \begin{cases} \{1/(1-r)\}[1 - \{v(0)/v(x)\}], & 0 \le x < m \\ 1, & m \le x \le 1 \end{cases} ,$$

the following relations hold:

$$F(0) = \{1/(1-r)\}[1 - \{v(0)/v(0)\}] = 0,$$

and

$$\begin{aligned} F(m) &= \{1/(1-r)\}[1 - \{v(0)/v(m)\}] \\ &< \{1/(1-r)\}[1 - \{rv(m)/v(m)\}] = 1. \end{aligned}$$

Putting α as

$$\alpha = 1 - \{1/(1-r)\}[1 - \{v(0)/v(m)\}],$$

we obtain

$$M_1(x, F) = \begin{cases} v(0), & 0 \le x < m \\ v(a) = rv(m), & x = m \\ rv(x) < rv(m) < v(0), & m < x \le 1 \end{cases} .$$

Thus if we consider the following *cdf* $F^*(x)$:

$$F^*(x) = \begin{cases} \{1/(1-\alpha)\}\{1/(1-r)\}[1 - \{v(0)/v(x)\}], & 0 \le x < m \\ 1, & m \le x \le 1 \end{cases} ,$$

$F^*(x)$ satisfies

$$F^*(0) = 0; \quad F^*(m) = 1,$$

and then

$$\begin{aligned} v(x)[1 - (1-r)F^*(x)] &= v(x)\{1/(1-\alpha)\}\{v(0)/v(x)\} \\ &= \{1/(1-\alpha)\}v(0), \quad 0 \le x < m. \end{aligned}$$

Hence we have

$$M_1(x,F) = \begin{cases} \{1/(1-\alpha)\}v(0), & 0 \le x < m \\ rv(m), & x = m \\ rv(x) < rv(m) < v(0), & m < x \le 1 \end{cases}.$$

After all Theorem 2 holds.

Theorem 2. Assume that $v(0) > rv(m)$, and let α be

$$\alpha = 1 - \{1/(1-r)\}[1 - \{v(0)/v(m)\}] > 0.$$

Then we consider the following *cdf* $F^*(x)$:

$$F^*(x) = \begin{cases} \{1/(1-\alpha)\}\{1/(1-r)\}[1 - \{v(0)/v(x)\}], & 0 \le x < m \\ 1, & m \le x \le 1 \end{cases}.$$

Then the pair of mixed strategy (F^*, F^*) constitutes a Nash equilibrium point of two person non-zero sum game given by (1) and (2). And the corresponding equilibrium values v_1 to Player I and v_2 to II are given as

$$v_1 = M_1(F^*, F^*) = \{1/(1-\alpha)\}v(0); \quad v_2 = M_2(F^*, F^*) = \{1/(1-\alpha)\}v(0),$$

respectively.

4 Noisy Game In this section we consider noisy game, that is, both of the two players are noisy players. Since a player is informed of his opponent's action time as soon as his opponent put product A on the market, we establish the pure strategy of Player I as $x \in [0,1]$. It means that Player I select point $x \in [0,1]$ in advance, and then if he can observe his opponent takes the action before x he acts at the time which maximizes the price $v(t)$, and conversely if he could not find Player II put the product A until time x he take his action at the time x. In a similar fashion we can establish the pure strategy of Player II as $y \in [0,1]$.

Then, the expected payoff kernels $M_1(x,y)$ to I and $M_2(x,y)$ to II are given as

$$M_1(x,y) = \begin{cases} v(x), & 0 \le x < y \\ r \max\limits_x v(x), & y \le x \le 1 \end{cases};$$

$$M_2(x,y) = \begin{cases} v(y), & 0 \le y < x \\ r \max\limits_y v(y), & x \le y \le 1 \end{cases}.$$

Related to the above payoff kernels, if we assume $0 \le x_1 \le m \le x_2 \le 1$ we get

$$M_1(x_1,y) = \begin{cases} v(x_1), & 0 \le x_1 < y \\ rv(m), & y \le x_1 \le 1 \end{cases}; \; M_1(x_2,y) = \begin{cases} v(x_2), & 0 \le x_2 < y \\ rv(y), & y \le x_2 \le 1 \end{cases},$$

and then Player I is forced to choose his action time x before m. Similarly, Player II has to choose his action time y before m. Therefore, we consider the non-zero sum two person game given by the following expected payoffs:

$$M_1(x,y) = \begin{cases} v(x), & 0 \le x < y \\ rv(m), & y \le x \le 1 \end{cases}; \tag{4}$$

$$M_2(x,y) = \begin{cases} v(y), & 0 \le y < x \\ rv(m), & x \le y \le 1 \end{cases}. \tag{5}$$

Observing (4), (5) and the analysis of silent game, we can suppose that the equilibrium point is determined by considering the relation of $v(0)$ and $rv(m)$.

First, we consider the case where $v(0) \le rv(m)$. Let a be the unique root of the equation $v(a) = rv(m)$ in the interval $[0, m]$. Then if Player II chooses pure strategy a, we have

$$M_1(x, a) = \begin{cases} v(x) < v(a) = rv(m), & 0 \le x < a \\ rv(a) < rv(m), & x = a \\ rv(m), & a < x \le m \\ rv(x) < rv(m), & m < x \le 1 \end{cases}. \tag{6}$$

Therefore, it is optimal for Player I to select point a. Similarly, if Player I chooses pure strategy a, we also obtain

$$M_2(a, y) = \begin{cases} v(y) < v(a) = rv(m), & 0 \le y < a \\ rv(a) < rv(m), & y = a \\ rv(m), & a < y \le m \\ rv(y) < rv(m), & m < y \le 1 \end{cases}. \tag{7}$$

The two relations of (6) and (7) means that the pair (a, a) cannot constitute Nash equilibrium, however, give the candidate of an equilibrium by means of limit for the response to avoid that both players act at the same time. Here, we consider the following mixed strategy.

For any $\varepsilon > 0$ we choose $\delta \in (0, m - a)$ which satisfies $v(a + \delta) - v(a) < \varepsilon$ and then define *cdf* $H_a(x)$ as

$$H_a(x) = \begin{cases} 0, & 0 \le x < a \\ \int_a^x (1/\delta) dt, & a \le x \le a + \delta \\ 1, & a + \delta < x \le 1 \end{cases}.$$

So we get the following relation on the expected payoff for *cdf* $H_a(x)$:

$$M_2(H_a(x), y) = \begin{cases} v(y) < rv(m), & 0 \le y < a \\ rv(m)\{(y - a)/\delta\} + v(y)\{(a + \delta - y)/\delta\}, & a \le y \le a + \delta \\ rv(m), & a + \delta < y \le m \\ rv(y) < rv(m), & m < y \le 1 \end{cases}.$$

For the case of $a \le y \le a + \delta$, since $v(a) = rv(m)$ we get

$$\begin{aligned} & rv(m)\{(y - a)/\delta\} + v(y)\{(a + \delta - y)/\delta\} \\ & \le \; rv(m)\{(y - a)/\delta\} + v(a + \delta)\{(a + \delta - y)/\delta\} \\ & \le \; rv(m)\{(y - a)/\delta\} + \{v(a) + \varepsilon\}\{(a + \delta - y)/\delta\} \\ & \le \; rv(m) + \varepsilon\{(a + \delta - y)/\delta\} \\ & \le \; rv(m) + \varepsilon. \end{aligned}$$

Thus we have

$$M_2(H_a(x), y) \le rv(m) + \varepsilon, \quad \text{for all } y \in [0, 1].$$

We also get

$$\begin{aligned} & rv(m)\{(y - a)/\delta\} + v(y)\{(a + \delta - y)/\delta\} \\ & \ge \; rv(m)\{(y - a)/\delta\} + v(a)\{(a + \delta - y)/\delta\} \\ & = \; rv(m) \quad \text{for all } y \in [a, a + \delta], \end{aligned}$$

the following inequality holds:

$$rv(m) \le M_2(H_a(x), y) \le rv(m) + \varepsilon, \quad \text{for all } y \in [a, a+\delta].$$

Similar arguments give

$$M_1(x, H_a(y)) \le rv(m) + \varepsilon, \quad \text{for all } x \in [0, 1]$$

and so

$$rv(m) \le M_1(x, H_a(y)) \le rv(m) + \varepsilon, \quad \text{for all } x \in [a, a+\delta].$$

After all, we have Theorem 3.

Theorem 3. Assume that $v(0) \le rv(m)$, and let a be the unique root of equation $v(a) = rv(m)$ in the interval $[0, m]$. Then for any $\varepsilon > 0$ we choose $\delta \in (0, m-a)$ which satisfies $v(a+\delta) - v(a) < \varepsilon$ and consider *cdf* $H_a(x)$ given as

$$H_a(x) = \begin{cases} 0, & 0 \le x < a \\ \int_a^x (1/\delta) dt, & a \le x \le a+\delta \\ 1, & a+\delta < x \le 1 \end{cases}.$$

So we have

$$\begin{aligned} M_1(x, H_a(y)) &\le rv(m) + \varepsilon, \quad \text{for all } x \in [0,1]; \\ M_2(H_a(x), y) &\le rv(m) + \varepsilon, \quad \text{for all } y \in [0,1] \end{aligned}$$

and

$$\begin{aligned} rv(m) \le M_1(x, H_a(y)) &\le rv(m) + \varepsilon, \quad \text{for all } x \in [a, a+\delta]; \\ rv(m) \le M_2(H_a(x), y) &\le rv(m) + \varepsilon, \quad \text{for all } y \in [a, a+\delta]. \end{aligned}$$

Next we consider the case where $v(0) > rv(m)$. As well as the previous case, we consider the following mixed strategy.

For any $\varepsilon > 0$ we choose $\delta \in (0, m)$ which satisfies $v(\delta) - v(0) < \varepsilon$, and then define *cdf* $H_0(x)$ as

$$H_0(x) = \begin{cases} \int_0^x (1/\delta) dt, & 0 \le x \le \delta \\ 1, & \delta < x \le 1 \end{cases}.$$

Then we get the following relation on the expected payoff for *cdf* $H_0(x)$:

$$M_2(H_0(x), y) = \begin{cases} rv(m)(y/\delta) + v(y)\{(\delta - y)/\delta\}, & 0 \le y \le \delta \\ rv(m) < v(0), & \delta < y \le m \\ rv(y) < rv(m) < v(0), & m < y \le 1 \end{cases}.$$

First we consider the case where $0 \le y \le \delta$, since $v(0) \ge rv(m)$ we get

$$\begin{aligned}
&rv(m)(y/\delta) + v(y)\{(\delta - y)/\delta\} \\
&\quad\le\quad rv(m)(y/\delta) + v(\delta)\{(\delta - y)/\delta\} \\
&\quad\le\quad rv(m)(y/\delta) + \{v(0) + \varepsilon\}\{(\delta - y)/\delta\} \\
&\quad\le\quad v(0) + \varepsilon\{(\delta - y)/\delta\} \\
&\quad\le\quad v(0) + \varepsilon,
\end{aligned}$$

and then the following relation holds:

$$M_2(H_0(x), y) \le v(0) + \varepsilon, \quad \text{for all } y \in [0, 1].$$

We also get

$$\begin{aligned}
&rv(m)(y/\delta) + v(y)\{(\delta - y)/\delta\} \\
&\quad\ge\quad rv(m)(y/\delta) + v(0)\{(\delta - y)/\delta\} \\
&\quad\ge\quad rv(m).
\end{aligned}$$

Then the following inequalities hold

$$rv(m) \le M_2(H_0(x), y) \le v(0) + \varepsilon, \quad \text{for all } y \in [0, \delta].$$

Almost same arguments give

$$M_1(x, H_0(y)) \le v(0) + \varepsilon, \quad \text{for all } x \in [0, 1]$$

and

$$rv(m) \le M_1(x, H_0(y)) \le v(0) + \varepsilon, \quad \text{for all } x \in [0, \delta].$$

Now then we suppose that both of the two players use *cdf* $H_0(x)$ as their mixed strategies, we have

$$\{rv(m) + v(0)\}/2 < \int_0^\delta M_2(H_0(x), y)(1/\delta)dy < \{rv(m) + v(\delta)\}/2.$$

And then the following inequalities hold:

$$\{v(0) + rv(m)\}/2 < M_2(H_0(x), H_0(y)) < \{v(0) + rv(m)\}/2 + \varepsilon.$$

Similar arguments give

$$\{v(0) + rv(m)\}/2 < M_1(H_0(x), H_0(y)) < \{v(0) + rv(m)\}/2 + \varepsilon.$$

After all we obtain Theorem 4.

Theorem 4. Assume that $v(0) > rv(m)$. Consider the following mixed strategy: For any $\delta \in (0, m)$ we define *cdf* $H_0(x)$ given by

$$H_0(x) = \begin{cases} \int_0^x (1/\delta)dt, & 0 \le x \le \delta \\ 1, & \delta < x \le 1 \end{cases}.$$

Then we have

$$\begin{aligned}
M_1(x, H_0(y)) \le v(0) + \varepsilon, &\quad \text{for all } x \in [0, 1]; \\
M_2(H_0(x), y) \le v(0) + \varepsilon, &\quad \text{for all } y \in [0, 1]
\end{aligned}$$

and

$$rv(m) \le M_1(x, H_0(y)) \le v(0) + \varepsilon, \quad \text{for all } x \in [0, \delta];$$
$$rv(m) \le M_2(H_0(x), y) \le v(0) + \varepsilon, \quad \text{for all } y \in [0, \delta].$$

Furthermore, both of the two players use *cdf* $H_0(x)$ as their mixed strategies, the following inequalities hold

$$\{v(0) + rv(m)\}/2 < M_1(H_0(x), H_0(y)) < \{v(0) + rv(m)\}/2 + \varepsilon;$$
$$\{v(0) + rv(m)\}/2 < M_2(H_0(x), H_0(y)) < \{v(0) + rv(m)\}/2 + \varepsilon.$$

5 Concluding Remarks We proposed and analyzed a new model on games of timing. According to the previous sections, we find the silent game is essentially different from the noisy game. If we observe the realistic phenomena, it is more interest for us to consider the noisy game, however, our results gives us ε-equilibrium but not Nash equilibrium.

We supposed the discount factor r is constant over the interval $[0, 1]$. Observing the real sale problem, it may be natural to generalize our model to the case where r is a function of the pass time $t \in [0, 1]$. We also have to consider more realistic models, even if it is complicate to formulate the models and difficult to analyze them.

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